Unit - I

Finite, Countable and uncountable Sets.

Defn:

Consider two sets A and B, and suppose that with each element x of A there is associated an element of B, which we denoted by $f(x)$.

Then f is said to be a function from A and B.

The set A is called the domain of f and the set of all values of f is called the range of f.

Defn:

If there exists a 1-1 mapping of A onto B we say that A and B can be put in 1-1 correspondence or A and B have the same cardinal ~~no~~ number.

i.e, A and B are equivalent and we write $A \sim B$.

This relation clearly has the following properties.

i) $A \sim B$ is a reflexive.

ii) If $A \sim B$ then $B \sim A$ is a symmetric.

iii) If $A \sim B$ and $B \sim C$ then $A \sim C$ is transitive.

Any relation with these properties is called an equivalence relation.

Defn:

For any positive integer n, let J_n be the set whose elements are integers $1, 2 \dots n$.

Let J be the set consisting of all positive integers. for any set A, we say:

i) A is finite if $A \sim J_n$ for some n

ii) A is infinite if A is not finite

iii) A is countable if $A \sim J_n$

iv) A is uncountable if A is neither finite nor countable.

v) A is atmost countable if A is finite or countable.

countable sets are sometimes called enumerable or denumerable.

Ex:

Let A be the set of integers. Then A is countable

i.e, $0, 1, -1, 2, -2 \dots$.

Defn:

Let f be a function and J be the set of all ~~postive~~ positive integers.

If $f(n) = x_n$ for $n \in J$, we denote the sequence f by the symbol $\{x_n\}$.

The values of f are called the terms of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a sequence in A.

Theorem : 1

Every infinite subset of a countable set is countable.

Proof:

Let A be a countable and $E \subset A$.

Let E be infinite, the element x of A in a sequence $\{x_n\}$ of distinct elements.

Construct a sequence $\{n_k\}$. Let n_1 be the smallest positive integer $\ni: x_n \in E$.

$n_1, n_2 \dots n_{k-1}$ $(k = 2, 3 \dots)$.

Let n_k be the smallest integer $n_{k-1} \ni: x_{n_k} \in E$

$f(k) = x_{n_k}$ $(k = 1, 2 \dots)$

we obtain a 1-1 correspondence between E and J

i.e) E is countable.

Hence the proof.

Defn:

Let A and Ω be sets, and suppose that with each element $\alpha (A)$ there is associated a subset of Ω which we denoted by E_α.

Defn:

The union of sets E_α is defined to be the set $S \ni : x \in S$ iff $x \in E_\alpha$ for at least one $\alpha \in A$. ie) $S = \bigcup_{\alpha \in A} E_\alpha$

Defn:

The intersection of the sets E_α is defined to be the set $P \ni : x \in P$ iff $x \in E_\alpha$ for every $\alpha \in A$.

ie, $P = \bigcap_{\alpha \in A} E_\alpha$.

Ex:

Let $E_1 = \{1, 2, 3\}$ and $E_2 = \{2, 3, 4\}$

$E_1 \cup E_2 = \{1, 2, 3, 4\}$

$E_1 \cap E_2 = \{2, 3\}$.

Theorem: 2

Let $\{E_n\}$, $n = 1, 2 \cdots$ be a sequence of countable sets, and put $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable.

Proof:

Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2 \cdots$

and consider the infinite array

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$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \end{array}$$

in which the element of E_n from the n^{th} row. The array contains all elements of S.

AS indicated by the arrows, these elements can be arranged in a sequence,

$\{x_{11}\}, \{x_{21}, x_{12}\}, \{x_{31}, x_{22}, x_{13}\}, \{x_{41}, x_{32}, x_{23}, x_{14}\}, \ldots \rightarrow (1)$

If any two of the sets E_n have elements in common, these will appear more than once in (1).

Hence there is a subset T of the sets of all positive integers $\rightarrow$: $S \sim T$.

S is ~~the~~ at most countable.

since $E_1 \subset S$ and E_1 is infinite, S is infinite and this countable.

Cor:

suppose A is atmost countable, and for every $\alpha \in A$, B_α is atmost countable.

$T = \bigcup_{\alpha \in A} B_\alpha$. Then T is atmost countable.

Theorem: 3

Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, a_2 \dots a_n)$ where $a_k \in A$ $(k = 1, \dots n)$, and the elements $a_1, a_2 \dots a_n$ need not be distinct. Then B_n is countable.

Proof:

B_1 is countable. $[\because B_1 = A]$

Suppose B_{n-1} is countable $(n = 2, 3 \dots)$. The elements of B_n are of the form (b, a), $b \in B_{n-1}$ and $a \in A$.

For every fixed b, the set of pairs (b, a) is equivalent to A. and countable.

Thus, B_n is the union of a countable set of countable sets. Then B_n is countable.

Cor:

The set of all rational numbers is countable.

Theorem: 4

Let A be the set of all sequences, whose elements are the digits 0 and 1. This set A is uncountable. The elements of A are sequences like $1, 0, 0, 1, 0, 1, 1, 1, \dots$.

Proof:

Let E be a countable subset of A, and

Let E consist of the sequence $S_1, S_2 \dots$.

If the n^{th} digit in S_n is 1, Let n^{th} digit of S be 0. and vice-versa, Then the sequence s differs from every member of E in atleast one place.

Hence $s \notin A$. But $s \in A$. So E is a proper subset of A.

Every countable subset of A is a proper subset of A.

It follows that A is uncountable.

METRIC SPACES

Defn:

A set X, is said to be a metric space if with any two points p and q of X there is associated a real number $d(p,q)$, called the distance from p to q, ∋:

i) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$

ii) $d(p,q) = d(q,p)$

iii) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

Any function with these three properties is called a distance function or a metric.

Ex:

R - real number system and C - complex number system are metric spaces ~~with~~ w.r.t $d(x,y) = |x-y|$.

Defn:

* Let the set of all real numbers x ∋: $a < x < b$. then (a,b) are called segment.
* Let the set of all real numbers x ∋: $a \leq x \leq b$. then $[a,b]$ are called interval.

Defn:

* If $x \in R^n$ and $r > 0$, the open ball B with center at x and radius r is defined to be the set of all $y \in R^n$ $\ni$: $|y - x| < r$

* A set $E \subset R^n$ is convex if $\lambda x + (1-\lambda) y \in E$. whenever $x \in E$, $y \in E$ and $0 < \lambda < 1$.

Defn:

Let X be a metric space, All points/and sets,

* A neighbourhood of a point p is a set $N_r(P)$ cosisting of all points q $\ni$: $d(p, q) < r$. The number r is called the radius of $N_r(P)$

*) A point p is a limit point of the set E if every neighbourhood of p contains a point $q \neq p$ $\ni$: $q \in E$.

*) If $p \in E$ and p is a not a limit point of E, then p is called an isolated point of E.

*) E is closed if every limit point of E is a point of E

*) A point p is an interior point of E if there is a neighbourhood N of p $\ni$: $N \subset E$.

*) E is open if every point of E is an interior point of E.

*) The complement of E (E^c) is the set of all points $p \in X$ $\ni$: $p \notin X$.

*) E is perfect E is closed and if every point of E is a limit point of E.

i) E is bounded if there is a real number M and a point $q \in X \ni d(p,q) < M \ \forall \ p \in E$

j) E is dense in X if every point of X is a limit point of E, or a point of E.

Theorem:

Every neighbourhood is an open set.

Proof:

Consider a neighbourhood $E = N_r(p)$

Let q be any point of E. Then there is a positive real number $h \ni d(p,q) = r - h$. $\forall$ points $s \ni d(q,s) < h$ we have

$$d(p,s) \le d(p,q) + d(q,s) < r - h + h = r$$

so that $s \in E$. Thus q is an interior point of E.

Theorem:

If p is limit point of a set E, then every neighbourhood of p contains infinitely many point of E.

Proof:

Let $N_r(p)$ is a neighbourhood which contains only a finite number of points of E.

Let $q_1, q_2 \cdot q_n$ be these points $N \cap E$. which are distinct from p. and put $r = \min\limits_{1 \le m \le n} d(p, q_m)$.

$r > 0$.

The neighbourhood $N_r(p)$ contains no point q of $E \rightarrow \because$ $q \ne p$, so p is not a limit point of E. This is $\Rightarrow\Leftarrow$

Hence the proof.

Cor: A finite point set has no limit points.

Thm

A set E is open iff its complement is closed.

proof:

suppose that E^c is closed. choose $x \in E$.

Then $x \notin E^c$ and x is not a limit point of E^c.

Hence, $\exists$ a neighbourhood $N_1(x) \ni: E^c \cap N = \phi$.

i.e.) $N \subset E$.

Thus x is an interior point of E and E is open.

Next, suppose E is open. Let x be a limit point of E^c

Then every $N(x) \subset E^c$, so that x is not an interior point of E.

since E is open. $\Rightarrow x \in E^c$. then E^c is closed.

Cor:

A set E is closed iff it complement is open.

Defn:

If X is a metric space, if $E \subset X$ and if E' denotes the set of all limit points of E in X. then the closure of E is the set $\overline{E} = E \cup E'$

Thm:

If X is a metric space and $E \subset X$ then

a) $\overline{E}$ is closed

b) $E = \overline{E}$ iff E is closed.

c) $\overline{E} \subset F$ for every closed set $F \subset X \ni: E \subset F$.

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Proof:

a) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E.

Hence p has a neighbourhood which doesnot intersect E. The complement of $\bar{E}$ is open.

Hence $\bar{E}$ is closed.

b) if $E = \bar{E}$ $\Leftrightarrow$ E is closed. if E is closed, then $E' \subset E$ $\therefore \bar{E} = E \cup E' = E$

(c) If F is closed and $E \subset F$

Hence $F \supset E'$. Thus $F \supset \bar{E}$.

COMPACT SETS

Defn:

Let E be an set of open cover in a metric space X a collection $\{G_\alpha\}$ of open subsets of X $\ni$: $E \subset \bigcup_\alpha G_\alpha$.

Defn.:

A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

Theorem:

Suppose $K \subset Y \subset X$ Then K is compact relative to X iff K is compact relative to Y.

Proof:

Let K is compact relative to X. and let $\{V_\alpha\}$ be a collections of sets, open relative to Y $\ni$: $K \subset \bigcup_\alpha V_\alpha$.

There are sets G_α, open relative to X ∋

$V_\alpha = Y \cap G_\alpha \; \forall \alpha$

K is compact relative to X, $K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$ for some $\alpha_1, \alpha_2 \dots \alpha_n$.

$K \subset Y$ we get $K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$.

Then K is compact relative to Y.

conversely,

suppose that K is compact relative to Y.

Let G_α be a collection of open subsets of X which covers K.

put $V_\alpha = Y \cap G_\alpha$.

Then $K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ for $\alpha_1, \alpha_2 \dots \alpha_n$ and $V_\alpha \subset G_\alpha$

$K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \Rightarrow K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$

Hence the proof.

Theorem:

Closed subsets of compact sets are compact.

proof:

suppose $F \subset K \subset X$. F is closed and K is compact.

Let $\{V_\alpha\}$ be an open cover of F.

If F^c is adjoined to $\{V_\alpha\}$. we obtain an open cover $\Omega(K)$. since K is compact.

There is a finite subcollection $\phi(\Omega)$ which covers K and hence F. If F^c is a member of ϕ. an open cover of F. A finite subcollection of $\{V_\alpha\}$ covers F.

Hence the proof.

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Theorem (Weierstrass)

Every bounded infinite subset of R^k has a limit point in R^k.

Proof:

Being bounded, the set E is a subset of a k-cell $I \subset R^k$. Every k cell is compact. I is compact, so E has a limit point in I.

If E is an infinite subset of a compact set K, then E has a limit point in K.

PERFECT SETS

Thm:

Let P be a nonempty perfect set in R^k. Then P is uncountable.

Proof:

Let P has a limit points, and P be infinite. Suppose P is countable, and denote the points $x_1, x_2 \ldots$ construct the sequence $\{V_n\}$ of neighbourhoods

Let V_1 be any neighbourhood of x_1. If V_1 consists of all $y \in R^k \ni : |y - x_1| \le r$. The closure $\bar{V}_1$ (V) is the set of all $y \in R^k \ni \cdot |y - x_1| \le r$

Suppose V_n has been constructed, so that $V_n \cap P$ is not empty.

every point of p is a limit point of P, there is a neighbourhood $V_{n+1} \ni$: $\overline{V}_{n+1} \subset V_n$, $x_n \notin \overline{V}_{n+1}$

$V_{n+1} \cap P$ is not empty

put $K_n = \overline{V}_n \cap P$ $\overline{V}_n$ is closed and bounded, $\overline{V}_n$ is compact

$x_n \notin K_{n+1}$ no point of P lies in $\bigcap_1^{\alpha} K_n$ since $K_n \subset P$.

$\Rightarrow \bigcap_1^{\infty} K_n$ is empty.

K_n is non empty and $K_n \supset K_{n+1}$

This is contradicts the theorem.

CONNECTED SETS

Defn:

Two subsets A and B of a metric space X are said to be separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. i.e) if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subset X$ is said to be connected if E is not a union of two non empty separated sets.

UNIT - 2

CONVERGENT SEQUENCES

Defn:

A sequence $\{p_n\}$ in a metric space X is said to converge if there is a point $p \in X$. For every $\varepsilon > 0$ there is an integer $N \ni ; n \geq N \Rightarrow d(p_n, p) < \varepsilon$.

If $\{p_n\}$ converges to p or p is the limit of $\{p_n\}$ and $p_n \to p$ or $\lim_{n \to \infty} p_n = p$.

If $\{p_n\}$ does not converge it is called diverge.

Theorem:

Let $\{p_n\}$ be a sequence in a metric space X.

a) $\{p_n\}$ converges to $p \in X$ iff ~~if~~ every neighbourhood of p contains all but finitely many of the terms of $\{p_n\}$.

b) if $p \in X$, $p' \in X$ and if $\{p_n\}$ converges to p and p', then $p' = p$

c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

d) If $E \subset X$ and if p is a limit point of E, there is a sequence $\{p_n\}$ in $\ni$: $p = \lim_{n \to \infty} p_n$.

Proof:

$p_n \to p$ and let V be a neighbourhood of p.

for some $\varepsilon > 0 \Rightarrow d(p, q) < \varepsilon$, $q \in X \Rightarrow q \in V$.

$\exists N \ni : n \geq N \Rightarrow d(p_n, p) < \varepsilon$

Thus $n \geq N \Rightarrow p_n \in V$.

conversely, suppose every neighbourhood of p contains all but finitely many of the p_n.

$\varepsilon > 0$, V be the set of all $q \in X \ni: d(p,q) < \varepsilon$

$\exists N \ni: p_n \in V$ if $n \geq N$.

$d(p_n, p) < \varepsilon$ if $n \geq N$. hence $p_n \to p$

b) Let $\varepsilon > 0$. $\exists$ integers $N, N' \ni: n \geq N \Rightarrow d(p_n, p) < \varepsilon/2$

$n \geq N' \Rightarrow d(p_n, p') \, d(p_n, p') < \varepsilon/2$

Hence $n \geq \max(N, N')$

$d(p, p') \leq d(p, p_n) + d(p, p') < \varepsilon$

$d(p, p') = 0$.

c) $p_n \to p$ $N \ni: n > N \Rightarrow d(p_n, p) < 1$

$r = \max\{1, d(p_1, p) \ldots d(p_N, p)\}$

Then $d(p_n, p) \leq r$ for $n = 1, 2, 3$

d, For each positive integer n,

$p_n \in E \ni: d(p_n, p) < 1/n$ $\varepsilon > 0 \ni: N_\varepsilon > 1$

If $n > N \Rightarrow d(p_n, p) < \varepsilon$

Hence $p_n \to p$

Defn:

Given a sequence $\{p_n\}$ consider a sequence $\{n_r\}$ of positive integers, $\ni: n_1 < n \ldots$ Then the sequence $\{p_n\}$ is called subsequence of $\{p_n\}$.

Defn:

A sequence $\{P_n\}$ in a metric space X is said to be a cauchy sequence if for every $\varepsilon > 0$ there is an integer N $\ni$: $d(P_n, P_m) < \varepsilon$ if $n \geq N$ and $m \geq N$

Defn:

Let E be a subset of a metric space X and S be the set of all real numbers of the form $d(p,q)$ with $p \in E$ and $q \in E$. The sup of S is called the diameter of E

Defn:

A metric space in which every cauchy sequence converges is said to be complete.

Defn:

A sequence $\{s_n\}$ of real numbers is said to be monotonically increasing if $s_n \leq s_{n+1}$ and monotonically decreasing if $s_n \leq s_{n+1}$ $n = 1, 2 \ldots$

Theorem:

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges iff [illegible] bounded.

Proof

[illegible] s_{n+1}. Let E be the range of $\{s_n\}$. [illegible] bounded and s be the least upper bound of E. [illegible] there is an integer $N \ni$: $s - \varepsilon < s_n \leq s$. [illegible] converges to s.

Hence the proof.

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UPPER AND LOWER LIMITS :

Defn:

Let $\{s_n\}$ be a sequence of real numbers ~~with~~ for every real M there is an integer $N \ni: n \geq N \Rightarrow s_n \geq M$. $s_n \to$

By if every real M there is an integer $N \ni: n \geq N \Rightarrow s_n \leq M$

$s_n \to$

Defn:

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers $x \ni: s_{n_i} \to x$ for some sub sequence $\{s_{n_k}\}$. This set E contains all sub sequential limits.

$s^* = \sup E$ and $s_* = \inf E$.

The numbers s^*, s_* are called the upper and lower limit of $\{s_n\}$. i.e., $\lim\limits_{n \to \infty} \sup s_n = s^*$.

Remark:

If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is some fixed number and if $s_n \to 0$ then $x_n \to 0$.

Theorem:

a) If $p > 0$ then $\lim\limits_{n \to \infty} \frac{1}{n^p} = 0$

b) If $p > 0$ then $\lim\limits_{n \to \infty} \sqrt[n]{p} = 1$

c) $\lim\limits_{n \to \infty} \sqrt[n]{\ } = 1$

d) If $p > 0$ and α is real then $\lim\limits_{n \to \infty} \frac{n^\alpha}{(1+p)^n} = 0$

e) if $|x| < 1$ then $\lim\limits_{n \to \infty} x^n = 0$.

Proof :

a) Take $n > \left(\frac{1}{\epsilon}\right)^{1/p}$

b) $p > 1$, $x_n = \sqrt[n]{p} - 1$, $x > 0$

$1 + n x_n \le (1 + x_n)^n = p$

$0 \le x_n \le \frac{p-1}{n}$

$x_n \to 0$. If $p = 1$, (b) is trivial and if $0 < p < 1$

c) $x_n = \sqrt[n]{n} - 1$ Then $x_n \ge 0$

$n = (1 + x_n)^n \ge \frac{n(n-1)}{2} x_n^2$

$0 \le x_n \le \sqrt{\frac{2}{n-1}}$ $(n \ge 2)$

d) Let k be an integer $\ni$: $k > \alpha$, $k > 0$ for $n > 2k$

$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$

$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k}$ $(n > 2k)$

[$\because$ $\alpha - k < 0$, $n^{\alpha - k} \to 0$ by (a)]

f) Take $\alpha = 0$ in d)

Theorem: [ROOT TEST]

Given Σa_n put $\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$

Then

a) if $\alpha < 1$ Σa_n converges

b) if $\alpha > 1$ Σa_n diverges

c) if $\alpha = 1$ that gives no information

Proof:

If $\alpha > 1$ choose β so that $\alpha < \beta < 1$ and N be an integer

$\rightarrow$: $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$ i.e., $n \geq N \rightarrow |a_n| < \beta^n$ [$\because 0 < \beta < 1$]

$\Sigma \beta^n$ converges.

convergence of Σa_n follows from the comparison Test

If $\alpha > 1$ then $\{n_k\} \rightarrow : \sqrt[n]{|a_{n_k}|} \rightarrow \infty$. $|a_n| < 1$ for n

$a_n \rightarrow 0$ convergence of Σa_n

consider $\Sigma \frac{1}{n}, \Sigma \frac{1}{n^2}$ for each of the series $\alpha = 1$

but the first diverges, the 2nd converges

Theorem [RATIO TEST]

The series Σa_n i) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

ii) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$ where n_0 is some fixed integer.

Proof:

In (i) holds $\beta < 1$ and N be an integer $\rightarrow$:

$|a_{N+1}| < \beta |a_N|$

$|a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|$ ...

.

$|a_{N+p}| < \beta^p |a_N|$

i.e.) $|a_n| < |a_N| \beta^{-N} \cdot \beta^n$

since $\Sigma \beta^n$ converges

If $|a_{N+1}| \geq |a_n|$ for $n \geq n_0$ i.e.) $a_n \rightarrow 0$ does not hold (b) follows.

Defn:

Given a sequence $\{a_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series.

Theorem: Addition and Multiplication of series

If $\sum a_n = A$ and $\sum b_n = B$ then $\sum (a_n + b_n) = A + B$ and $\sum c a_n = c A$ for any fixed c.

Proof:

Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$

$$A_n + B_n = \sum_{k=0}^{n} (a_k + b_k)$$

$\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$

$$\lim_{n\to\infty} (A_n + B_n) = A + B.$$

UNIT - III

LIMIT OF FUNCTIONS

Defn:

Let X and Y be metric spaces, suppose $E \subset X$, f maps E into Y and p is a limit point of E.

We write $f(x) \to q$ as $x \to p$ or $\lim_{n\to\infty} f(x) = q$

Cor:

If f has a limit at p, this limit is unique

Defn:

Consider two complex functions f and g both defined on E.

If f and g maps E into R^k, we define $f+g$ and $f-g$ by $(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

and if λ is a real number $(\lambda f)(x) = \lambda f(x)$

Remark:

If f and g map E into R^k, then

$\lim_{x \to p} (f g)(x) = A B$

Defn:

If f is continuous at every point of E, then f is said to be continuous on E.

f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

Defn:

A mapping f of a set E into R^k is said to be bounded if there is a real number $M \ni: |f(x)| \leq M$ $\forall x \in E$.

Theorem:

Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Proof:

Let $\{V_\alpha\}$ be an open cover of $f(X)$.

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Since f is continuous. $f^{-1}(V_\alpha)$ is open since X is compact.

$\alpha_1, \alpha_2 \dots \alpha_n \ni : X \subset f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n}) \longrightarrow (1)$

since $f(f^{-1}(E)) \subset E \;\forall\; E \subset Y$

$(1) \Rightarrow f(X) \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$.

Hence The proof.

Theorem:

Suppose f is continuous real function on a compact metric space X. and $M = \sup\limits_{P \in X} f(P)$

$m = \inf\limits_{P \in X} f(P)$. Then $\exists$ points $p, q \in X \ni : f(p) = M$ and

$f(q) = m$.

Proof:

$f(X)$ is closed and bounded set of real numbers.

hence $f(x)$ contains $M = \sup f(x)$ and

$m = \inf f(x)$.

Theorem: Continuity and connectedness:

If f is a continuous mapping of a metric X into a metric space Y and if E is a connected subset of X, then $f(E)$ is connected.

Proof:

Assume that $f(E) = A \cup B$, where A and B are non empty separated subsets of Y.

$G = E \cap f^{-1}(A)$ $H = E \cap f^{-1}(B)$ then $E = G \cup H$.

and neither G nor H is empty.

Since $A \subset \bar{A}$ $G \subset f^{-1}(\bar{A})$, since f is continuous.

$\bar{G} \subset f^{-1}(\bar{A}) \Rightarrow f(\bar{G}) \subset (\bar{A})$

$f(H) = B$ and $\bar{A} \cap B$ is empty. and $\bar{G} \cap H$ is empty.

Thus G and H are separated.

Defn:

Let f be real on (a, b). Then f is said to be monotonically increasing on (a, b) if $a < x < y < b$ $\Rightarrow f(x) \leq f(y)$ if $f(x) \geq f(y)$ then f is monotonically decreasing.

Defn:

Let f be defined on $[a, b]$ for any $x \in [a, b]$ from the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x) \text{ and define}$$

$$f'(x) = \lim_{t \to x} \phi(t).$$

Theorem:

Let f be defined on $[a,b]$ if f is differentiable at a point $x \in [a,b]$ then f is continuous at x

Proof:

As $t \to x$ $\quad f(t) - f(x) = \dfrac{f(t) - f(x)}{t - x}(t - x) \to f'(x) \cdot 0$

The converse of the theorem is not true

Defn:

Let f be a real function defined on a metric space X. we say that f has a local maximum at a point $p \in X$, if $\exists\ \delta > 0 \ni f(q) \le f(p)\ \forall\ q \in X$ with $d(p,q) < \delta$.

UNIT-4

Differentiation

The derivative of a real function

Def:

Let f be defined (and real valued) on $[a,b]$. For any $x \in [a,b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x}, \ [(a<t<b), t \neq x]$$

and define $f'(x) = \lim_{t \to x} \phi(t)$

If f' is defined at a point x, we say that f is differentiable at x.

If f' is defined at every point of set $E \subset [a,b]$. we say that f is differentiable on E.

Thm: 1

(-Imp) Let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x.

Proof

As $t \to x$,

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot t - x \quad \text{[By known theorem]}$$

$$= f'(x)(t-x)$$

$$= f'(x)(x-x)$$

$$f(t) - f(x) = 0$$

The converse of the theorem is not true.

Thm : 2

Suppose f and g are defined on $[a,b]$.

And are differentiable at a point $x \in [a,b]$.

Then $f+g$, fg & f/g are differentiable at x,

and a) $(f+g)'(x) = f'(x) + g'(x)$

b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

c) $(f/g)'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

In (c), we assume of course that $g(x) \neq 0$.

Proof

a) Let f and g are defined on $[a,b]$.

By $f+g$, we need the function which assigns to each point x on $[a,b]$. The number $f(x) + g(x)$.

|||ly, we define $f'(x) + g'(x)$.

Clearly. $(f+g)'(x) = f'(x) + g'(x)$.

b) Let $h = fg$. As $t \to x$

Then, $h(t) - h(x) = f(t)g(t) - f(x)g(x) + f(t)g(x) - f(t)g(x)$

$h(t) - h(x) = f(t)\left[g(t) - g(x)\right] + g(x)\left[f(t) - f(x)\right]$

$\div (t-x)$ on b.s and note that $f(t) \to f(x)$ as $t - x$

$$\frac{h(t) - h(x)}{t-x} = f(x)\left[\frac{g(t) - g(x)}{t-x}\right] + g(x)\left[\frac{f(t) - f(x)}{t-x}\right]$$

$h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$ $\quad \because h = fg$

$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$

c) Next, let $h = f/g$

$$h(t) - h(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)}$$

Dividing $t-s$ on b.s,

$$\frac{h(t)-h(x)}{t-x} = \frac{1}{g(t)g(x)}\left[\frac{f(t)g(x) - f(x)g(t) + f(x)g(x) - f(x)g(x)}{t-x}\right]$$

$$= \frac{1}{g(t)g(x)}\left[g(x)\left(\frac{f(t)-f(x)}{t-x}\right) - f(x)\left(\frac{g(t)-g(x)}{t-x}\right)\right]$$

$$h'(x) = \frac{1}{g^2(x)}\left[g(x)f'(x) - f(x)g'(x)\right] \quad \because \text{as } t \to x$$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Eg: The derivative of any constant is clearly zero. If f is defined by $f(x) = x$, Then $f'(x) = 1$. Repeated application of (b) & (c). Then shows that x^n is differentiable And that its derivative is nx^{n-1}, for any integer n. Thus every polynomial is differentiable.

Theorem : 3 [Chain rule differentiation]

Suppose f is continuous on $[a,b]$, $f'(x)$ exist at some point $x \in [a,b]$, G is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$ $(a \le t \le b)$ then h is differentiable at x and

$h'(x) = g'(f(x)) f'(x)$.

Proof

Let $y = f(x)$

By the definition of derivative, we have

$$f(t) - f(x) = (t-x)[f'(x) + u(t)] \rightarrow ①$$

$$g(s) - g(y) = (s-y)[g'(y) + v(s)] \rightarrow ②$$

where $t \in [a,b]$, $s \in I$ and $u(t) \rightarrow 0$ as $t \rightarrow x$, $v(s) \rightarrow 0$, $s \rightarrow y$.

Let $s = f(t)$

By using ② & ①

$$h(t) - h(x) = g(f(t)) - g(f(x))$$

$$g(s) - g(y) = (s-y)[g'(y) + v(s)]$$

$$= [f(t) - f(x)] \cdot [g'(y) + v(s)]$$

$$h(t) - h(x) = (t-x) \cdot [f'(x) + u(t)] \cdot [g'(y) + v(s)]$$

or, if $t \neq x$,

$$\frac{h(t) - h(x)}{t-x} = [f'(x) + u(t)] \cdot [g'(y) + v(s)]$$

$$h'(x) = f'(x)\, g'(f(x)) \qquad \because y = f(x)$$

Examples : 1

1) Let f be defined by $f(x) = \begin{cases} x \sin 1/x & (x \neq 0) \\ 0 & (x=0) \end{cases}$

Soln

We can apply the above theorem ② & ③, whenever $x \neq 0$ and obtain

$$f'(x) = \sin 1/x + x \cos 1/x \, (-1/x^2)$$

$$= \sin 1/x - \frac{1}{x} \cos 1/x$$

At $x=0$, These theorem do not apply any longer, since $\frac{1}{x}$ is not defined there, we applied directly to the definition : for $t \neq 0$,

$$\frac{f(t) - f(0)}{t-0} = \sin 1/t$$

As $t \to 0$, these doesn't tends to any limit, so that $f'(x)$ doesn't exist.

Eg: 2 Let f be defined by $f(x) = \begin{cases} x^2 \sin 1/x & (x \neq 0) \\ 0 & x=0 \end{cases}$

Soln

As above we obtain

$$f'(x) = \sin 1/x \cdot 2x + x^2 \cos 1/x \cdot (-1/x^2)$$

$$f'(x) = 2x \sin 1/x - \cos 1/x$$

At $x=0$, we appeal to the definition, and obtain

$$\left| \frac{f(t) - f(0)}{t-0} \right| = |t \sin 1/t| \leq |t| \quad (t \neq 0)$$

Letting $t \to 0$, we see that $f'(0) = 0$.

Thus f is differentiable at all points x,

but f' is not continuous function.

Since, $\cos 1/x$ doesn't tend to a limit as $x \to 0$.

Mean value theorem:

Definition: Local maximum:

Let f be a real function defined on a metric space X. We say that f has a <u>local maximum</u> at a point $p \in X$ if there exists ~~this~~ $\delta > 0 \ni: f(q) \leq f(p) \ \forall \ q \in X$ with [distance ~~of~~ p, q.] $[d(p,q)] < \delta$.

Local minimum:

Let f be a real function defined on a metric space X. We say that f has a <u>local minimum</u> at a point $p \in X$ if there exists $\delta > 0 \ni: f(q) \geq f(p)$ $\forall \ q \in X$ with $d(p,q) > \delta$.

Thm: A

local maximum $\Rightarrow f'(x) = 0$

Let f be defined on $[a,b]$; if f has a local maximum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$ [For local minimum, the statement also proof]

Proof

Choose δ in accordance with the above definition, so that

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$a < x - \delta < x < x + \delta < b$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0$$

Letting $t \to x$, we see that

$$f'(x) \geq 0 \longrightarrow (1)$$

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0$$

i.e) $f'(x) \leq 0 \rightarrow (2)$

From (1) & (2), $f'(x) = 0$

Rolle's thm:

1) f is defined on $[a,b]$

2) f is differentiable on (a,b)

Then, $a < \xi < b$, $f'(x) = \dfrac{f(b) - f(a)}{b - a}$

5/8/19

Thm 5 [Generalised mean value theorem]

If f and g are continuous real function on $[a,b]$ which are differentiable in (a,b). Then there is a point $x \in (a,b)$ at which

$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x) \quad \text{(or)}$$

$$\frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\Rightarrow \frac{f'(x)}{g'(x)} = \frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}}$$

Proof

Put $h(t) = [f(b) - f(a)]\, g(t) - [g(b) - g(a)]\, f(t)$

Then h is continuous on $[a,b]$, h is differentiable in (a,b) and

$$h(a) = [f(b) - f(a)]\, g(a) - [g(b) - g(a)]\, f(a)$$

$$= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a)$$

$$h(a) = f(b)g(a) - g(b)f(a)$$

$h(b) = [f(b) - f(a)]\, g(b) - [g(b) - g(a)]\, f(b)$

$= \cancel{f(b)g(b)} - f(a)g(b) - \cancel{g(b)f(b)} + g(a)f(b)$

$h(b) = g(a)f(b) - f(a)g(b)$

To prove the thm, we have to show that

$h'(x) = 0$ for some $x \in (a,b)$.

(i) If h is constant, this holds for every $x \in (a,b)$.

(ii) If $h(t) > h(a)$ for some $t \in (a,b)$. Let x be a point on $[a,b]$ at which h attains its maximum by theorem 4 shows that $h'(x) = 0$.

(iii) If $h(t) < h(a)$ for some $t \in (a,b)$. let x be a point on $[a,b]$ at which h attains its minimum by theorem 4 shows that $h'(x) = 0$.

[Where local minimum also proved then condition $h'(x) = 0$]

Thm: 6 cauchy's mean value theorem

If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), Then there is a point $x \in (a,b)$ at which $f(b) - f(a) = (b-a) f'(x)$.

Proof $g(x) = x \quad g'(x) = 1, \quad g(a) = a, \quad g(b) = b$

Put $h(t) = [f(b) - f(a)]\, g(t) - [g(b) - g(a)]\, f(t)$

Then h is continuous on $[a,b]$ & h is differentiable on (a,b) and

$h(a) = [f(b)g(a) - \cancel{f(a)g(a)} - g(b)f(a) + \cancel{g(a)f(a)}]$

$= f(b)\, g(a) - g(b)\, b$

$h(a) = af(b) - bf(a)$

$h(b) = [f(b) - f(a)]\,g(b) - [g(b) - g(a)]\,f(b)$

$= f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b)$

$= g(a)f(b) - f(a)g(b)$

$= af(b) - f(a)b$

$h(a) = h(b)$

To prove the thm, We have to show that

$h'(x) = 0$ for some $x \in (a,b)$.

i) If h is constant, this holds for every $x \in (a,b)$

By cauchy's mean value thm,

$$h'(x) = \frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Take, $g(x) = x$, $g'(x) = 1$

$g(a) = a$, $g(b) = b$

$$h'(x) = \frac{f'(x)}{1} = \frac{f(b) - f(a)}{b - a}$$

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Eg:

1) Find the value of c by using Cauchy's mean value theorem for $f(x) = \sqrt{x}$, $g(x) = 2x+1$ in $[1, 4]$.

Soln

Gn: $f(x) = \sqrt{x}$ | $g(x) = 2x+1$

$f'(x) = \frac{1}{2\sqrt{x}}$ | $g'(x) = 2$

$f(a) = f(1) = \sqrt{1} = 1$ | $g(a) = g(1) = 3$

$f(b) = f(4) = \sqrt{4} = 2$ | $g(b) = g(4) = 9$

$$\frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{2\sqrt{x}}}{2} = \frac{2-1}{9-3}$$

$$\frac{1}{4\sqrt{x}} = \frac{1}{6} \Rightarrow 6 = 4\sqrt{x} \Rightarrow \sqrt{x} = \frac{6}{4} = \frac{3}{2}$$

$$\sqrt{x} = \frac{3}{2} \Rightarrow x = \frac{9}{4} = 2.25 \in [1, 4].$$

Eg:2

Find the value of c by using Cauchy's mean value thm for $f(x) = \sin x$, $g(x) = \cos x$ in $[-\pi/2, 0]$.

Soln

Gn: $f(x) = \sin x$ | $g(x) = \cos x$

$f'(x) = \cos x$ | $g'(x) = -\sin x$

$f(-\pi/2) = \sin(-\pi/2) = -1$ | $g(-\pi/2) = \cos(-\pi/2) = 0$

$f(0) = \sin(0) = 0$ | $g(0) = \cos(0) = 1$

$$\frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\cos x}{-\sin x} = \frac{0+1}{1-0} = \frac{1}{1}$$

Reciprocal $\cancel{-\cot x = 1/1 \Rightarrow \cot x = -1/1 \Rightarrow x = (\cot)^{-1}(-1)}$

$$\frac{-\sin x}{\cos x} = 1 \Rightarrow -\tan x = 1 \Rightarrow x = \tan^{-1}(-1)$$

$$x = -\pi/4$$

Thm: 7
~~Let~~ suppose f be differentiable in (a, b)

a) If $f'(x) \geq 0 \ \forall \ x \in (a, b)$, then f is monotonically increasing.

b) If $f'(x) = 0 \ \forall \ x \in (a, b)$, then f is constant.

c) If $f'(x) \leq 0 \ \forall \ x \in (a, b)$, then f is monotonically decreasing.

Proof

a) Choose $[x_1, x_2]$ where $x_1, x_2 \in (a, b)$

Applying Lagrange's mean value theorem,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) \text{ for some } x \in (x_1, x_2)$$

Given $f'(x) \geq 0 \ \forall \ x \in (x_1, x_2)$

$\Rightarrow f(x_2) \geq f(x_1)$ for $x_2 > x_1$

$\Rightarrow f$ is monotonically increasing

b) Given $f'(x) = 0$ for every $x \in (x_1, x_2)$.

$\therefore f$ is a constant.

c) Choose $[x_1, x_2]$ where $x_1, x_2 \in (a, b)$

Applying Lagrange's mean value theorem

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) \text{ for some } x \in (x_1, x_2)$$

Given $f'(x) \leq 0 \ \forall \ x \in (x_1, x_2)$

$\Rightarrow f(x_2) \leq f(x_1)$ for $x_2 < x_1$

$\Rightarrow f$ is monotonically decreasing.

Continuity of derivatives

Thm 4.8

Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$

Proof

Put $g(t) = f(t) - \lambda t$

Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a,b)$ and $g'(b) > 0$; so that $g(t_2) < g(b)$ for some $t_2 \in (a,b)$

Hence g attains its minimum on $[a,b]$ at some point x such that $a < x < b$ (By thm 5)

$g'(x) = 0$.

$\therefore f'(x) = \lambda$

$g(a) = f(t) - \lambda a$

$g'(x) = f'(x) - \lambda$

$0 = f'(x) - \lambda$

$\boxed{f'(x) = \lambda}$

L'Hospital's Rule $\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Thm: 4.9

Suppose f and g are real and differentiable in (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$.

Suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$

(i) If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$,

(or) if $g(x) \to +\infty$ as $x \to a$.

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then $\dfrac{f(x)}{g(x)} \to A$ as $x \to a$.

Proof

We first consider the case in which

$-\infty \leq A < +\infty$.

choose a real number q such that $A < q$;

and then choose r such that $A < r < q$.

By $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a$ there is a point

$c \in (a, b)$ such that $a < x < c$ implies

$$\frac{f'(x)}{g'(x)} < r \to ①$$

$$\frac{\cancel{f(x) - f(y)}}{\cancel{g(x) - g(y)}} = \frac{\cancel{f'(t)}}{\cancel{g'(t)}} \cancel{< r}.$$

If $a < x < y < c$, then cauchy's mean value

theorem shows that there is a point $t \in (x, y)$

such that $\dfrac{f(x) - f(y)}{g(x) - g(y)} = \dfrac{f'(t)}{g'(t)} < r \to ②$

suppose, $f(x) \to 0$ and $g(x) \to 0$. letting $x \to a$ in

②, we see that

$$\frac{f(y)}{g(y)} \leq r < q \quad (a < y < c)$$

Next, $g(x) \to +\infty$ as $x \to a$.

keeping y is fixed in ②, we can choose a point

$c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$

if $a < x < c_1$.

Multiplying ② by $\frac{g(x)-g(y)}{g(x)}$, we obtain

$$\frac{f(x)-f(y)}{g(x)-g(y)} \times \frac{g(x)-g(y)}{g(x)} < r \times \frac{g(x)-g(y)}{g(x)}$$

$$\frac{f(x)-f(y)}{g(x)} < r - r\,\frac{g(y)}{g(x)} \quad (a<x<c_1)$$

$$\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} < r - r\,\frac{g(y)}{g(x)}$$

$$\frac{f(x)}{g(x)} < r - r\,\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \quad (a<x<c_1) \rightarrow ③$$

If we let $x \to a$ in ③ and $g(x) \to +\infty$ shows that there is a point $c_2 \in (a, c_1)$ such that

$$\frac{f(x)}{g(x)} < q \quad (a<x<c_2) \rightarrow ④$$

Summing of, ② and ④ shows that for any q subject only to the condition, $A < q$, there is a point c_2 such that $\frac{f(x)}{g(x)} < q$ if $a<x<c_2$.

Eg: $\lim_{x\to 0} \frac{\sin x}{x}$.

Soln $\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$ form

By applying L'hospital rule,

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Derivatives of Higher order:

Definition:

If f has a derivative f' on an interval, and its f' is itself differentiable and we denote the derivative of f' by f''.

Continue in this manner, we obtain the functions $f, f', f'', \cdots f^n$ each of which is the derivative of the preceeding one. f^n is called the n^{th} derivative, (or) the derivative of order n, of f.

6/8/19 Taylor's Theorem:

Let f be a function defined on $[a, a+h]$ such that

i) f^{n-1} is defined on $[a, a+h]$ and

ii) f^{n-1} is continuous on $[a, a+h]$

iii) $f^{(n)}(t)$ exists for every $t \in (a, a+h)$

Then there exists a real number p in $0 \leq p \leq n$ such that $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + f^{n-1}(a)\frac{h^{n-1}}{(n-1)!} + \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h)$

Proof

Let $g(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + (a+h-x)^p A$ $\to$ (1)

Put $x = a$,

$$g(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$
$$+ h^p \cdot A$$

Put $g(a) = g(a+h)$, $\rightarrow$ ②

$$g(a+h) = f(a+h)$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$
$$+ h^p \cdot A \rightarrow ③$$

RHS of equ ① have $f, f', f'' \cdots f^{n-1}$ are continuous in $[a, a+h]$. Also ~~a function~~ have $(a+h-x)$ continuous in $[a, a+h]$.

$\therefore$ g is continuous in $[a, a+h]$

$f^n(x)$ is exist, g is also differentiable in $(a, a+h)$; Also $g(a) = g(a+h)$ [②]

$\therefore$ g satisfy Rolle's theorem.

$\therefore$ minimum θ is in $(0, 1)$.

Therefore, $g'(a+\theta h) = 0 \rightarrow$ ④

Now, $g'(x) = f'(x) - f'(x) + (a+h-x) f''(x) + \frac{1}{2!} 2(a+h-x)$

① ⇒

$$(-1) f''(x) + \frac{(a+h-x)^2}{2!} f'''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x)$$
$$- p(a+h-x)^{p-1} \cdot A$$

$$= (a+h-x) f''(x) - (a+h-x) f''(x) + \frac{(a+h-x)^2}{2!} f'''(x) + \cdots$$

① ←

$$+ \frac{(a+h-x)^{(n-1)}}{(n-1)!} f^{n}(x) - p(a+h-x)^{p-1} \cdot A$$

$$g'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - PA(a+h-x)^{P-1} \longrightarrow (5)$$

Put $x = a + \theta h$ in (5)

$$g'(a+\theta h) = \frac{(a+h-a-\theta h)^{n-1}}{(n-1)!} f^n(a+\theta h) - PA(a+h-a-\theta h)^{P-1}$$

$$= \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - PAh^{P-1}(1-\theta)^{P-1}$$

From (4).

$$0 = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - PAh^{P-1}(1-\theta)^{P-1}$$

$$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) = PAh^{P-1}(1-\theta)^{P-1}$$

$$A = \frac{\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)}{P h^{P-1}(1-\theta)^{P-1}}$$

$$A = \frac{h^{n-P}(1-\theta)^{n-P}}{P(n-1)!} f^n(a+\theta h)$$

∴ Substitute A in (3).

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x)$$

$$+ h^P \cdot \frac{h^{n-P}(1-\theta)^{n-P}}{P(n-1)!} f^n(a+\theta h)$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x)$$

$$+ \frac{h^n(1-\theta)^{n-P}}{P(n-1)!} f^n(a+\theta h)$$

1) Find the value of $f(6)$. Given that $f(4) = 125$, $f'(4) = 75$, $f''(4) = 30$, $f'''(4) = 6$ and all other higher derivatives of $f(x)$ at $x = 4$ are zero.

Soln

Taylor's Thm:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$x = 4$; $h = 2$

As 4^{th} derivative and other higher derivatives of $f(x)$ are zero at $x = 4$.

$$f(4+2) = f(4) + 2f'(4) + \frac{2^2}{2!} f''(4) + \frac{2^3}{3!} f'''(4) + \dots$$

$$f(6) = 125 + 2 \times 75 + 2 \times 30 + \frac{4}{3} \times 6$$

$$f(6) = 341$$

Derivation of vector valued function:

Thm

Suppose f is a continuous mapping on $[a,b]$ into $\mathbb{R}^K$ (Euclidean Space) and f is differentiable in (a,b). Then there exists $x \in (a,b)$ such that $|f(b) - f(a)| \leq (b-a)|f'(x)|$

Proof

Put $z = f(b) - f(a)$, and define

$$\phi(t) = z \cdot f(t), \quad (a \leq t \leq b)$$

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Then ϕ is a real-valued continuous function on $[a,b]$ which is differentiable in (a,b).

The mean value theorem shows therefore that

$$\phi(b) - \phi(a) = (b-a)\,\phi'(x)$$

$$\phi(b) - \phi(a) = (b-a)\,z\,f'(x) \longrightarrow ①$$

for some $x \in (a,b)$

On the other hand

$$\phi(b) - \phi(a) = z\cdot f(b) - z\cdot f(a)$$

$$\phi(b) - \phi(a) = z\,(f(b) - f(a))$$

$$= z\cdot z$$

$$\phi(b) - \phi(a) = |z|^2 \longrightarrow ②$$

From ① & ②

$$|z|^2 = (b-a)\,z\,f'(x)$$

By Schwarz inequality

$$|z|^2 \leq (b-a)|z|\,|f'(x)|$$

$$|f(b) - f(a)| \leq (b-a)\,|f'(x)|$$

UNIT-5

RIEMANN - STIELTJES INTEGRAL

Definition and Existence of the integral

Definition:

Let $[a,b]$ be a given interval. By a partition P of $[a,b]$, we mean a finite set of points $x_0, x_1, \ldots x_n$, where

$$a = x_0 \le x_1 \le \ldots \le x_{n-1} \le x_n = b$$

We write, $\Delta x_i = x_i - x_{i-1} \quad (i = 1, 2 \ldots n)$

Now, suppose f is a bounded real function defined on $[a,b]$. Corresponding to each partition P of $[a,b]$, we put

Sup = GLB
Inf = LUB

$$M_i = \sup f(x), \quad (x_{i-1} \le x \le x_i)$$

$$m_i = \inf f(x), \quad (x_{i-1} \le x \le x_i)$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

and finally,

$$\int_a^{-b \text{ (upper bound)}} f\,dx = \inf U(P,f) \longrightarrow ①$$

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$$\int_{a}^{b} f\,dx = \sup L(P,f) \longrightarrow ②$$
(Lower bound)

The equation ① & ② are called upper and lower integrals of f over $[a,b]$ respectively.

If the upper and lower integrals are equal, we say that f is Riemann integrable on $[a,b]$, we write $f \in \mathcal{R}$ (i.e, the $\mathcal{R}$ denotes the set of Riemann integrable functions) and we denote the common value of equation ① & ② by

$$\int_{a}^{b} f\,dx \quad (\text{or}) \text{ by } \int_{a}^{b} f(x)\,dx$$

the Riemann integrable of f on $[a,b]$.

This is ~~not~~ the Riemann integrable of f on $[a,b]$.

Since, f is bounded, there exist two numbers m and M $(a \le x \le b)$,

Hence, for every P

$$m(b-a) \le L(P,f) \le U(P,f) \le M(a,b),$$

So that, the numbers $L(P,f)$ and $U(P,f)$ form a bounded set.

This shows that the upper and lower integrals defined for every bounded functions f.

Definition:

Let α be a monotonically increasing function on $[a,b]$. (Since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded ~~that~~ on $[a,b]$) Corresponding to each partition P of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

It is clear that $\Delta\alpha_i \geq 0$. For any real function f which is bounded on $[a,b]$.

we put $\quad U(P,f,\alpha) = \sum_{i=1}^{n} M_i \, \Delta\alpha_i$,

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \, \Delta\alpha_i$$

(Where M_i and m_i have the same meaning as)

Where $M_i = \sup \alpha(x)$, $\alpha(x_{i-1}) \leq \alpha(x) \leq \alpha(x_i)$

$m_i = \inf \alpha(x)$, $\alpha(x_{i-1}) \leq \alpha(x) \leq \alpha(x_i)$

$$\int_a^b f \, d\alpha = \inf U(P,f,\alpha) \longrightarrow (3)$$

$$\int_a^b f \, d\alpha = \sup L(P,f,\alpha) \longrightarrow (4)$$

If (3) and (4) are equal, we denote their common value by $\int_a^b f \, d\alpha$ (or) $\int_a^b f(x) \, d\alpha(x)$

$\hookrightarrow (5)$

f with r. to α over a, b.

This is the Riemann-Stieltjes integral f with respect to α over a, b.

If ⑤ exist, ~~the~~ i.e) if ③ and ④ are equal, we say that f is integrable with respect to α, in the Riemann Sence and write $f \in \mathcal{R}$.

Definition

The partition p^* is a refinement of P if p^* is a super set of P $(p^* \supset P)$ (i.e) ~~the~~ ~~partition~~ if every point of P is a point of P^*)

Given two partitions, P_1 & P_2, we say that p^* is their common refinement if $P^* = P_1 \cup P_2$.

Theorem: 1

If P^* is the refinement of P, then

Text

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

and

$$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

Proof

i) $L(P, f, \alpha) \leq L(P^*, f, \alpha)$

Suppose first that P^* contains just one point more than P.

Let this extra point be x^*, Suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P.

put $\omega_1 = \inf f(x) \quad (x_{i-1} \le x \le x^*)$

$\omega_2 = \inf f(x) \quad (x^* \le x \le x_i)$

Clearly, $\omega_1 \ge m_i$ and $\omega_2 \ge m_i$ where as before

$m_i = \inf f(x) \quad (x_{i-1} \le x \le x_i)$

Hence, $L(P^*, f, \alpha) - L(P, f, \alpha)$

$= \omega_1[\alpha(x^*) - \alpha(x_{i-1})] + \omega_2[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})]$

$= (\omega_1 - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (\omega_2 - m_i)[\alpha(x_i) - \alpha(x^*)]$

$L(P^*, f, \alpha) - L(P, f, \alpha) \ge 0$

$L(P^*, f, \alpha) \ge L(P, f, \alpha)$

ii) $U(P^*, f, \alpha) \le U(P, f, \alpha)$

Put $W_1 = \sup f(x) \quad (x_{i-1} \le x \le x^*)$

$W_2 = \sup f(x) \quad (x^* \le x \le x_i)$

Clearly, $W_1 \le M_i$ and $W_2 \le M_i$, where as before

$M_i = \sup f(x), (x_{i-1} \le x \le x_i)$

Hence, $U(P^*, f, \alpha) - U(P, f, \alpha)$

$= -W_1[\alpha(x^*) - \alpha(x_{i-1})] - W_2[\alpha(x_i) - \alpha(x^*)] + M_i[\alpha(x_i) - \alpha(x_{i-1})]$

$= (M_i - W_1)[\alpha(x^*) - \alpha(x_{i-1})] + (M_i - W_2)[\alpha(x_i) - \alpha(x^*)]$

Ⓐ $U(P^*, f, \alpha) - U(P, f, \alpha) \leq 0$

$$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

Theorem: 2

$$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha$$

Proof

Let P^* be the common refinement of two partitions P_1 and P_2

By above theorem,

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$$

Hence, $L(P_1, f, \alpha) \leq U(P_2, f, \alpha) \rightarrow ①$

If P_2 is fixed and the sup is taken over all P_1, equn. ① gives

$$\underline{\int_a^b} f\, d\alpha \leq U(P_2, f, \alpha) \rightarrow ②$$

From ② by taking the infimum for overall P_2, it gives

$$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha$$

Theorem: 3

$f \in R(\alpha)$ on ~~closed~~ $[a, b]$ iff for every $\varepsilon > 0$ there exist a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

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Proof $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \rightarrow$ (A)

For every P, we have

$$L(P,f,\alpha) \leq \underline{\int} f\,d\alpha \leq \overline{\int} f\,d\alpha \leq U(P,f,\alpha)$$

By (A) $\Rightarrow$ $0 \leq \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha < \varepsilon$

Hence, if (A) can be satisfied for $\varepsilon > 0$, we have

$$\overline{\int} f\,d\alpha = \underline{\int} f\,d\alpha$$

$\therefore f \in \mathcal{R}(\alpha)$

Conversely, suppose $f \in \mathcal{R}(\alpha)$, and let $\varepsilon > 0$ be given. Then there exist partition P_1 and P_2 such that

$$U(P_2,f,\alpha) - \int f\,d\alpha < \varepsilon/2 \rightarrow (1)$$

$$\int f\,d\alpha - L(P_1,f,\alpha) < \varepsilon/2 \rightarrow (2)$$

We choose P to be the common refinement of P_1 and P_2.

Then the previous thm (1), together with (1) and (2) shows that

$$U(P,f,\alpha) \leq U(P_2,f,\alpha) < \int f\,d\alpha + \varepsilon/2$$
$$< L(P_1,f,\alpha) + \varepsilon/2 \leq L(P,f,\alpha) + \varepsilon$$

$$\therefore U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon.$$