MATHEMATICAL ECONOMICS

by

Dr. VELU SURESH KUMAR, MA., M.Phil., Ph.D., MBA.,
Assistant Professor & Head
PG & Research Department of Economics
H.H. The Rajah's College (Autonomous),
Pudukkottai, Tamil Nadu – 622 001

Unit – I

DIFFERENTIAL CALCULUS AND INTEGRATION

Rules of Differentiation

1. **Function in power terms**

   Let us suppose that the functional relationship between x and y is of the form: $y = ax^n$, where a is some constant term.

   In this case $dy/dx = anx^{n-1}$

Example

Differentiate

(a) $y = 2x^4$ (b) $y = 10x$

Solution

(a) To differentiate $2x^4$ we first differentiate x^4 to get $4x^3$ and then multiply by 2. Hence

$$\text{if } y = 2x^4 \quad \text{then} \quad \frac{dy}{dx} = 2(4x^3) = 8x^3$$

(b) To differentiate $10x$ we first differentiate x to get 1 and then multiply by 10. Hence

$$\text{if } y = 10x \quad \text{then} \quad \frac{dy}{dx} = 10(1) = 10$$

2. **Differentiation of a constant**

   Suppose the functional relationship is of the form y = C

   In this case $dy/dx = 0$

Example

1. Differentiate

   (a) $y = 4x^3$ (b) $y = 2/x$

The constant rule can be used to show that

constants differentiate to zero

To see this, note that the equation

$$y = c$$

is the same as

$$y = cx^0$$

because $x^0 = 1$. By the constant rule we first differentiate x^0 to get $0x^{-1}$ and then multiply by c. Hence

$$\text{if } y = c \quad \text{then} \quad \frac{dy}{dx} = c(0x^{-1}) = 0$$

3. Differentiation of Sum

Example

Differentiate

(a) $y = x^2 + x^{50}$ **(b)** $y = x^3 + 3$

Solution

(a) To differentiate $x^2 + x^{50}$ we need to differentiate x^2 and x^{50} separately and add. Now

x^2 differentiates to $2x$

and

x^{50} differentiates to $50x^{49}$

so

$$\text{if } y = x^2 + x^{50} \quad \text{then} \quad \frac{dy}{dx} = 2x + 50x^{49}$$

(b) To differentiate $x^3 + 3$ we need to differentiate x^3 and 3 separately and add. Now

x^3 differentiates to $3x^2$

and

3 differentiates to 0 (constants differentiate to zero)

so

$$\text{if } y = x^3 + 3 \quad \text{then} \quad \frac{dy}{dx} = 3x^2 + 0 = 3x^2$$

4. Differentiation of Difference

Example

Differentiate

(a) $y = x^5 - x^2$ (b) $y = x - \frac{1}{x^2}$

Solution

(a) To differentiate $x^5 - x^2$ we need to differentiate x^5 and x^2 separately and subtract. Now

x^5 differentiates to $5x^4$

and

x^2 differentiates to $2x$

so

$$\text{if } y = x^5 - x^2 \quad \text{then} \quad \frac{dy}{dx} = 5x^4 - 2x$$

(b) To differentiate $x - \frac{1}{x^2}$ we need to differentiate x and $\frac{1}{x^2}$ separately and subtract. Now

x differentiates to 1

and

$\frac{1}{x^2}$ differentiates to $-\frac{2}{x^3}$

x^{-2} differentiates to $-2x^{-3}$

so

$$\text{if } y = x - \frac{1}{x^2} \quad \text{then} \quad \frac{dy}{dx} = 1 - \left(-\frac{2}{x^3}\right) = 1 + \frac{2}{x^3}$$

Example

Differentiate

(a) $y = 3x^5 + 2x^3$ **(b)** $y = x^3 + 7x^2 - 2x + 10$ **(c)** $y = 2\sqrt{x} + \frac{3}{x}$

Solution

(a) The sum rule shows that to differentiate $3x^5 + 2x^3$, we need to differentiate $3x^5$ and $2x^3$ separately and add. By the constant rule

$3x^5$ differentiates to $3(5x^4) = 15x^4$

and

$2x^3$ differentiates to $2(3x^2) = 6x^2$

so

$$\text{if } y = 3x^5 + 2x^3 \quad \text{then} \quad \frac{dy}{dx} = 15x^4 + 6x^2$$

With practice you will soon find that you can just write the derivative down in a single line of working by differentiating term by term. For the function

$$y = 3x^5 + 2x^3$$

we could just write

$$\frac{dy}{dx} = 3(5x^4) + 2(3x^2) = 15x^4 + 6x^2$$

(b) So far we have only considered expressions comprising at most two terms. However, the sum and difference rules still apply to lengthier expressions, so we can differentiate term by term as before. For the function

$$y = x^3 + 7x^2 - 2x + 10$$

we get

$$\frac{dy}{dx} = 3x^2 + 7(2x) - 2(1) + 0 = 3x^2 + 14x - 2$$

(c) To differentiate

$$y = 2\sqrt{x} + \frac{3}{x}$$

we first rewrite it using the notation of indices as

$$y = 2x^{1/2} + 3x^{-1}$$

Differentiating term by term then gives

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)x^{-1/2} + 3(-1)x^{-2} = x^{-1/2} - 3x^{-2}$$

which can be written in the more familiar form

$$\frac{1}{\sqrt{x}} - \frac{3}{x^2}$$

5. Differentiation of a Product

$$\text{If } y = uv \quad \text{then} \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example

Differentiate

(a) $y = x^2(2x + 1)^3$ (b) $x\sqrt{(6x + 1)}$ (c) $y = \dfrac{x}{1 + x}$

Solution

(a) The function $x^2(2x + 1)^3$ involves the product of two simpler functions, namely x^2 and $(2x + 1)^3$, which we denote by u and v, respectively. (It does not matter which function we label u and which we label v. The same answer is obtained if u is $(2x + 1)^3$ and v is x^2. You might like to check this for yourself later.) Now if

$$u = x^2 \quad \text{and} \quad v = (2x + 1)^3$$

then

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = 6(2x + 1)^2$$

where we have used the chain rule to find dv/dx. By the product rule,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= x^2[6(2x + 1)^2] + (2x + 1)^3(2x)$$

The first term is obtained by leaving u alone and multiplying it by the derivative of v. Similarly, the second term is obtained by leaving v alone and multiplying it by the derivative of u.

(b) The function $x\sqrt{(6x+1)}$ involves the product of the simpler functions

$$u = x \quad \text{and} \quad v = \sqrt{6x+1} = (6x+1)^{1/2}$$

for which

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = \frac{1}{2}(6x+1)^{-1/2} \times 6 = 3(6x+1)^{-1/2}$$

where we have used the chain rule to find dv/dx. By the product rule,

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= x[3(6x+1)^{-1/2}] + (6x+1)^{1/2}(1)\\ &= \frac{3x}{\sqrt{(6x+1)}} + \sqrt{(6x+1)}\end{aligned}$$

If desired, this can be simplified by putting the second term over a common denominator

$$\sqrt{(6x+1)}$$

To do this we multiply the top and bottom of the second term by $\sqrt{6x+1}$ to get

$$\frac{(6x+1)}{\sqrt{(6x+1)}}$$

$$\sqrt{(6x+1)} \times \sqrt{(6x+1)} = 6x+1$$

Hence

$$\frac{dy}{dx} = \frac{3x + (6x+1)}{\sqrt{(6x+1)}} = \frac{9x+1}{\sqrt{(6x+1)}}$$

(c) At first sight it is hard to see how we can use the product rule to differentiate

$$\frac{x}{1+x}$$

since it appears to be the quotient and not the product of two functions. However, if we recall that reciprocals are equivalent to negative powers, we may rewrite it as

$$x(1+x)^{-1}$$

It follows that we can put

$$u = x \quad \text{and} \quad v = (1+x)^{-1}$$

which gives

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = -(1+x)^{-2}$$

where we have used the chain rule to find dv/dx. By the product rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x[-(1+x)^{-2}] + (1+x)^{-1}(1)$$

$$= \frac{-x}{(1+x)^2} + \frac{1}{1+x}$$

If desired, this can be simplified by putting the second term over a common denominator

$$(1+x)^2$$

To do this we multiply the top and bottom of the second term by $1 + x$ to get

$$\frac{1+x}{(1+x)^2}$$

Hence

$$\frac{dy}{dx} = \frac{-x}{(1+x)^2} + \frac{1+x}{(1+x)^2} = \frac{-x+(1+x)}{(1+x)^2} = \frac{1}{(1+x)^2}$$

6. Differentiation of a Quotient

$$\text{If } y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\,du/dx = v\,dv/dx}{v^2}$$

Example

Differentiate

(a) $y = \dfrac{x}{1+x}$ **(b)** $y = \dfrac{1+x^2}{2-x^3}$

Solution

(a) In the quotient rule, u is used as the label for the numerator and v is used for the denominator, so to differentiate

$$\frac{x}{1+x}$$

we must take

$$u = x \quad \text{and} \quad v = 1 + x$$

for which

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = 1$$

By the quotient rule

$$\frac{dy}{dx} = \frac{v du/dx - u du/dx}{v^2}$$

$$= \frac{(1 + x)(1) - x(1)}{(1 + x^2)}$$

$$= \frac{1 + x - x}{(1 + x^2)}$$

$$= \frac{1}{(1 + x^2)}$$

(b) The numerator of the algebraic fraction

$$\frac{1 + x^2}{2 - x^3}$$

is $1 + x^2$ and the denominator is $2 - x^3$, so we take

$$u = 1 + x^2 \quad \text{and} \quad v = 2 - x^3$$

for which

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = -3x^2$$

By the quotient rule

$$\frac{dy}{dx} = \frac{v du/dx - u dv/dx}{v^2}$$

$$= \frac{(2 - x^3)(2x) - (1 + x^2)(-3x^2)}{(2 - x^3)^3}$$

$$= \frac{4x - 2x^4 + 3x^2 + 3x^4}{(2 - x^3)^3}$$

$$= \frac{x^4 + 3x^2 + 4x}{(2 - x^3)^3}$$

Criteria for Maxima and Minima

	Maximum	Minimum
Necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Sufficient condition	$\frac{dy}{dx} = 0\,;\ \frac{d^2y}{dx^2} < 0$	$\frac{dy}{dx} = 0\,;\ \frac{d^2y}{dx^2} > 0$

Example Sum:

Investigate the maxima and minima of the function $2x^3 + 3x^2 - 36x + 10$.

Solution :

Let y $= 2x^3 + 3x^2 - 36x + 10$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 6x^2 + 6x - 36 \qquad \text{.............(1)}$$

$$\frac{dy}{dx} = 0 \Rightarrow 6x^2 + 6x - 36 = 0$$

$$\Rightarrow x^2 + x - 6 = 0$$

$$\Rightarrow (x+3)(x-2) = 0$$

$$\Rightarrow x = -3, 2$$

Again differentiating (1) with respect to x, we get

$$\frac{d^2y}{dx^2} = 12x + 6$$

when $x = -3$, $\frac{d^2y}{dx^2} = 12(-3) + 6 = -30 < 0$

$\therefore$ It attains maximum at $x = -3$

$\therefore$ Maximum value is $y = 2(-3)^3 + 3(-3)^2 - 36(-3) + 10 = 91$

when $x = 2$, $\frac{d^2y}{dx^2} = 12(2) + 6 = 30 > 0$

$\therefore$ It attains minimum at $x = 2$

$\therefore$ Minimum value is $y = 2(2)^3 + 3(2)^2 - 36(2) + 10 = -34$

PARTIAL DERIVATIVES

In differential calculus, so far we have discussed functions of one variable of the form $y = f(x)$. Further one variable may be expressed as a function of several variables. For example, production may be treated as a function of labour and capital and price may be a function of supply and demand. In general, the cost or profit depends upon a number of independent variables, for example, prices of raw materials, wages on labour, market conditions and so on. Thus a dependent variable y depends on a number of independent variables $x_1, x_2, x_3 .. x_n$. It is denoted by $y = f(x_1, x_2, x_3 .. x_n)$ and is called a function of *n* variables.

Definition

Let $u = f(x, y)$ be a function of two independent variables *x* and *y*. The derivative of $f(x, y)$ with respect to *x*, keeping *y* constant, is called partial derivative of *u* with respect to *x* and is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x or u_x. Similarly we can define partial derivative of *f* with respect to *y*.

Thus we have

$$\frac{\partial f}{\partial x} = \underset{\Delta x \to 0}{\text{Lt}} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

provided the limit exists.

(Here y is fixed and Δx is the increment of x)

Also $$\frac{\partial f}{\partial y} = \underset{\Delta y \to 0}{\text{Lt}} = \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limit exists.

(Here x is fixed and Δy is the increment of y)

Example Sums

It $u(x, y) = 1000 - x^3 - y^2 + 4x^3y^6 + 8y$, find each of the following.

(i) $\frac{\partial u}{\partial x}$ **(ii)** $\frac{\partial u}{\partial y}$ **(iii)** $\frac{\partial^2 u}{\partial x^2}$ **(iv)** $\frac{\partial^2 u}{\partial y^2}$ **(v)** $\frac{\partial^2 u}{\partial x \partial y}$ **(vi)** $\frac{\partial^2 u}{\partial y \partial x}$

Solution:

$$u(x, y) = 1000 - x^3 - y^2 + 4x^3y^6 + 8y$$

(i)

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(1000 - x^3 - y^2 + 4x^3y^6 + 8y)$$

$$= 0 - 3x^2 - 0 + 4(3x^2)y^6 + 0$$

$$= -3x^2 + 12x^2y^6$$

(ii)

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(1000 - x^3 - y^2 + 4x^3y^6 + 8y)$$

$$= 0 - 0 - 2y + 4x^3(6y^5) + 8$$

$$= -2y + 24x^3y^5 + 8$$

(iii)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(-3x^2 + 12x^2y^6)$$

$$= -6x + 12(2x)y^6$$

$$= -6x + 24xy^6$$

(iv)

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(-2y + 24x^3y^5 + 8)$$

$$= -2 + 24x^3(5y^4) + 0$$

$$= -2 + 120x^3y^4$$

(v) $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial}\right)$$

$$= \frac{\partial}{\partial x}(-2y + 24x^3y^5 + 8)$$

$$= 0 + 24(3x^2)y^5 + 0$$

$$= 72x^2y^5$$

(vi) $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(-3x^2 + 12x^2y^6)$$

$$= 0 + 12x^2(6y^5) = 72x^2y^5$$

Integration

When a function f(x) is known we can differentiate it to obtain its derivative $\frac{df}{dx}$. The reverse process is to obtain the function f(x) from knowledge of its derivative. This process is called integration.

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UNIT - II

THEORY OF CONSUMER BEHAVIOUR

UTILITY ANALYSIS

A consumer demands a good or a service. He demands a good because it gives him utility.

Meaning of Utility

The term utility in economics is used to denote that quality in a commodity or service by virtue of which our wants are satisfied. In other words, want – satisfying power of a good is called utility.

Utility means satisfaction which a consumer derives from commodities and services by purchasing different units of money.

Definitions

- According to Jevons, "Utility refers to abstract quality whereby an object serves our purpose.
- In the words of Hibdon, "Utility is the quality of good to satisfy a want."
- According to Mrs. Robinson, "Utility is the quality in commodities that makes individuals wants to buy them".

Concepts of Utility

1) **Initial Utility:** The utility derived from the first unit of commodity is called initial utility. It is obtained from the consumption of the first unit of a commodity. It is always positive.

2) **Total Utility**: The aggregate of utility obtained from the consumption of different units of a commodity, is called Total utility.

TUx = f (Qx)

TUx = total utility of x is a function (f) of quantity of commodity

3) **Marginal Utility:** The change that takes place in the total utility by the consumption of an additional unit of commodity is called marginal utility.

MUnth = Tn – Tn-1 or MU = change TU/ change Q

MUnth	= Marginal utility of nth unit.
Tn	= Total utility of n units.
Tn-1	= Total utility of n-1 units
Change TU	= change in total utility

Marginal utility can be:

a) **Positive Marginal Utility:** If by consuming additional units of commodity, total utility goes on increasing, then marginal utility of these units will be positive.

b) **Zero Marginal Utility:** If the consumption of additional unit of commodity causes no change in the total utility, it means the marginal utility of additional unit is zero.

c) **Negative Marginal Utility:** If the consumption of an additional unit of a commodity causes fall in total utility, it means the marginal utility is negative.

RELATION BETWEEN TOTAL UTILITY AND MARGINAL UTILITY

Total utility is the summation of the marginal utilities of different units of a commodity.

$$\textbf{TU} = \sum \textbf{MU}$$

Total utility is summation of Marginal utility

Table:

Quantity	Total Utility	Marginal Utility	Description
0	0	-	**Initial Utility**
1	8	8 – 0 = 8	
2	14	14 – 8 = 6	**Positive Utility**
3	18	18 – 14 = 4	
4	20	20 – 18 = 2	
5	20	20 – 20 = 0	**Zero Utility**
6	18	18 – 20 = -2	**Negative Utility**

Table shows that:

a) As more and more units of commodity is consumed, the marginal utility derived from each successive unit goes on diminishing. But the total utility increases up to a limit.

b) Marginal utility of the first four units being positive, the total utility goes on increasing. Thus as long as the marginal utility of the commodity remains positive, total utility goes on increasing.

c) Marginal utility of the fifth unit is zero. In this situation total utility (20) will be maximum. This situation also represents point of saturation.

d) Marginal utility of the sixth unit is negative. As a result of it, total utility of six units of the commodity falls from 20 to 18 units.

CARDINAL AND ORDINAL UTILITY

Cardinal utility means satisfaction that can be measured in numbers such as 1, 2, and 3.

While ordinal utility refers to satisfaction cannot be measureable in numbers.

The concept of cardinal utility was used by **Marshal** to define Consumer's Equilibrium. Cardinal Utility means consumer could measure the satisfaction derived by the consumption of any goods or services in terms of number and unit of that measurement is Utils or the Money.

Whereas ordinal utility means giving the rank to the utility derived by the consumption of goods and services. This concept was given by **J.R. Hicks**. This is more realistic and better than cardinal utility.

UTILITY FUNCTION

It is a function which states that an individual's utility depends upon the goods he consumes and their amounts. Mathematically,

U = f(x1, x2, …………xn)

where x1 and x2 are the amounts of the goods consumed.

INDIFFERENCE CURVE

An Indifference curve is the locus of all those combination of any two goods which give the same level of satisfaction to the consumer. i.e., he will be indifference between that combination and he does not matter if any combination he gets.

Indifference Schedule

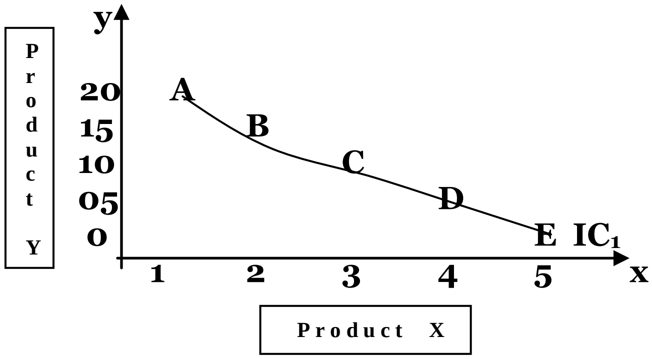
Combinations	Goods (X)	Goods (Y)	Level of satisfactions
A	1	18	S
B	2	13	S
C	3	9	S
D	4	6	S
E	5	4	S

S = same level of Satisfaction.

In the above schedule there are 5 combinations of 2 goods (X) and (Y). But all are achieved combinations of (X) and (Y). The consumer is indifferent between them. It can be explained in further detail as:

To get one more units of X the consumer prefer to give up 5 units or Y. The gain in utility of one additional unit of X will exactly compensated the consumer by the loss of 5 units of Y. Thus the total level of satisfaction from (1X + 18Y) is equal to (2X + 13Y). Similarly, the total utility or the level of satisfaction from (2X + 13Y) is equal to (3X + 9Y) and so on. Since all these combinations gives the same level of satisfaction they are also known as Iso-utility combination.

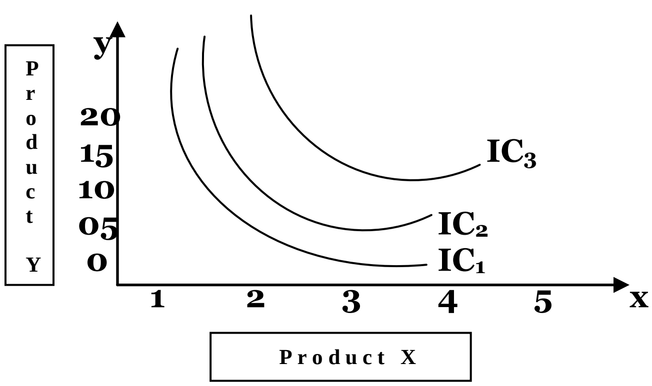
Indifference schedule in Indifference curve as shown below:



In the above figure, X axis represents product X and Y axis represents product Y. IC1 is the Indifference curve. All combinations of the goods X & Y represented by points A,B,C,D & E on the Indifferent curve will be equally preferable to the consumer. As these goods gives him the same level of satisfactions.

INDIFFERENCE MAP

An indifference map is consists of a set of indifference curve drawn together. It shows the scale of preference of consumer for different combinations of any two goods.



In the indifferent map given above, all the combination of two goods represented by the curve IC_1 will give the consumer the same level of satisfaction. But the level of satisfaction will be less than those given by IC_2 and IC_3 etc. Higher and higher indifference curve represents higher & higher level of satisfaction as compared to lower one. Therefore Indifference curve in a indifference map are labeled in an ascending order such as IC_0, IC_1, IC_2, IC_3, IC_{10}, IC_{100} ……… ICn.

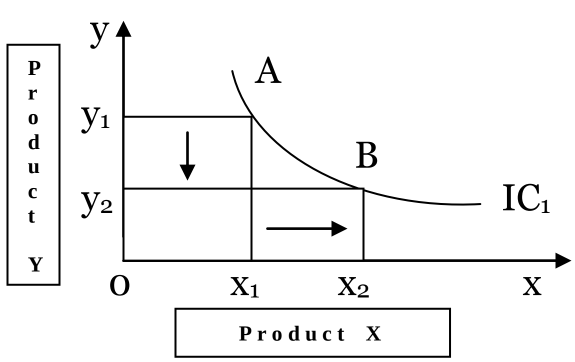
PROPERTIES OF INDIFFERENCE CURVE

The indifference curves possess certain characteristics which are also called as properties. The important properties are:

1. Indifference curve must slopes downwards from left to right.
2. Indifference curve must be convex to the origin.
3. No two Indifference curves should intersect.

Let us see in detail one by one.

1. Indifference curve must slopes downwards from left to right

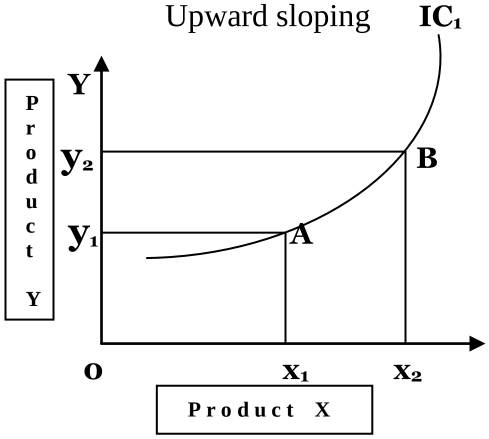


Indifference curves slopes downwards from left to right indicating that as the quantity of commodity X increases, the amount of commodity Y should fall in order that the level of satisfaction from every combination should remain the same.

In the above figure, where the Indifference curve (IC_1) slopes downwards from left to right, shows that as the consumer moves from point A to B on (IC_1), consuming more of commodity X [i.e. from OX_1 to OX_2] and less of commodity Y [i.e. from OY_1 to OY_2], level of satisfaction remains the same.

Let us see weather, the indifference curve can slope upwards from left to right or it can be horizontal or vertical as shown in the following figure

a. Upward sloping



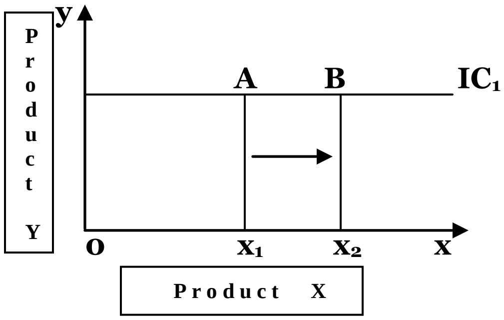
In the upward sloping Indifference curve, when the consumer prefers OX_1 quantity of commodity X he prefers OY_1 quantity of commodity Y at the point A. When the consumer tends to prefer OX_2 quantity of commodity X he prefers OY_2 quantity of commodity Y at the point B. From the diagram it is very clear that,

$$OX_1 < OX_2 \ \&$$

$$OY_1 < OY_2$$

Therefore, the satisfaction derived from the combination of goods X and Y at point B is greater than the satisfaction derived from the combination of goods X and Y at point A. This is happening, because the consumer moves from A to B on IC_1 consuming more of commodity X [i.e. OX_1 to OX_2] and more of commodity Y [i.e. from OY_1 to OY_2].

b. Horizontal

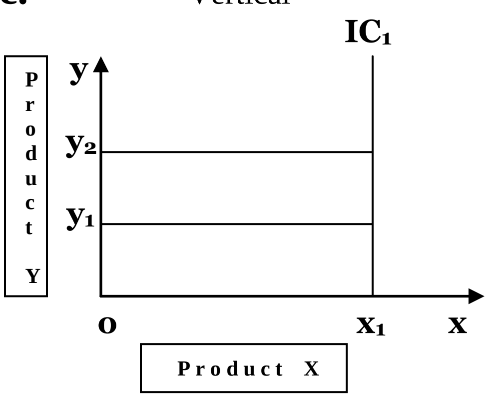


In the horizontal slopping of Indifference curve, when the consumer prefers OX_1 quantity of commodity X as well as OX_2 quantity of commodity X he prefers the same quantity of Y (i.e.) OY_1. From the diagram, it is very clear that,

$$OX_1 < OX_2 \quad \&$$
$$OY_1 = OY_1$$

Therefore, the satisfaction derived from the combination of goods X and Y at point B is greater than the satisfaction derived from the combination of goods X and Y at point A. This is because the consumer moves from A and B on (IC1) Consuming more of commodity X (i.e. OX_1 to OX_2) and same level of commodity Y (i.e. from OY_1 to OY_1).

c. Vertical



In the vertical sloping Indifference curve, when the consumer prefers OY_1 quantity of commodity Y as well as OY_2 quantity of commodity Y he prefers the same quantity of commodity X (i.e.) OX_1. From the diagram, it is very clear that,

$$OX_1 = OX_1 \quad \&$$

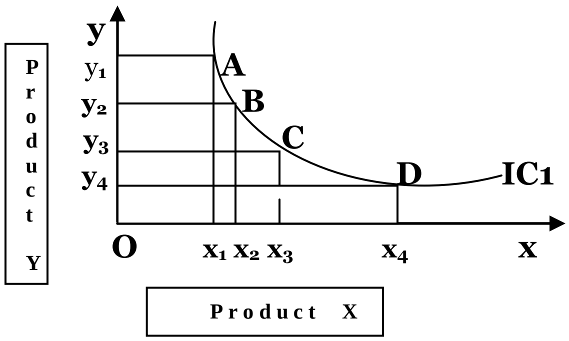
$$OY_1 < OY_2$$

Therefore, the satisfaction derived from the combination of goods X and Y at point B is greater than the satisfaction derived from the combination of goods X and Y at point A. This is because the consumer move from A to B on (IC_1) Consuming same level of commodity X (i.e. from OX_1 to OX_1) and more of commodity Y (i.e. from OY_1 to OY_2).

2. Indifference curve must be Convex to the origin

The convexity of an Indifference curve is explained by the law of diminishing marginal rate of substitution. Marginal rate of substitution between X and Y is the quantity of good Y which the consumer is willing to give up for every additional unity of X, so that the level of satisfaction remains the same, from all the successive combinations.

Combinations	Goods (X)	Goods (Y)	MRSxy	Level of satisfactions
A	1	18	—	S
B	2	13	5:1	S
C	3	9	4:1	S
D	4	6	3:1	S
E	5	4	2:1	S

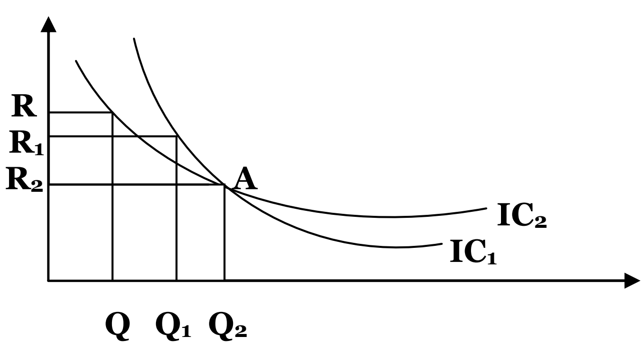


Convexity implies that the consumer is willing to give up less of good Y to obtain a little more of good X. This means, a diminishing slope (△y/△x) of the indifference curve.

A rational consumer gives less significance to an extra unit of a commodity with a large stock and more significance to a unit of a commodity with a smaller stock. As the consumer moves down the indifference curve, quantity or X becomes larger and that of Y becomes smaller. In order to be at the same level of satisfaction, the consumer will sacrifice less and less of Y in exchange of X. So MRS of X for Y will diminish as the consumer gets more and more of X. Only then the subsequent combinations will give the consumer an equal level of satisfaction. Hence indifference curves are convex to the origin.

3. No two Indifference curve intercept with each other

In order to prove that two indifference curve do not intercept with each other. Let us draw two Indifference curve (IC_1 & IC_2) intercepting with each other at point A, as shown in the diagram.



Each indifferent curve represents a particular level of satisfaction to the consumer, which is different from other Indifference Curve representing different level of satisfaction. If two indifference curves intercepts (as shown in the above diagram) it will corresponding to B and C, managed to equal at some other point A. But it is logically meaningless and unacceptable proportion.

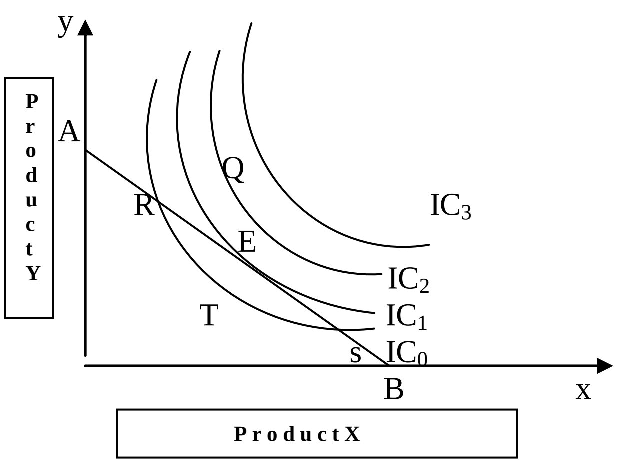
CONSUMER’S EQUILIBRIUM

“A consumer is said to be equilibrium when he gets maximum level of satisfaction by spending his limited income on purchase of any two goods”. A rational consumer will therefore attempt to reach the highest possible indifferent curve and try to obtain maximum level of satisfaction by spending his limited income.

Consumer equilibrium can be explained by making the following assumption:-

1. A consumer has a scale or preference for different combination of any two goods and it remain constant throughout the analysis.
2. A consumer has a fixed amount of income to be spend on any two goods and he is spent his entire income on the purchase of the two goods and does not save any part of his income.
3. Prices per unit of two goods X and Y are given and remain constant throughout the analysis.
4. The two goods are perfectly divisible and substitutable to some extent.
5. All the units of goods are homogeneous.
6. Consumer is a rational person & attempts to get maximum level of satisfaction.

In order to explain consumer equilibrium under Indifference curve analysis, we have to draw the Indifference Map and the Budget/price line together as shown in the figure below.



A rational consumer will try to reach the highest possible Indifference curve given his income and price per unit of the two goods x and y. The consumer will not be equilibrium below the price line because he will not be spending his entire income and he will not get maximum level of satisfaction. On the other hand all the combination of x and y represented by the IC_2 and IC_3 are ruled out because his income is not sufficient to reach any point on the IC_2 and IC_3.

The consumer equilibrium should be somewhere in the Budget line neither below nor above. ‘E’ is the equilibrium point. The consumer will not be equilibrium at any point on the Budget line above the point E because MRSxy is greater. Similarly, he will not be equilibrium at any point below Equilibrium point E on the Budget line because MRSxy is lesser.

BUDGET LINE/PRICE LINE

Price line represents different combination of any two goods X and Y which the consumer can actually purchase. Assuming the fixed income of the consumer and price per unit of X and Y is given.

MARGINAL RATE OF SUBSTITUTION

The concept marginal rate of substitution is an important tool of indifference curve analysis of demand. It is the rate at which a consumer is ready to give up one good in exchange for another good so that his level of satisfaction remains the same.

Marginal rate of substitution is a measure of how much of a commodity a consumer will give up to get one or more units of another commodity, while maintaining the same level of satisfaction.

ELASTICITY OF DEMAND

Elasticity 'η' of the function $y = f(x)$ at a point x is defined as the limiting case of ratio of the relative change in y to the relative change in x. Symbolically,

$$\eta = \frac{x}{y} \cdot \frac{dy}{dx}$$

(i) Price elasticity of demand

Price elasticity of demand is the degree of responsiveness of quantity demanded to a change in price.

If x is demand and p is unit price of the demand function $x = f(p)$, then the elasticity of demand with respect to the price is defined as

$$\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$$

(ii) Price elasticity of supply

Price elasticity of supply is the degree of responsiveness of quantity supplied to a change in price.

If *x* is supply and *p* is unit price of the supply function $x = g(p)$, then the elasticity of supply with respect to the price is defined as

$$\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$$

Some important results on price elasticity

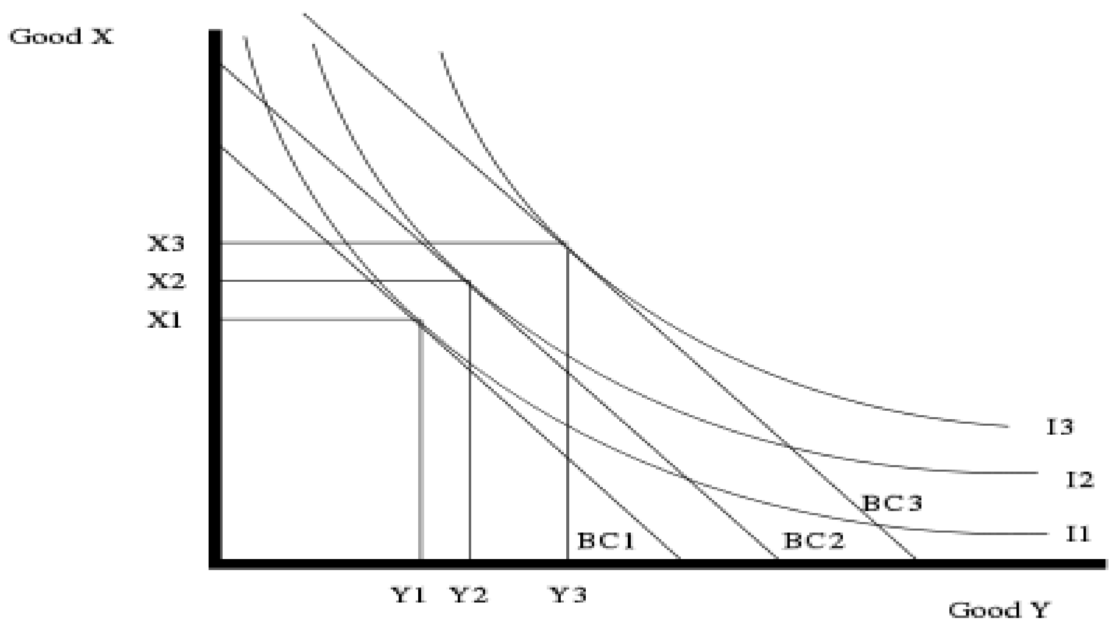
(i) If $|\eta| > 1$, then the quantity demand or supply is said to be elastic.

(ii) If $|\eta| = 1$, then the quantity demand or supply is said to be unit elastic.

(iii) If $|\eta| < 1$, then the quantity demand or supply is said to be inelastic.

Income Effect

Income effect is the effect on the quantity demanded exclusively as a result of change in money income, while prices of the goods remain the same.



Price Effect

Price effect is the effect on the quantity demanded exclusively as a result of change in the price of one of the goods, while his income and the price of the other good remain the same.

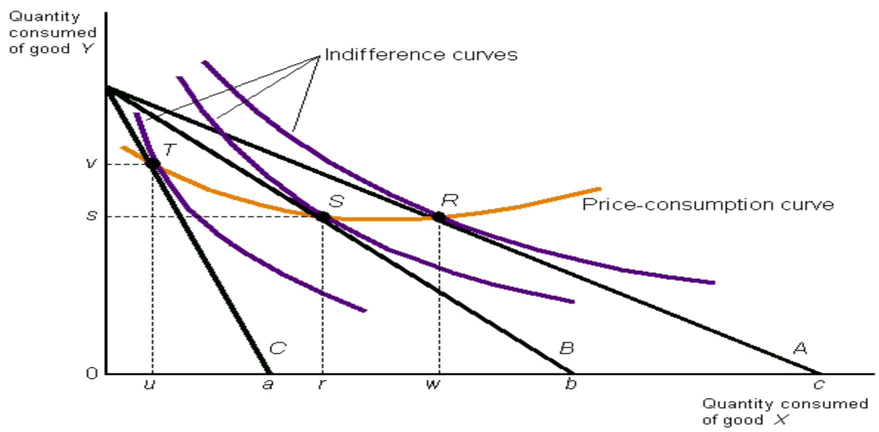
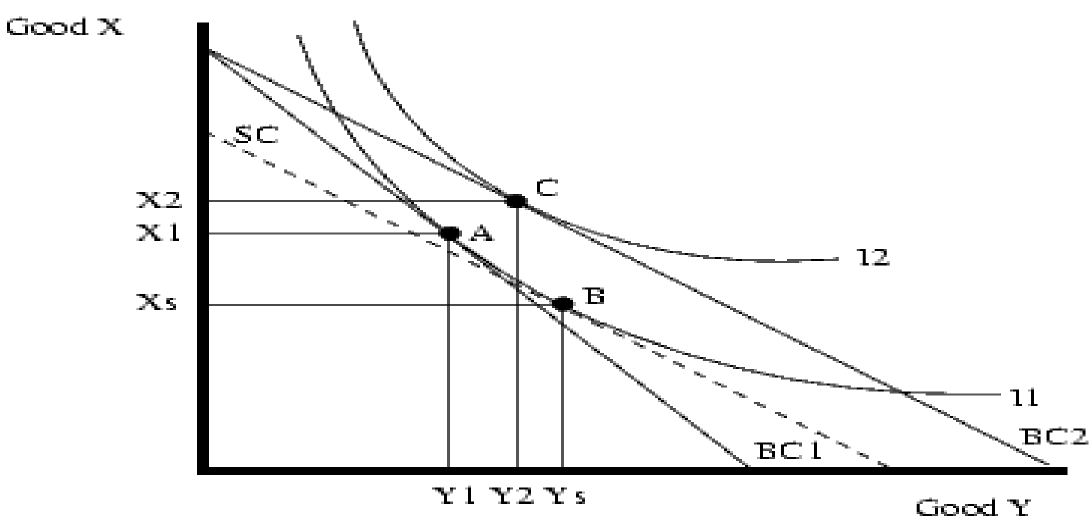


Figure 4.4 EFFECTS OF CHANGES IN THE PRICE OF GOOD X ON CONSUMER EQUILIBRIUM

Substitution Effect

Substitution effect is the effect observed with changes in the relative price of goods alone, while the real income remains the same.



Example Sum

1. **Find the elasticity of demand for the functions *x* = 100 – *p* – *p*2 when *p* = 5.**

Solution :

$$x = 100 - p - p^2$$

$$\frac{dx}{dp} = -1 - 2p.$$

$$\text{Elasticity of demand } \eta_d = -\frac{p}{x}\frac{dx}{dp}$$

$$= -\frac{p(-1-2p)}{100 - p - p^2} = \frac{p + 2p^2}{100 - p - p^2}$$

$$\text{When } p = 5,\ \eta_d = \frac{5 + 50}{100 - 5 - 25}$$

$$= \frac{55}{70} = \frac{11}{14}$$

2. If $y = \dfrac{1 - 2x}{2 + 3x}$, obtain the values of η when $x = 0$ and $x = 2$.

Solution :

We have $y = \dfrac{1-2x}{2+3x}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{(2+3x)(-2)-(1-2x)(3)}{(2+3x)^2}$$

$$= \frac{-4-6x-3+6x}{(2+3x)^2} = \frac{-7}{(2+3x)^2}$$

$$\eta = \frac{Ey}{Ex} = \frac{x}{y} \ \frac{dy}{dx}$$

$$= \frac{x(2+3x)}{(1-2x)} \times \frac{-7}{(2+3x)^2}$$

$$\eta = \frac{-7x}{(1-2x)\ (2+3x)}$$

when $x = 0$, $\eta = 0$

when $x = 2$, $\eta = \dfrac{7}{12}$

****

UNIT - III

THEORY OF THE FIRM

PRODUCTION FUNCTION

Production function relates physical output of a production process to physical inputs or factors of production. A production function can be expressed in a functional form such as

$$Q = f(X_1, X_2, X_3, \ldots, X_n)$$

where

Q is the quantity of output and $X_1, X_2, X_3, \ldots, X_n$ are the quantities of factor inputs (such as capital, labour, land or raw materials).

LINEAR HOMOGENEOUS PRODUCTION FUNCTION

When all the inputs are increased in the same proportion, the production function is said to be homogeneous. The degree of production function is equal to one. This is known as linear homogeneous production function. Mathematically, this form of production function is expressed as

$$nQ = f(nL, nK)$$

This production function also implies constant returns to scale. That is if L and K are increased by n-fold, the output Q also increases by n-fold.

COBB-DOUGLAS PRODUCTION FUNCTION

Charles W. Cobb and Paul H. Douglas studied the relationship of inputs and outputs and formed an empirical production function, popularly known as Cobb-Douglas production function. Originally, C-D production function applied not to the production process of an individual firm but to the whole of the manufacturing production.

The Cobb-Douglas production function is expressed by

$$Q = AL^{\alpha}K^{\beta}$$

where

Q is output and L and A' are inputs of labour and capital respectively. A, α and β are positive parameters where $\alpha > 0$, $\beta > 0$. The equation tells that output depends directly on L and K and that part of output which cannot be explained by L and K is explained by A which is the 'residual', often called technical change.

The marginal products of labour and capital are the functions of the parameters A, α and β and the ratios of labour and capital inputs. That is,

$MP_L = \partial Q/\partial L = \alpha AL^{\alpha-1}K^{\beta}$

$MP_K = \partial Q/\partial K = \beta AL^{\alpha}K^{\beta-1}$

In other words, this function characterizes the returns to scale thus:

$\alpha + \beta > 1$: Increasing returns to scale

$\alpha + \beta = 1$: Constant returns to scale

$\alpha + \beta < 1$: Decreasing returns to scale.

Although the C-D production function is a multiplicative type and is non-linear in its general form, it can be transferred into linear function by taking it in its logarithmic form. That is why, this function is also known as log linear function, which is

$$\text{Log } Q = \log A + a \log L + p \log K$$

It is easier to compute C-D function when expressed in log linear form.

Properties of C-D Production Function

1. There are constant returns to scale.
2. Elasticity of substitution is equal to one.
3. A and p represent the labour and capital shares of output respectively.
4. A and p are also elasticities of output with respect to labour and capital respectively.
5. If one of the inputs is zero, output will also be zero.
6. The expansion path generated by C-D function is linear and it passes through the origin.
7. The marginal product of labour is equal to the increase in output when the labour input is increased by one unit.
8. The average product of labour is equal to the ratio between output and labour input.
9. The ratio α/β measures factor intensity. The higher this ratio, the more labour intensive is the technique and the lower is this ratio and the more capital intensive is the technique of production.

Importance of C-D Production Function

1. It suits to the nature of all industries.
2. It is convenient in international and inter-industry comparisons.
3. It is the most commonly used function in the field of econometrics.
4. It can be fitted to time series analysis and cross section analysis.
5. The function can be generalised in the case of 'n' factors of production.
6. The unknown parameters 'a' and 'p' in the function can be easily computed.
7. It becomes linear function in logarithm.
8. It is more popular in empirical research.

Limitations of C-D Production Function

1. The function includes only two factors and neglects other inputs.
2. The function assumes constant returns to scale.
3. There is the problem of measurement of capital which takes only the quantity of capital available for production.
4. The function assumes perfect competition in the factor market which is unrealistic.
5. It does not fit to all industries.
6. It is based on the substitutability of factors and neglects complementarily of factors.
7. The parameters cannot give proper and correct economic implication.

Cost Function

Normally total cost consists of two parts. (i) Variable cost and (ii) fixed cost. Variable cost is a single - valued function of output, but fixed cost is independent of the level of output.

Let $f(x)$ be the variable cost and k be the fixed cost when the output is x units. The total cost function is defined as $C(x) = f(x) + k$, where x is positive. Note that $f(x)$ does not contain constant term.

We define Average Cost (AC), Average Variable Cost (AVC), Average Fixed Cost (AFC), Marginal Cost (MC), and Marginal Average Cost (MAC) as follows:

(i) Average Cost (AC) = $\frac{f(x)+k}{x} = \frac{\text{Total Cost}}{\text{Output}}$

(ii) Average Variable Cost (AVC) = $\frac{f(x)}{x} = \frac{\text{Variable Cost}}{\text{Output}}$

(iii) Average Fixed Cost (AFC) = $\frac{k}{x} = \frac{\text{Fixed Cost}}{\text{Output}}$

(iv) Marginal Cost (MC) = $\frac{d}{dx}\,C(x) = C'(x)$

(v) Marginal Average Cost (MAC) = $\frac{d}{dx}\,(AC)$

Revenue Function

Let x units be sold at Rs. p per unit. Then the total revenue R(x) is defined as R(x) = px, where p and x are positive.

Average revenue (AR) = $\frac{\text{Total revenue}}{\text{quantity sold}} = \frac{px}{x} = p$

(i.e Average revenue and price are the same)

Marginal revenue (MR) = $\frac{d}{dx}\,(R) = R'(x)$

Profit Function

The profit function P(x) is defined as the difference between the total revenue and the total cost. *i.e.* P(x) = R(x) - C(x).

Some standard results

If $P(x) = R(x) - C(x)$ be the profit function, then

(i) Average profit $[AP] = \frac{Total\ Profit}{Output} = \frac{P(x)}{x}$

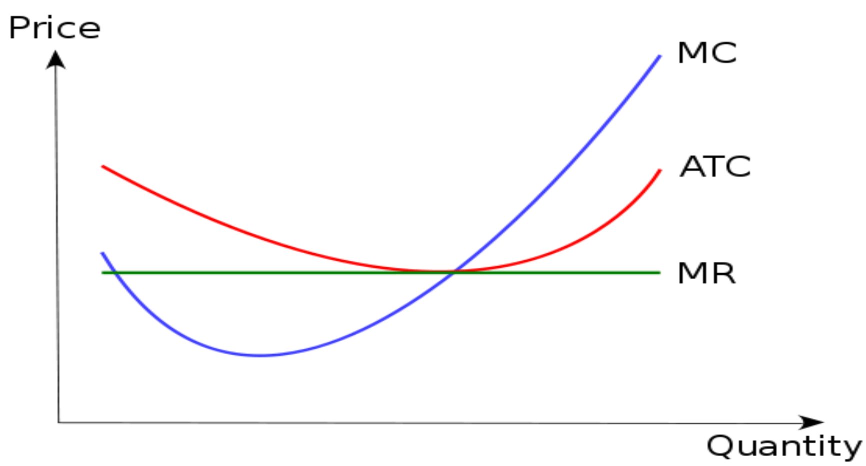
(ii) Marginal profit $[MP] = \frac{dP}{dx} = \frac{d}{dx}(P(x)) = P'(x)$

(iii) Marginal average profit $[MAP] = \frac{d}{dx}(AP) = AP'(x)$

(iv) Profit $[P(x)]$ is maximum when $MR = MC$

Relationship between Average and Marginal Cost

Average cost and marginal cost impact one another as production fluctuate:



1. When the average cost declines, the marginal cost is less than the average cost.
2. When the average cost increases, the marginal cost is greater than the average cost.
3. When the average cost stays the same (is at a minimum or maximum), the marginal cost equals the average cost.

Relationship among Marginal revenue [MR], Average revenue [AR] and Elasticity of demand [ηd].

We know that $R(x) = px$

i.e., $R = px$

and $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

Now, $MR = \frac{d}{dx}(R)$

$$= \frac{d}{dx}(px) = p + x\frac{dp}{dx}$$

$$= p\left[1 + \frac{x}{p} \cdot \frac{dp}{dx}\right]$$

$$= p\left[1 + \frac{1}{\frac{p}{x} \cdot \frac{dx}{dp}}\right]$$

$$= p\left[1 - \frac{1}{-\frac{p}{x} \cdot \frac{dx}{dp}}\right]$$

$$= p\left[1 - \frac{1}{\eta_d}\right]$$

i.e. $MR = AR\left[1 - \frac{1}{\eta_d}\right]$ (or)

$$\eta_d = \frac{AR}{AR - MR}$$

Example Sums

1. The total cost function for the production of *x* units of an item is given by

$$C(x) = \frac{1}{3}x^3 + 4x^2 - 25x + 7.$$

Find (i) Average cost (ii) Average variable cost (iii) Average fixed cost (iv) Marginal cost and (v) Marginal Average cost.

Solution:

$$C(x) = \frac{1}{3}x^3 + 4x^2 - 25x + 7$$

(i) Average cost (AC) $= \frac{C}{x}$

$$= \frac{1}{3}x^2 + 4x - 25 + \frac{7}{x}$$

(ii) Average variable cost (AVC) $= \frac{f(x)}{x}$

$$= \frac{1}{3}x^2 + 4x - 25$$

(iii) Average fixed cost (AFC) $= \frac{k}{x}$

$$= \frac{7}{x}$$

(iv) Marginal cost (MC) $= \frac{dC}{dx}$ (or) $\frac{d}{dx}(C(x))$

$$= \frac{d}{dx}\left[\frac{1}{3}x^3 + 4x^2 - 25x + 7\right]$$

$$= x^2 + 8x - 25$$

(v) Marginal Average cost (MAC) $= \frac{d}{dx}[AC]$

$$= \frac{d}{dx}\left[\frac{1}{3}x^2 + 4x - 25 + \frac{7}{x}\right]$$

$$= \frac{2}{3}x + 4 - \frac{7}{x^2}$$

2. The total cost C of making x units of product is $C = 0.00005x^3 - 0.06x^2 + 10x + 20{,}000$. Find the marginal cost at 1000 units of output.

Solution

$C = 0.00005x^3 – 0.06x^2 + 10x + 20{,}000$

$$\text{Marginal Cost } \frac{dC}{dx} = (0.00005)(3x^2) – (0.06)\,2x + 10$$
$$= 0.00015\,x^2 – 0.12x + 10$$
$$\text{when } x = 1000$$
$$\frac{dC}{dx} = (0.00015)(1000)^2 – (0.12)(1000) + 10$$
$$= 150 - 120 + 10 = 40$$

At $x = 1000$ units, Marginal Cost is Rs. 40

3. The total cost C in Rupees of making x units of product is $C(x) = 50 + 4x + 3\sqrt{x}$. Find the marginal cost of the product at 9 units of output.

Solution:

$$C(x) = 50 + 4x + 3\sqrt{x}$$

$$\text{Marginal cost } (MC) = \frac{dC}{dx} = \frac{d}{dx}[C(x)]$$
$$= \frac{d}{dx}\left[50 + 4x + 3\sqrt{x}\right] = 4 + \frac{3}{2\sqrt{x}}$$
$$\text{When } x = 9, \quad \frac{dC}{dx} = 4 + \frac{3}{2\sqrt{9}}$$
$$= 4\frac{1}{2} \text{ (or) ₹}4.50$$

$\therefore$ MC is ₹ 4.50 , when the level of output is 9 units. ∎

UNIT - IV

MARKET EQUALIBRIUM

PERFECT COMPETITION

A type of market where there is large number of buyers and sellers and no buyer or seller influences the market individually.

Features of Perfect Competition

1. Large number of buyers and sellers: Perfect competition is a market where there are large number of buyers and sellers. This feature indicates that both the buyers and sellers do n0t have any major control over the market and they cannot individually influence the market. Thus, it means that quantity supplied by a single seller is so small that it does not affect the market supply and the price of the commodity produced by hid. Similarly, quantity demanded by a single buyer does not influence the total demand and the price of the commodity.

2. Homogeneous products: A commodity produced by different producers is exactly identical in respect of quality, size, price, etc. So a seller has no excuse to charge a higher price for his commodity. The buyer also need not discriminate between the sellers.

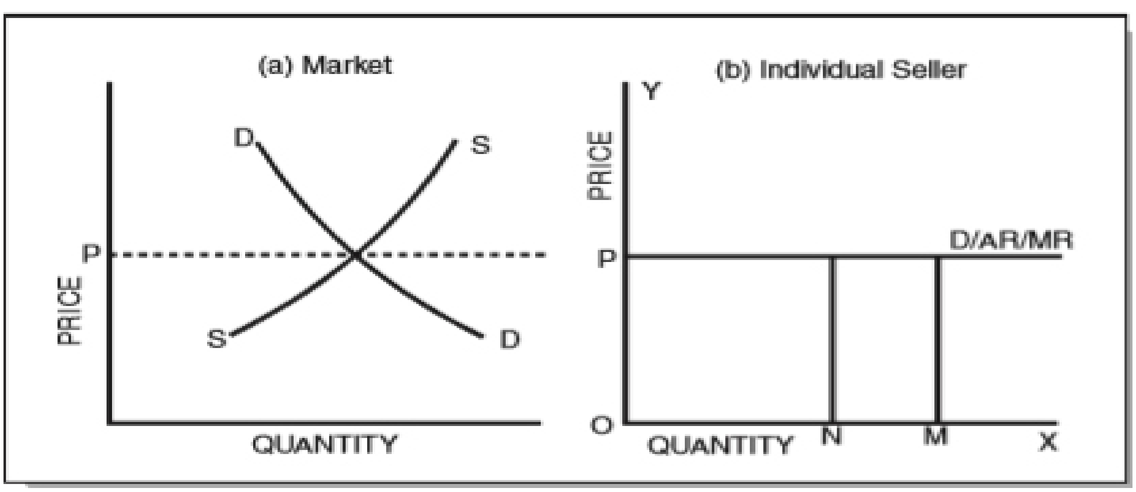
3. Complete Market information: According to this feature both the buyers and sellers must have the complete knowledge of market, regarding price, demand and supply situations in the market.

4. Free entry and exist: Perfect competition allows free entry and exist for the sellers of the commodity under consideration. The sellers are free to enter the market at any time as per their wish and they also can quit the market whenever they want. There are no legal restrictions on the closing down of the firm.

5. Perfect mobility of factors of production: It is an important feature of the perfectly competitive markets that all the factors of production like labor and capital are perfectly mobile, both geographically and occupationally. If labor and capital are move from one place to another as per the requirement of the market they are mobile geographically. If labor and capital move from one type of job or occupation to other type of job easily which means they are mobile occupationally.

6. No transport Cost: Perfect competition assumes that there is a absence of transport cost. This is mainly because the seller will have no excuse of transport cost to charge a different price.

Price and Output Determination under Perfect Competition



MONOPOLY

Monopoly is a type of market in which there is only one seller producing a commodity having no close substitute.

Features of Monopoly Market

1. **Single Seller**: In this type of market, there is only one seller producing a particular commodity.

2. **No Substitutes**: Monopoly not only implies a single seller but it also means a single seller producing a commodity having no close substitutes. If the substitutes are available, there will be a competition among the firms. Monopoly means a complete absence of competition. So under monopoly, the commodity has no close substitutes.

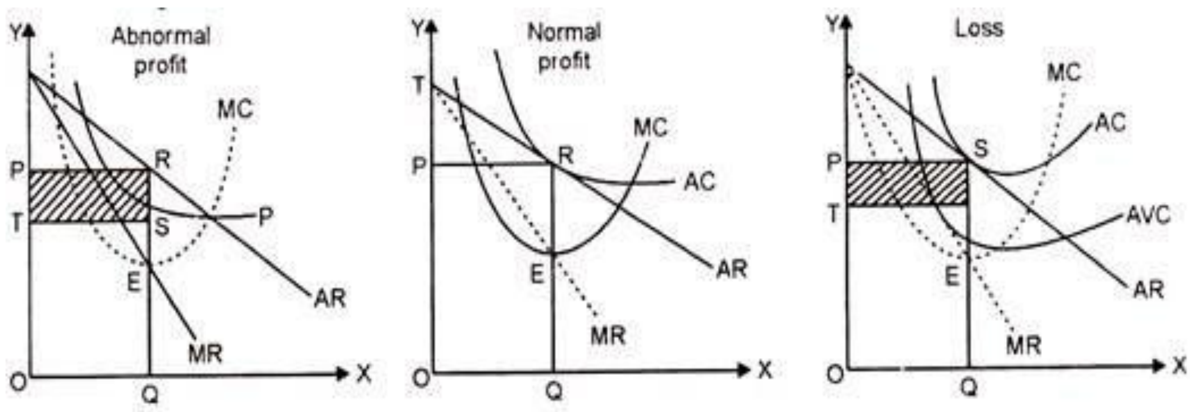
3. **No distinction between a Firm and Industry**: Since there is only one seller of a commodity, there is only one firm producing that commodity in the market. So there is no distinction between the concepts of industry and firm under monopoly.

4. **No free Entry and Exist**: In the monopoly market, there are strong barriers to the entry of a new firm in the market. This prevents new firms from entering the market and so there is only one firm producing that commodity.

5. **Large number of Buyers**: Under monopoly there are large number of buyers in the market who compete with one another.

6. **Downward sloping demand curve**: The demand curve of the monopoly firm slopes downward indicating that the monopolist can maximize sales only by reducing the price.

Price and Output Determination under Monopoly



PRODUCTION

MONOPOLISTIC COMPETITION

Monopolistic competition refers to a market where many sellers sell similar but differentiated product to a large number of buyers. In a monopolistic competition market, many monopolistic firms compete with each other by producing same but differentiated products. For example, companies selling toothpaste products like Colgate, Pepsodent, Close-up, etc. fall under Monopolistic competition.

Features of Monopolistic Competition

1. **Large number of Sellers**: In a monopolistic competition market, there is large number of sellers. Hence no single seller can control the

market supply. Each seller follows his own course of action. In other words each seller is independent.

2. **Product Differentiation**: Product differentiation is the most important feature of monopolistic competition. Since all sellers sell the products which are perfect substitutes for each other, they go for product differentiation. Every seller makes efforts to show that his product is superior to other product. This differentiation is done through Advertisement, brands, trademarks, designs, packaging, color etc. Thus the products are not homogeneous under monopolistic competition.

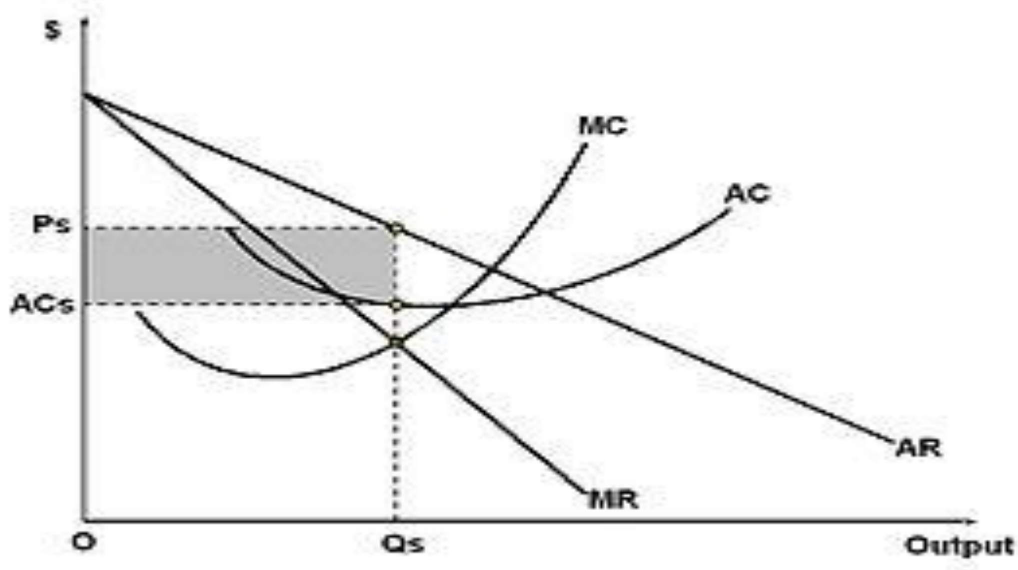
3. **Selling Costs**: One of the unique features of monopolistic competition is its selling cost. Selling cost is the cost incurred by the seller on sales promotion activities like advertisement, salesman's service etc. Selling cost enables the seller to persuade buyers to buy their products than products from other sellers.

4. **Large Number of Buyers**: There is large number of buyers in a monopolistic competition market. Thus the buyers purchase goods by choice and not by chance.

5. **Free Entry and Exist**: There is free entry and exist of firms under monopolistic competition. There are no barriers for the firm to enter. Since each firm produces a product which is little different from others, there is no possibility of more firms entering the market.

6. **Competition**: Competition under monopolistic market is more as all the firms sell close substitutes. But the competition is in two dimensional:
   1. Price competition under which the firms were competes with each other by reducing their products price.
   2. Non-price competition under which they compete through advertisement and sales promotion activities, etc.

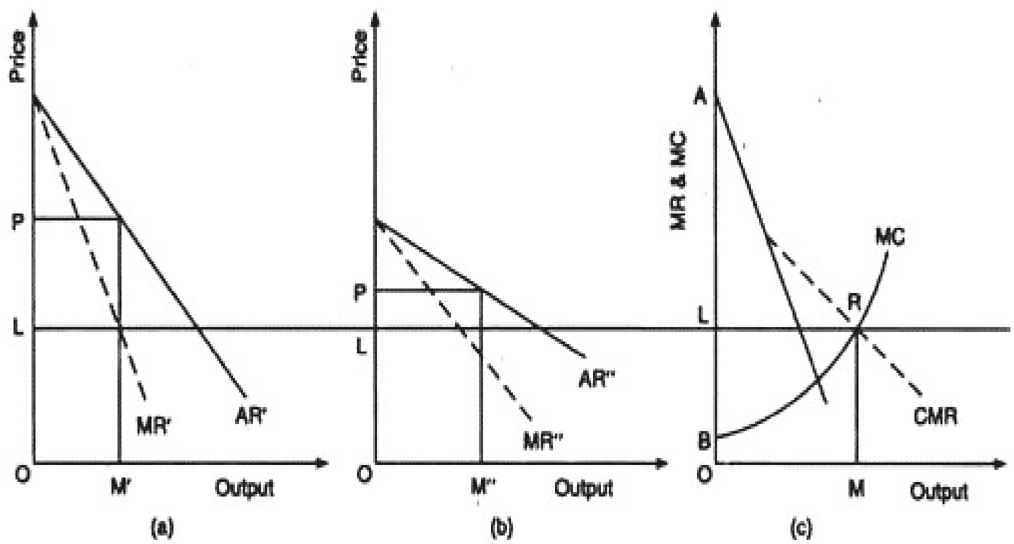
Price Determination under Monopolistic competition



DISCRIMINATING MONOPOLY

Discriminating monopoly' or 'price discrimination' occurs when a monopolist charges the same buyer different prices for the different units of a commodity, even though these units are in fact homogeneous.

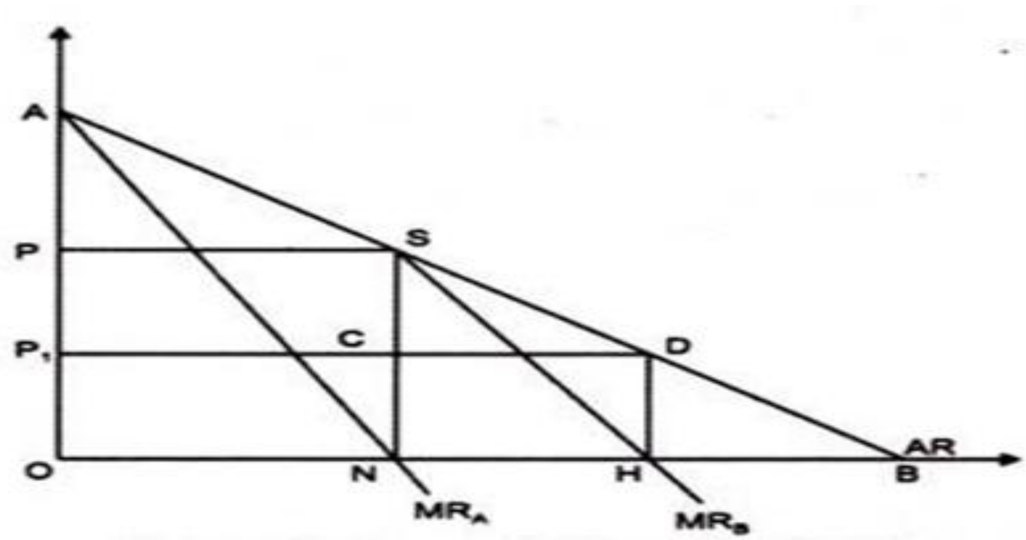
Price Determination under Discriminating Monopoly



Duopoly

Duopoly refers to a situation where there are just two sellers in a market. A duopoly can also refer to a situation where a market is dominated by two sellers. There may be more than two sellers in the market but the supply of the product is controlled by just two of them.

Price and Output Determination under Cournot's Duopoly Model



Example Sums

1. **A firm has the cost function C = 100 + 015x² and faces the demand function given by p = 3. Find the profit maximizing output and price.**

Solution:

Profit :

$\pi = TR - TC$

$TR = p \times q/y$

$= 3 \times x$

$= 3x.$

$= 3x - (100 + 0.015x^2)$

$= 3x - 100 - 0.015x^2$

$$\frac{d\pi}{dx} = 3 - 0.015(2)x^{2-1}$$

$= 3 - 0.030x$

$$\frac{d^2\pi}{dx^2} = -0.030$$

FOC

$$\frac{d\pi}{dx} = 0$$

$3 - 0.030x = 0$

$-0.030x = -3$

$$x = \frac{-3}{-0.030} = \frac{3}{0.030} = \frac{3}{\frac{3}{100}}$$

$$= \frac{3}{3} \times 100$$

$$x = 100$$

SOC

$$\frac{d^2\pi}{dx^2} = -0.030 < 0$$

∴ The firm maximize its profit at $x = 100$.

Profit Maximization.

$$\pi = 3x - 100 - 0.015x^2$$

At $x = 100$

$$= 3(100) - 100 - 0.015(100)^2$$

$$= 300 - 100 - 0.015(10000)$$

$$= 200 - 150$$

$$= 50$$

2) Find the profit maximizing output & price given the dema[nd] function $q = 200 - 10p$ & $AC = 10 +$ [...] What is the maximum profit.

Sol

Given: $q = 200 - 10p$

$$10p = 200 - q$$

$$p = \frac{200 - q}{10}$$

$$= \frac{200}{10} - \frac{q}{10}$$

$$= 20 - \frac{q}{10}$$

Revenue

$$R = pq$$

$$= \left(20 - \frac{q}{10}\right) q$$

$$= 20q - \frac{q^2}{10}$$

$$C = AC \times q$$

$$= \left(10 + \frac{q}{25}\right) q$$

$$= 10q + \frac{q^2}{25}$$

Profit

$$\pi = R - C$$

$$= \left(20q - \frac{q^2}{10}\right) - \left(10q + \frac{q^2}{25}\right)$$

$$= 20q - \frac{q^2}{10} - 10q - \frac{q^2}{25}$$

$$= 10q - \frac{q^2}{10} - \frac{q^2}{25}$$

$$= 10q - \frac{7q^2}{50}$$

$$-\frac{q^2}{10} - \frac{q^2}{25}$$

$$= \frac{-5q^2 - 2q^2}{50}$$

$$= \frac{-7q^2}{50}$$

$$\frac{d\pi}{dq} = 10 - \frac{14q}{50}$$

$$= 10 - \frac{7q}{25}$$

$$\frac{d^2\pi}{dq^2} = -\frac{7}{25}$$

<u>IOC</u>

$$\frac{d\pi}{dq} = 0$$

$$10 - \frac{7q}{25} = 0$$

$$250 - 7q = 0$$

$$7q = 250$$

$$q = \frac{250}{7}$$

<u>IIOC</u>

$$\frac{d^2\pi}{dq^2} = -7/25 < 0$$

$\Rightarrow \pi$ is max at $q = 250/7$

<u>Price</u>

$$P = 20 - \frac{q}{10}$$

$$= 20 - \frac{\frac{250}{7}}{10}$$

$$= 20 - \frac{250}{7} \times \frac{1}{10}$$

$$= 20 - \frac{250}{70}$$

$$= \frac{1150}{70} = 16.4$$

$$\frac{20}{1} - \frac{250}{70}$$

$$\frac{1400 - 250}{70}$$

$$\frac{1150}{70} = 16.4$$

Profit

$$\pi = 10q - \frac{7q^2}{50}$$

$$= 10\left(\frac{250}{7}\right) - \frac{7}{50}\left(\frac{250}{7}\right)^2$$

$$= \frac{2500}{7} - \frac{7}{50}\left(\frac{62500}{49}\right)$$

$$= \frac{2500}{7} - \frac{1250}{7}$$

$$= \frac{1250}{7}$$

$$= 178.5$$

$$\frac{2500}{7} - \frac{1250}{7}$$

$$\frac{2500 - 1250}{7}$$

$$= \frac{1250}{7}$$

3) $x = \frac{600-p}{8}$; $C = x^2 + 78x + 2500$.

$8x = 600 - p$

$600 - p = 8x$

$-p = 8x - 600$

$p = -8x + 600$

$= 600 - 8x$

<u>Revenue</u>

$R = px$

$= (600 - 8x)x$

$= 600x - 8x^2$

<u>Profit</u>

$\pi = R - C$

$= (600x - 8x^2) - (x^2 + 78x + 2500)$

$= 600x - 8x^2 - x^2 - 78x - 2500$

$= 522x - 9x^2 - 2500$

$$\frac{d\pi}{dx} = 522 - 18x$$

$$\frac{d^2\pi}{dx^2} = -18$$

FOC

$$\frac{d\pi}{dx} = 0$$

$$522 - 18x = 0$$

$$18x = 522$$

$$x = \frac{522}{18}$$

$$= 29$$

SOC

$$\frac{d^2\pi}{dx^2} = -18 < 0$$

$\Rightarrow \pi$ is max at

$x = 29$

Price

$$P = 600 - 8x$$

$$= 600 - 8(29)$$

$$= 600 - 232$$

$$= 368$$

Profit

$$\pi = 522x - 9x^2 - 2500$$

$$= 522(29) - 9(29)^2 - 2500$$

$$= 15138 - 7569 - 2500$$

$$= 5069$$

UNIT – V

LINEAR PROGRAMMING PROBLEM

LINEAR PROGRAMMING (LP)

Linear Programming Problem (LPP) is a mathematical technique which is used to optimize (maximize or minimize) the objective function with the limited resources.

Mathematically, the general linear programming problem (LPP) may be stated as follows.

Maximize or Minimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to the conditions (constraints)

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq (or = or \geq) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq (or = or \geq) b_2$$

$$\ldots \quad \ldots. \quad \ldots. \quad \ldots. \quad \ldots$$

$$\ldots \quad \ldots. \quad \ldots. \quad \ldots. \quad \ldots$$

$$a_{m1}x1 + a_{m2}x_2 + \ldots + a_{mn}x_n (\leq or = or \geq) b_m$$

$$x_1, x_2 \ldots\ldots. x_n \geq 0$$

SOME USEFUL DEFINITIONS

Objective function

A function $Z = c_1x_1 + c_2x_2 + \ldots. c_nx_n$ which is to be optimized (maximized or minimized) is called objective function.

Decision Variable

The decision variables are the variables, which has to be determined x_j, $j = 1,2,3,\ldots,n$, to optimize the objective function.

Constraints

There are certain limitations on the use of limited resources called constraints.

$$\sum_{j=1}^{n} a_{ij}x_j \leq (or = or \geq) b_i,\ i = 1,2,3,\ldots, m \text{ are the constraints.}$$

Solution

A set of values of decision variables x_j, $j=1,2,3,\ldots,n$ satisfying all the constraints of the problem is called a solution to that problem.

Feasible solution

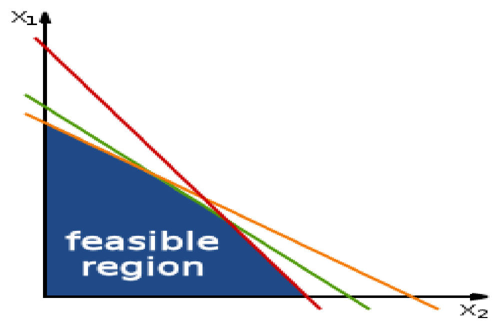
A set of values of the decision variables that satisfies all the constraints of the problem and non-negativity restrictions is called a feasible solution of the problem.

Optimal Solution

Any feasible solution which maximizes or minimizes the objective function is called an optimal solution.

Feasible Region

The common region determined by all the constraints including non-negative constraints $xj \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem.



CHARACTERISTICS OF LP

All linear programming problems must have following five characteristics:

1. **Objective function**

   There must be clearly defined objective which can be stated in quantitative way. In business problems the objective is generally profit maximization or cost minimization.

2. **Constraints**

   All constraints (limitations) regarding resources should be fully spelt out in mathematical form.

3. **Non-negativity**

   The value of variables must be zero or positive and not negative. For example, in the case of production, the manager can decide about any particular product number in positive or minimum zero, not the negative.

4. **Linearity**

   The relationships between variables must be linear. Linear means proportional relationship between two ‘or more variable, i.e., the degree of variables should be maximum one.

5. **Finiteness**

   The number of inputs and outputs need to be finite. In the case of infinite factors, to compute feasible solution is not possible.

ASSUMPTIONS

1. There are a number of constraints or restrictions- expressible in quantitative terms.
2. The prices of input and output both are constant.
3. The relationship between objective function and constraints are linear.
4. The objective function is to be optimized i.e., profit maximization or cost minimization.

ADVANTAGES

LP has been considered an important tool due to following reasons:

1. LP makes logical thinking and provides better insight into business problems.
2. Manager can select the best solution with the help of LP by evaluating the cost and profit of various alternatives.
3. LP provides an information base for optimum allocation of scarce resources.
4. LP assists in making adjustments according to changing conditions.
5. LP helps in solving multi-dimensional problems.

USES AND APPLICATIONS

LP technique is applied to a wide variety of problems listed below:

1. Optimizing the product mix when the production line works under certain specification;
2. Securing least cost combination of inputs;
3. Selecting the location of Plant;
4. Deciding the transportation route;
5. Utilizing the storage and distribution centres;
6. Proper production scheduling and inventory control;
7. Solving the blending problems;
8. Minimizing the raw materials waste;
9. Assigning job to specialized personnel.

The fundamental characteristic in all such cases is to find optimum combination of factors after evaluating known constraints. LP provides solution to business managers by understanding the complex problems in clear and sound way. The basic problem before any manager is to decide the manner in which limited resources can be used for profit maximization and cost minimization. This needs best allocation of limited resources - for this purpose linear programming can be used advantageously.

LIMITATIONS

LP approach suffers from the following limitations also:

1. This technique could not solve the problems in which variables cannot be stated quantitatively.

2. In some cases, the results of LP give a confusing and misleading picture. For example, the result of this technique is for the purchase of 1.6 machines.
3. It is very difficult to decide whether to purchase one or two- machine because machine can be purchased in whole.
4. LP technique cannot solve the business problems of non-linear nature.
5. The factor of uncertainty is not considered in this technique.
6. This technique is highly mathematical and complicated.
7. If the numbers of variables or contrains involved in LP problems are quite large, then using costly electronic computers become essential, which can be operated, only by trained personel.
8. Under this technique to explain clearly the objective function is difficult.

Example Sums:

1. Solve the linear programming graphically.

Minimize $C = 60m + 90n$

Subject to $30m + 48n \geq 450$

$24m + 48n \geq 480$

$m, n \leq 0$

Solution

$30m + 30n = 450$ ……………….. (1)

$24m + 48n = 480$ ……………….. (2)

To determine two points on the straight line $30m + 30n = 450$ ………(1)

Let $m = 0$

$30(0) + 30n = 450$

$30n = 450$

$n = 450/30 = 15$

⇨ (0, 15) is a point on the line (1)

Let $n = 0$

$30m + 30(0) = 450$

$30m = 450$

$m = 450/30 = 15$

⇨ (15, 0) is a point on the line (1)

To determine two points on the straight line $24m + 48n = 480$ ………(2)

Let $m = 0$

$24(0) + 48n = 480$

$48n = 480$

$n = 480/48 = 10$

⇨ (0, 10) is a point on the line (2)

Let $n = 0$

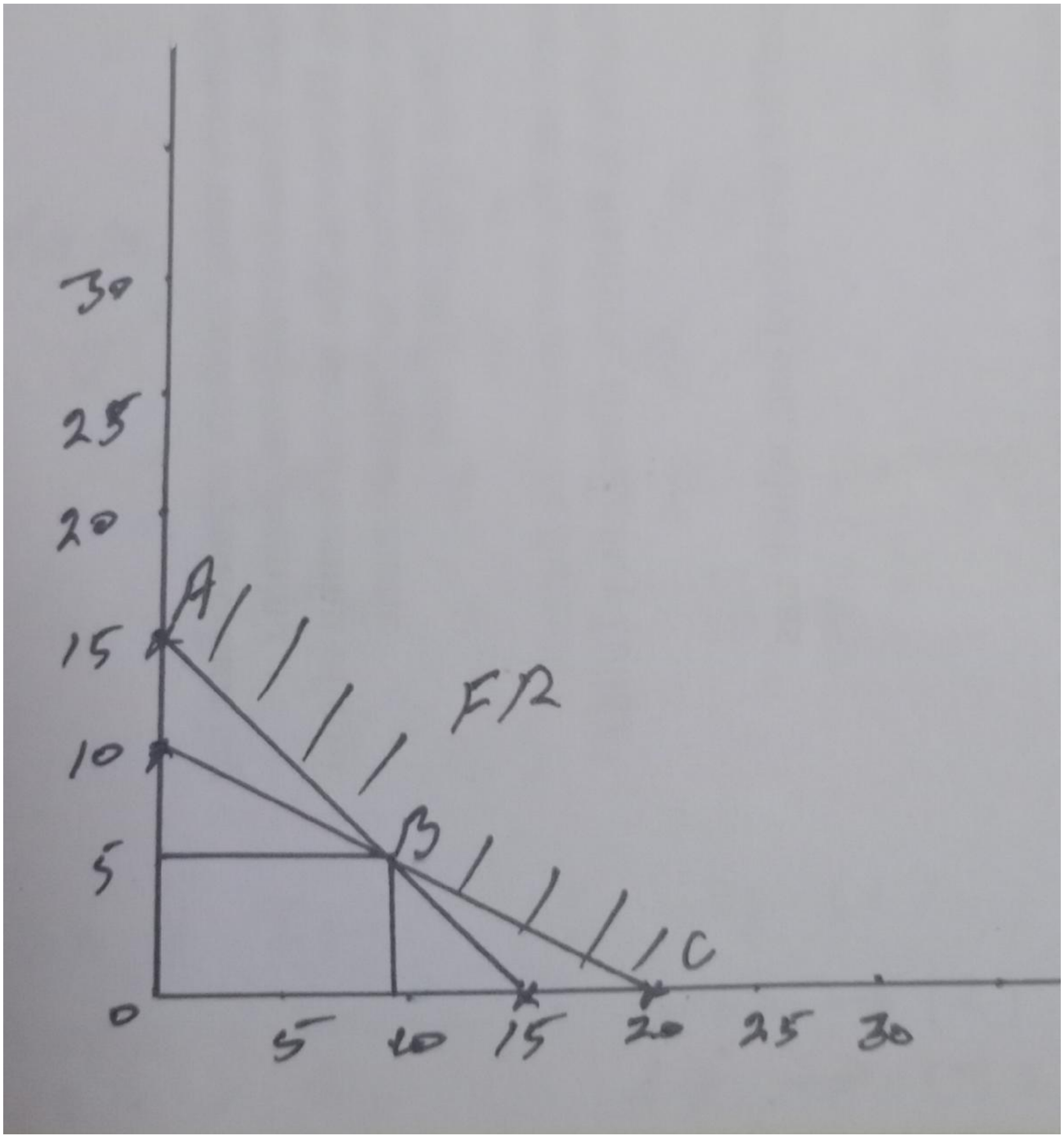
$24m + 48(0) = 480$

$24m = 480$

$m = 480/24 = 20$

⇨ (20, 0) is a point on the line (2)

Graph



Feasible Region	C = 60m + 90n	Value
A (0, 15)	60(0) + 90(15)	1350
B (10, 5)	60(10) + 90(5)	1050
C (20, 0)	60(20) + 90(0)	1200

Optimal Solution

m = 10, n = 5

C = 60(10) + 90(5)

C = 1050

2. Solve the linear programming graphically.

Maximize Z = 6x + 5y

Subject to $3x + 3y \leq 18$

$500x + 1000y \leq 4000$

$12x + 8y \leq 48$

$x, y \geq 0$

Solution

$3x + 3y \leq 18$ ……………… (1)

$500x + 1000y \leq 4000$ ……………… (2)

$12x + 8y \leq 48$ ……………… (3)

To determine two points on the straight line 3x + 3y = 18 ……… (1)

Let x = 0

3(0) + 3y = 18

3y = 18

y = 18/3 = 6

⇨ (0, 6) is a point on the line (1)

Let y = 0

3x + 3(0) = 18

3x = 18

x = 18/3 = 6

⇨ (6, 0) is a point on the line (1)

To determine two points on the straight line 500x + 1000y = 4000 …… (2)

Let x = 0

500(0) + 1000y = 4000

1000y = 4000

y = 4000/1000 = 4

⇨ (0, 4) is a point on the line (2)

Let y = 0

500x + 1000(0) = 4000

500x = 4000

x = 4000/500 = 8

(8, 0) is a point on the line (2)

To determine two points on the straight line 12x + 8y = 48 ……… (3)

Let x = 0

12(0) + 8y = 48

8y = 48

y = 48/8 = 6

⇨ (0, 6) is a point on the line (3)

Let y = 0

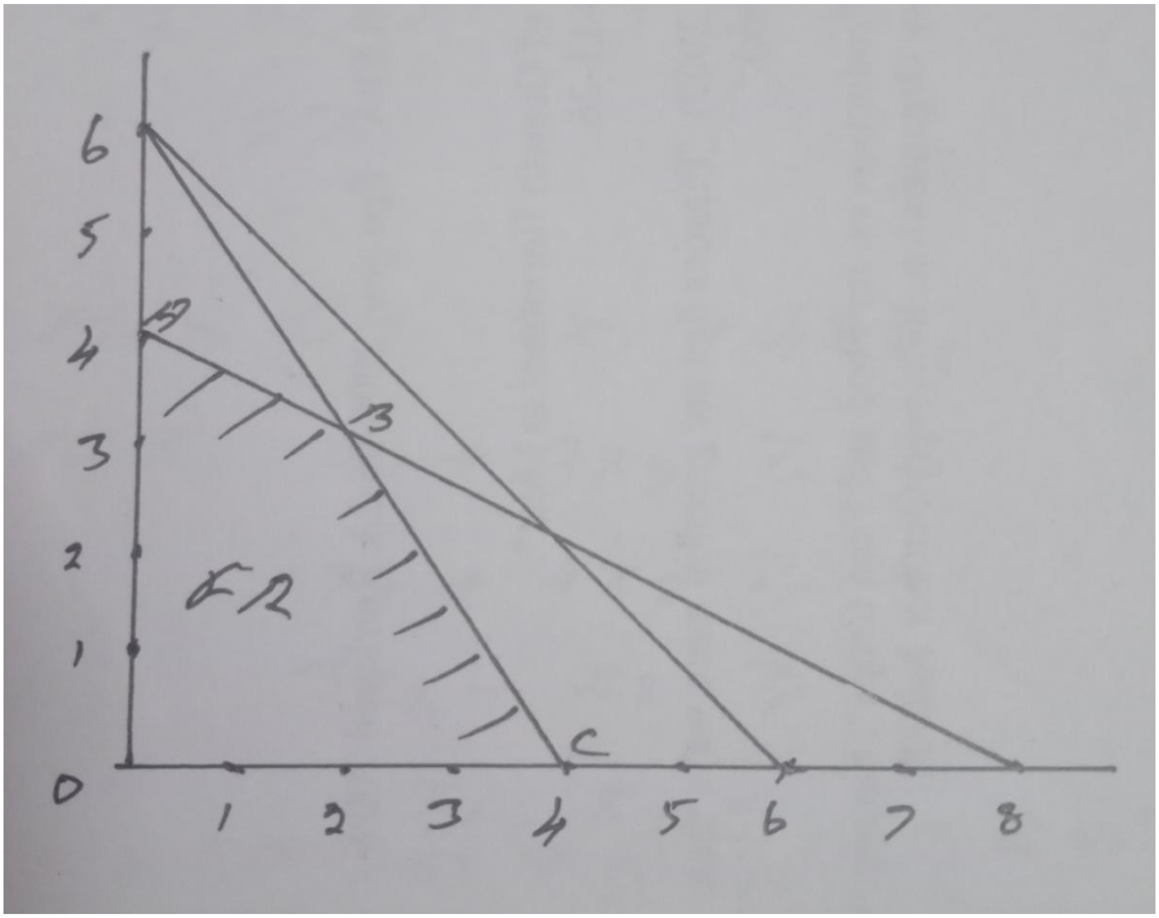
12x + 8(0) = 48

12x = 48

x = 48/12 = 4

(4, 0) is a point on the line (3)

Graph



Feasible Region	Z = 6m + 5y	Value
A (0, 4)	6(0) + 5(4)	20
B (2, 3)	6(2) + 5(3)	27
C (4, 0)	6(4) + 5(0)	24

Optimal Solution

x = 2, y = 3

Z = 6(2) + 5(3)

Z = 27

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