Class	: I M.Sc (Applied Mathematics)
Subject Code	: 18PMT
Subject	: Graph Theory - Unit V - Directed Graph
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GRAPH THEORY - UNIT - V - DIRECTED GRAPH

orientation—a directed graph. Formally, a *directed graph* D is an ordered triple $(V(D), A(D), \psi_D)$ consisting of a nonempty set $V(D)$ of *vertices*, a set $A(D)$, disjoint from $V(D)$, of *arcs*, and an *incidence function* ψ_D that associates with each arc of D an ordered pair of (not necessarily distinct) vertices of D. If a is an arc and u and v are vertices such that $\psi_D(a) = (u, v)$, then a is said to *join u to v*; u is the *tail* of a, and v is its *head*. For convenience, we shall abbreviate 'directed graph' to *digraph*. A digraph D' is a *subdigraph of D* if $V(D') \subseteq V(D)$, $A(D') \subseteq A(D)$ and $\psi_{D'}$ is the restriction of ψ_D to $A(D')$. The terminology and notation for subdigraphs is

With each digraph D we can associate a graph G on the same vertex set; corresponding to each arc of D there is an edge of G with the same ends. This graph is the *underlying graph* of D. Conversely, given any graph G, we can obtain a digraph from G by specifying, for each link, an order on its ends. Such a digraph is called an *orientation of G*.

edge

Just as with graphs, digraphs have a simple pictorial representation. A digraph is represented by a diagram of its underlying graph together with arrows on its edges, each arrow pointing towards the head of the corresponding arc. A digraph and its underlying graph are shown in figure 10.1.

Every concept that is valid for graphs automatically applies to digraphs too. Thus the digraph of figure 10.1a is connected and has no cycle of length three because its underlying graph (figure 10.1b) has these properties. However, there are many concepts that involve the notion of orientation, and these apply only to digraphs.

G – graph (V, E)

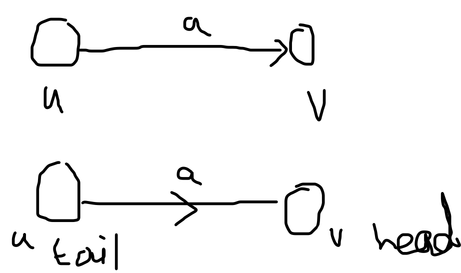
D – digraph

$D = (V, A, \psi_D)$

$\psi_D : A \to V \times V$

$a \in A$

$\psi_D(a) = (u, v)$



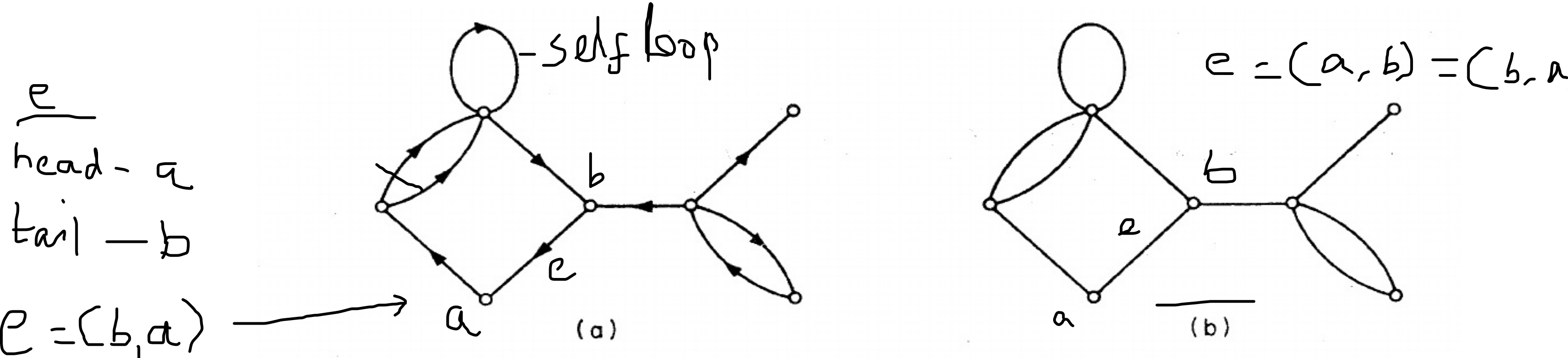


Figure 10.1. (*a*) A digraph *D*; (*b*) the underlying graph of *D*

A *directed walk* in *D* is a finite non-null sequence $W = (v_0, a_1, v_1, \ldots, a_k, v_k)$, whose terms are alternately vertices and arcs, such that, for $i = 1, 2, \ldots, k$, the arc a_i has head v_i and tail v_{i-1}. As with walks in graphs, a directed walk $(v_0, a_1, v_1, \ldots, a_k, v_k)$ is often represented simply by its vertex sequence $(v_0, v_1, \ldots, v_k)$. A *directed trail* is a directed walk that is a trail; *directed paths*, *directed cycles* and *directed tours* are similarly defined.

No edge appears more than once in a directed walk. A vertex may appear more than once in a walk.

Vertices with which a diected walk begins and ends are called its terminal vertices.

It is also possible for a directed walk start and end on the same vertex. Such a directed walk is called a Closed directed walk; Otherwise it is called a Open directed walk.

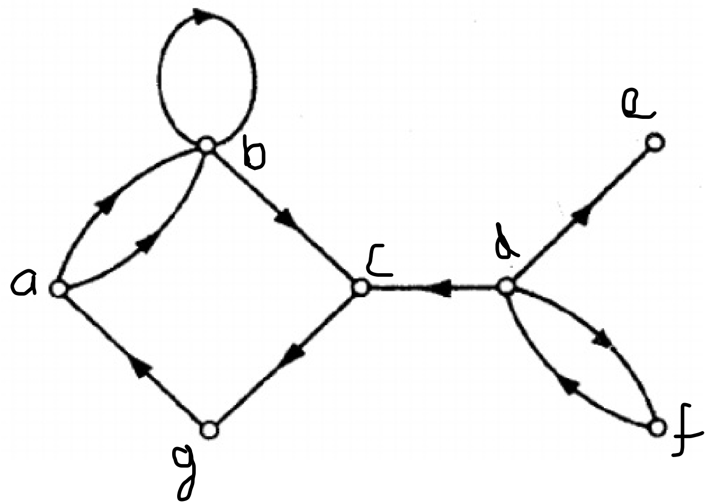
An Open directed walk is also referred as a directed trail.

An Open directed walk in which no vertex appears more than once is called a directed path.
Note that self loop can be included in a directed walk but not in a directed path.

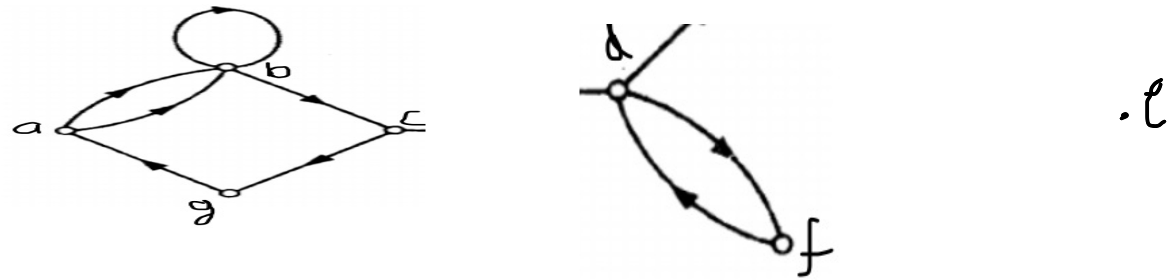
The number of edges in a directed path is called the length of the directed path.

A closed directed walk in which no vertex except the initial and final vertices appears more than once is called a directed cycle.

Example:



Directed walk or Open walk: (d, f, d, e); (d, c, g, a, b, b,c)
Directed Closed walk: (a, b, b, c, g, a); (d, f, d);
Directed Path: (c, g, a, b); (d, c, g, a)
Directed cycle: (d, f, d); (a, b, c, g, a)
The vertex a is reachable from c
The vertices a and c; a and b; and g are diconnected.
The vertex c and d are not diconnected.
similarly, the vertex d and e are not di connected.



The digraph is not diconnected and it has three dicomponents.

If there is a directed (u, v)-path in D, vertex v is said to be *reachable* from vertex u in D. Two vertices are *diconnected* in D if each is reachable from the other. As in the case of connection in graphs, diconnection is an equivalence relation on the vertex set of D. The subdigraphs $D[V_1], D[V_2], \ldots, D[V_m]$ induced by the resulting partition $(V_1, V_2, \ldots, V_m)$ of $V(D)$ are called the *dicomponents* of D. A digraph D is *diconnected* if it has exactly one dicomponent. The digraph of figure 10.2a is not diconnected; it has the three dicomponents shown in figure 10.2b.

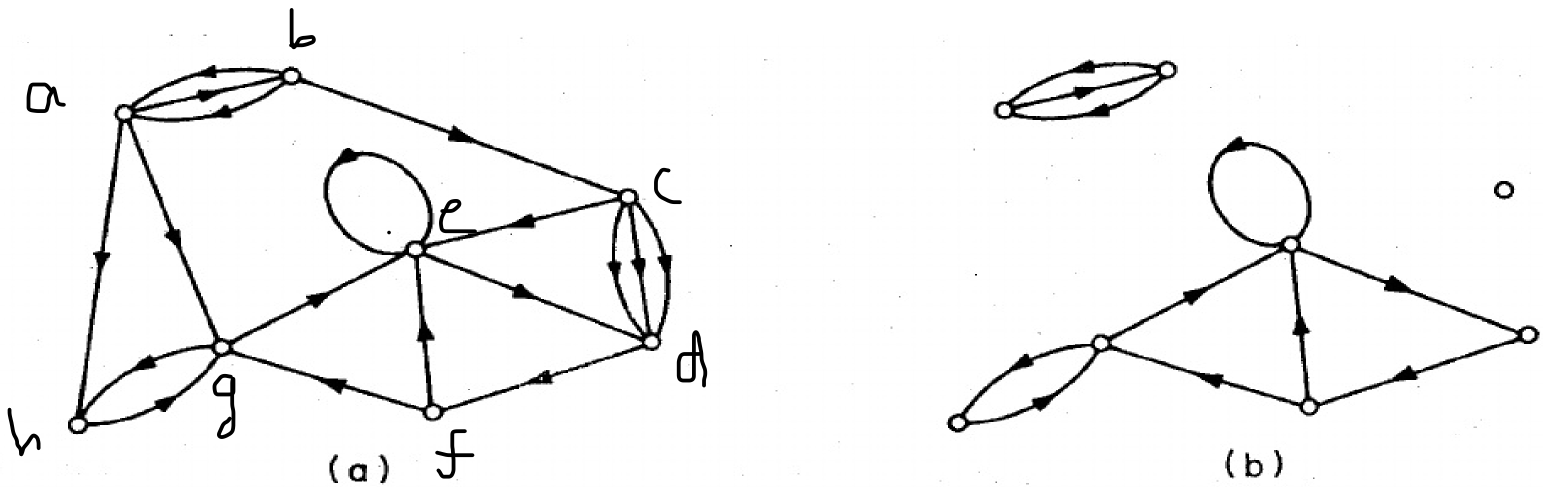
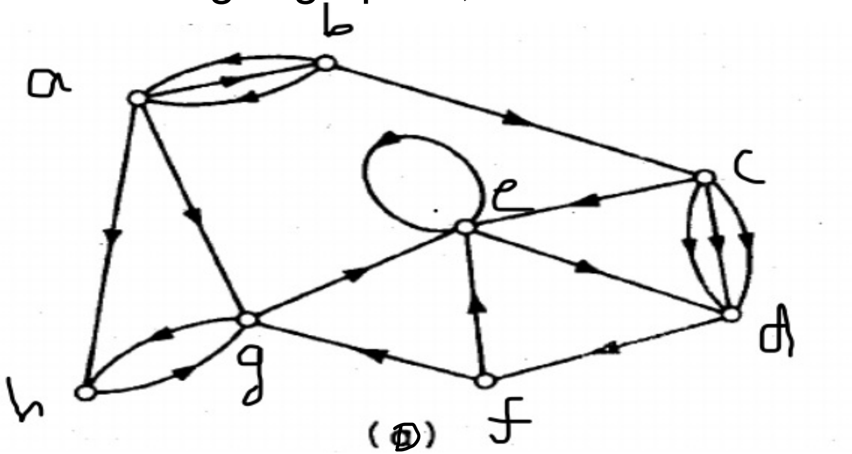
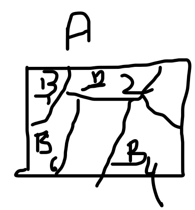


Figure 10.2. (a) A digraph D; (b) the three dicomponents of D

the other. As in the case of connection in graphs, diconnection is an equivalence relation on the vertex set of D. The subdigraphs $D[V_1], D[V_2], \ldots, D[V_m]$ induced by the resulting partition $(V_1, V_2, \ldots, V_m)$ of $V(D)$ are called the *dicomponents* of D. A digraph D is

Consider, the following Digraph D,





$B_i \cap B_j = \phi, \ i \neq j$

$\bigcup_i B_i = A$

In The above figure,

D[a] = { a, b}
D[b] = { a, b}
D[c] = { }
D[d] = { d, f, e, g, h}
D[e] = { d, e, f, g, h}
D[f] = { d, e, f, g, h}
D[g] = { d, e, f, g, h}
D[h] = { d, e, f, g, h}

That is,

D[a], diconnection of a in V(D) is called the equivalence class.

Any two Equivalence classes are either disjoint or identical.

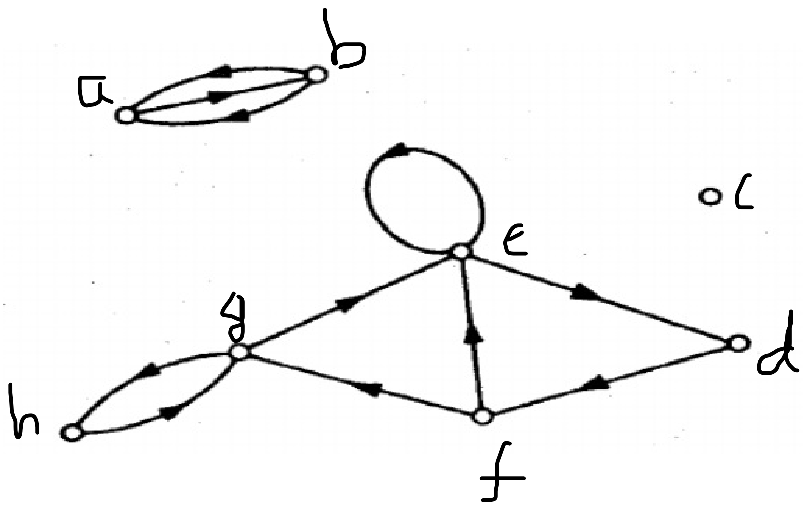
Hence, all the equivalence classes of D forms a partition in V(D).

Here, D[a] = D[b]

D[c] = { }

D[d] = D[e] = D[f] = D[g] = D[h]

Here, (a, c, d) is called dicomponent.



The Three dicomponents of D.

For any Digraph D,

Define the relation R on V(D) by a $\sim$ b if a is reachable from b and vice versa.

Then the relation R is called diconnected relation.

Clearly R is an equivalence relation.

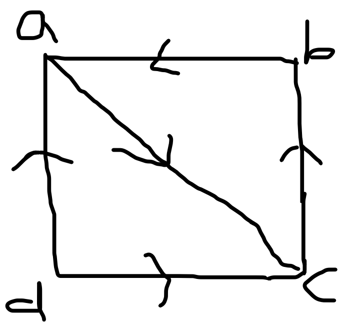
For example,

Consider the Previous digraph D,then

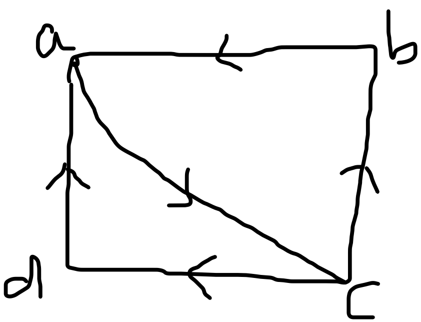
Let R = { d, e, f, g, h }

Then, R is an equivalence relation.

The *indegree* $d_D^-(v)$ of a vertex v in D is the number of arcs with head v; the *outdegree* $d_D^+(v)$ of v is the number of arcs with tail v. We denote the minimum and maximum indegrees and outdegrees in D by $\delta^-(D)$, $\Delta^-(D)$, $\delta^+(D)$ and $\Delta^+(D)$, respectively. A digraph is *strict* if it has no loops and no two arcs with the same ends have the same orientation.

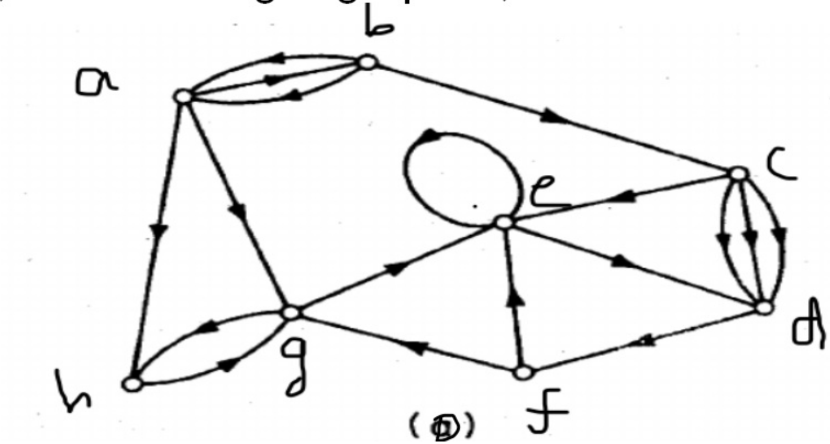


A Strict digraph with two dicomponents.



A diconnected digraph

Consider, the following Digraph D,



$d_D^-(a) = 2$

$d^-(b) = 1$

$d^-(c) = 1$

$d^-(d) = 4$

$d^-(e) = 4$

$d^-(f) = 1$

$d^-(g) = 3$

$d^-(h) = 2$

$\delta^-(D) = 1$

$\Delta^-(D) = 4$

$d_D^+(a) = 3$

$d_D^+(b) = 3$

$d_D^+(c) = 4$

$d_D^+(d) = 1$

$d_D^+(e) = 2$

$d_D^+(f) = 2$

$d_D^+(g) = 2$

$d_D^+(h) = 1$

$\delta^+(D) = 1$

$\Delta^+(D) = 4$