Rate of Convergence: **UNIT-I Transcendal & polynomial equations**

An iterative method is said to be of order p or has the rate of convergence p if is the largest positive real numbers for which there exists a finite constant $c (\neq 0)$ such that

$$|\epsilon_{k+1}| \leq c\,|\epsilon_k|^p, \text{ where } \epsilon_k = x_k - \xi$$

is the error in the k^{th} iterative.

Newton-Raphson's Method:

Consider the equation $f(x)=0$ & ξ is the root.

The Newton-Rapson's iterative formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \longrightarrow (1)$$

Taylors Series:

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

(1) becomes

$$\epsilon_{k+1} + \xi = \epsilon_k + \xi - \frac{f(\xi + \epsilon_k)}{f'(\xi + \epsilon_k)}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{f(\xi) + \frac{\epsilon_k}{1!} f'(\xi) + \frac{\epsilon_k^2}{2!} f''(\xi) + \frac{\epsilon_k^3}{3!} f'''(\xi) + \cdots}{f'(\xi) + \frac{\epsilon_k}{1!} f''(\xi) + \frac{\epsilon_k^2}{2!} f'''(\xi)}$$

ξ is the root of $f(x)=0$ of hence $f(\xi)=0$.

$$\epsilon_{k+1} = \epsilon_k - \frac{\left[\epsilon_k + \frac{\epsilon_k^2}{2}\frac{f''(\xi)}{f'(\xi)} + \frac{\epsilon_k^3}{6}\frac{f'''(\xi)}{f'(\xi)} + \cdots\right]}{\left[1 + \epsilon_k \frac{f''(\xi)}{f'(\xi)} + \frac{\epsilon_k^2}{2}\frac{f'''(\xi)}{f'(\xi)} + \cdots\right]}$$

$$= \epsilon_k - \left[\epsilon_k + \frac{\epsilon_k^2}{2}\frac{f''(\xi)}{f'(\xi)} + \frac{\epsilon_k^3}{6}\frac{f'''(\xi)}{f'(\xi)} + \cdots\right]$$

$$\left[1 + \epsilon_k \frac{f''(\xi)}{f'(\xi)} + \frac{\epsilon_k^2}{2}\frac{f'''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

$$= \frac{\epsilon_k^2}{2}\frac{f''(\xi)}{f'(\xi)} + O(\epsilon_k^3)$$

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\epsilon_k^2 + O(\epsilon_k^3)$$

Neglecting the higher power of ϵ_k^2, we get

$$\epsilon_{k+1} = C\,\epsilon_k^2 \text{ where } C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$$

Thus, the Newton - Raphson method has second order convergence.

Method of Complex roots:

The equation $f(z) = 0 \longrightarrow (1)$ where $z = x + iy$, x & y are independent variable & z is a dependent variable & $i = \sqrt{-1}$ is called a complex equation.

(1) can also be written as

$$f(z) = u(x,y) + iv(x,y) = 0 \longrightarrow (2) \text{ where}$$

$u(x,y)$, $v(x,y)$ are the real & imaginary parts of $f(z)$ respectively.

The problem of finding the roots of (1) is equivalent to determining the roots of two simultaneous equation.

$$u(x,y) = 0 \ \& \ v(x,y) = 0 \longrightarrow ③$$

Assume that (x_k, y_k) is an initial, approximation to a Solution of ③ then the exact solution is $(x_k + \Delta x, \ y_k + \Delta y)$

Solving ③ and we obtain,

$$x_{k+1} = x_k - \frac{u(x_k, y_k)\, u_x(x_k, y_k) + v(x_k y_k) . v_x (x_k y_k)}{u_x^2(x_k, y_k) + v_x^2(x_k, y_k)}$$

$$y_{k+1} = y_k - \frac{v(x_k, y_k) u_x(x_k, y_k) - u(x_k, y_k) . v_x(x_k, y_k)}{u_x^2(x_k, y_k) + v_x^2(x_k, y_k)}$$

The solution of equation ③ is

$$z = (x_k + \Delta x, \ y_k + \Delta y) = (x_{k+1}, y_{k+1}).$$

obtain Complex roots of the equation $f(z) = z^3 + 1 = 0$.

Sln: Given : $f(z) = z^3 + 1 = 0$

Let $z = x + iy$

$$u(x,y) + iv(x,y) = u + iv = (x+iy)^3 + 1 = 0$$
$$= x^3 - iy^3 + 3x^2 iy - 3xy^2 + 1$$
$$= (x^3 - 3xy^2 + 1) + i(3x^2y - y^3)$$

Equating real & Imaginary parts,

$u(x,y) = u = x^3 - 3xy^2 + 1$ $\quad V = 3x^2y - y^3$

$u_x(x,y) = u_x = 3x^2 - 3y^2$ $\quad V_x = 6xy$

By using Newton-Rapson's method, let the initial approximations be $Z_0 = (\overset{x_0}{0.25}, \overset{y_0}{0.25})$

Then $u_k = u(x_k, y_k) \Rightarrow u_0 = 0.96875$ $\quad V_k = V(x_k, y_k) \Rightarrow V_0 = 0.03125$

$u_{xk} = u_x(x_k, y_k) \Rightarrow u_{x0} = 0$ $\quad V_{xk} = V_x(x_k, y_k) \Rightarrow V_{x0} = 0.375$

Now,

$$x_1 = x_0 - \frac{u_0 u_{x0} + V_0 V_{x0}}{u_{x0}^2 + V_{x0}^2}$$

$$= 0.25 - \frac{0 + 0.01171875}{0.140625} = 0.16666667$$

$$y_1 = y_0 - \frac{V_0 u_{x0} - u_0 V_{x0}}{u_{x0}^2 + V_{x0}^2}$$

$$= 0.25 - \frac{0 - (0.96875 \times 0.375)}{0.375^2} = 2.83333333$$

$|||^{ly}$ Proceed for all values (x_k, y_k) $k = 0, 1, 2 \dots$

the root of equation is $(0.5, 0.86602675)$.

$u_1 = u(x_1, y_1) = -3.00925909$ $\quad v_1 = -22.5092592$

$u_{x_1} = -23.9999995$ $\quad v_{x_1} = 2.833321667$

$x_2 = 0.15220386$

$y_2 = 1.89374012$

$u_2 = -0.63399807$ $\quad v_2 = -6.65981738$

$u_{x_2} = -10.68925688$ $\quad v_{x_2} = 1.72940734$

$x_3 = 0.19263478$

$y_3 = 1.27724302$

$u_3 = 0.06438423$ $\quad v_3 = -1.94144165$

$u_{x_3} = -4.782724472$ $\quad v_{x_3} = 1.47624857$

$x_4 = 0.31932162$

$y_4 = 0.91041857$

$u_4 = 0.23853940$ $\quad v_4 = -0.47611530$

$u_{x_4} = -2.18068703$ $\quad v_{x_4} = 1.7442980$

$x_5 = 0.49252907$

$y_5 = 0.83063184$

$u_5 = 0.10001992$ $\quad V_5 = 0.03140237$

$u_{x5} = -1.34209311$ $\quad V_{x5} = 2.45466197$

$x_6 = 0.49983160$

$y_6 = 0.86738608$

$u_6 = -0.00328408$ $\quad V_6 = -0.00248396$

$u_{x6} = -1.50758095$ $\quad V_{x6} = 2.60128183$

$x_7 = 0.49999870$

$y_7 = 0.86602675.$

The root of the given equation is $(0.5, 0.86602675)$

BIRGE- VITA METHOD:

Let $P_n(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots a_{n-1} x + a_n \longrightarrow ①$

$P_n(x) = 0$ ——— be the polynomial equation.

In this method, we seek to determine a real numbers p such that $(x-p)$ is a factors of the polynomial equation ①. If we divide $P_n(x)$ by the factor $(x-p)$ then we set a quotient $Q_{n-1}(x)$ of degree $n-1$.

$Q_{n-1}(x) = x^{n-1} + b_1 \; x^{n-2} + b_{n-2}x + b_{n-1}$ and the reminder R, Then, we have

$$P_n(x) = (x-p)\,Q_{n-1}(x) + R \longrightarrow (2)$$

The value of R depends on P. starting with an initial approximation P_0 to P_1, we use some iterative method to improve the values of p such that $R(P) \simeq 0 \longrightarrow (3)$.

This is a single equation in one unknown and the Newton- Raphson method can be applied to improve the assumed value P_0, The Newton Raphson method becomes

$$P_{k+1} = P_k - \frac{P_n(P_k)}{P_n'(P_k)}, \quad k = 0,1,2\ldots$$

Equation (2) becomes.

$$P_n(x) = (x-p)\; Q_{n-1}(x) + R$$

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = (x-p)(x^{n-1} + b_1 x^{n-2} + \ldots$$
$$\ldots + b_{n-2}x + b_{n-1}) + R$$

Comparing like terms.

$a_1 = b_1 - p$; $b_1 = a_1 + p$

$a_2 = b_2 - pb_1$; $b_2 = a_2 + pb_2$

$\vdots$

$a_k = b_k - pb_{k-1}$; $b_k = a_k + pb_{k-1}$

$\vdots$

$a_n = R - pb_{n-1}$; $R = a_n + pb_{n-1}$

that is, $b_k = a_k + p b_{k-1}$, $k = 1, 2 \ldots n \longrightarrow (4)$

From equⁿ (2) we have,

$$P_n(p) = R = b_n \longrightarrow (5)$$

To determine R_n, Diff (4) w.r.t p

and obtain

$$\frac{db_n}{dp} = b_{k-1} + p\frac{db_{k-1}}{dp}$$

Let $\frac{db_k}{dp} = C_{k-1}$ then

$$C_{k-1} = b_{k-1} + p C_{k-2}$$

(i.e) $C_k = b_k + p C_{k-1}$, $k = 1, 2 \ldots n-1$

$c_0 = 1$

Diff equⁿ (5) w.r.t p

$$\frac{dP_n}{dp} = P_n'(p) = \frac{dR}{dp} = \frac{db_n}{dp} = C_{n-1}$$

Then $\boxed{p_{k-1} = p_k - \frac{b_n}{C_{n-1}}}$ $k = 0, 1, 2 \ldots$

This method is called Birge-Vita method.

The calculations of the coefficients b_k and C_k can be carried out as given below.

p	1	a_1	a_2	....	a_{n2}	a_{n-2}	a_n
		p	pb_1	....	pb_{n-3}	pb_{n-2}	pb_{n-1}
	1	b_1	b_2	...	b_{n-2}	b_{n-1}	$\boxed{b_n = R}$
p		p	pc_1	...	pc_{n-3}	pc_{n-2}	
	1	c_1	c_2	...	c_{n-2}	$\boxed{c_{n-1} = \frac{dR}{dP}}$	

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(i) Problem: Use synthestic division and perform two iterations by Birge - Vita method to find the smallest +ve root of the equation.

$x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$. let $P_0 = 0.5$

Soln:

Here $n = 4$ $P_0 = 0.5$

0.5	1	-3	3	-3	2
		0.5	-1.25	0.875	-1.0625
	1	-2.5	1.75	-2.125	$\boxed{0.9375 = b_4}$
0.5		0.5	-1	0.375	
	1	-2	0.75	$\boxed{-1.750 = c_3}$	

$$P_1 = P_0 - \frac{b_4}{c_3} = 0.5 + \frac{0.9375}{1.750} = 1.0356$$

$$\begin{array}{r|rrrrr}
1.0356 & 1 & -3 & 3 & -3 & 2 \\
 & & 1.0356 & -2.0343 & 1.0001 & -2.0711 \\
\hline
1.0356 & 1 & -1.9644 & 0.9657 & -1.9999 & \boxed{-0.0711} = b_4 \\
 & & 1.0356 & -0.9619 & 0.0039 & \\
\hline
 & 1 & -0.9288 & 0.0038 & \boxed{-1.9960} = c_3 &
\end{array}$$

$$P_2 = P_1 - \frac{b_4}{c_3} = 0.999979$$

The exact root is 1.0.

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ii) $x^3 - 11x^2 + 32x - 22 = 0, \quad P = 0.5$

Soln: Here $n = 3 \qquad P_0 = 0.5$

$$\begin{array}{r|rrrr}
0.5 & 1 & -11 & 32 & -22 \\
 & & 0.5 & -5.25 & 13.375 \\
\hline
0.5 & 1 & -10.5 & 26.75 & \boxed{-8.625} = b_4 \\
 & & 0.5 & -5 & \\
\hline
 & 1 & -10.0 & \boxed{21.75} = c_3 &
\end{array}$$

$$P_1 = P_0 - \frac{b_4}{c_3} = 0.5 + \frac{8.625}{21.75}$$

$$\boxed{P_1 = 0.896552}$$

0.896552	1	−11	32	−22
		0.896552	−9.058267	20.568457
	1	−10.103448	22.941733	$\boxed{-1.431543}$ = b_4
0.896552		0.896552	−8.254461	
	1	−9.206892	$\boxed{14.687272}$ = C_3	

$$P_2 = P_1 - \frac{b_4}{C_3} = 0.896552 + \frac{1.431543}{14.687272}$$

$$\boxed{P_2 = 0.994020}$$

The root is 0.994020

iii) $x^5 - x + 1 = 0$, $P = -1.5$.

Soln: Here $n = 5$ $P_0 = -1.5$

−1.5	1	0	0	0	−1	1
		−1.5	2.25	−3.375	5.0625	−6.09375
	1	−1.5	2.25	−3.375	4.0625	$\boxed{-5.09375}$ = b_4
−1.5		−1.5	4.5	−10.125	20.25	
	1	−3	6.75	−13.5	$\boxed{24.3125}$ = C_3	

$$P_1 = -1.5 + \frac{5.09375}{24.3125}$$

$$\boxed{P_1 = -1.290488}$$

-1.290488	1	0	0	0	-1	1
		-1.290488	1.665359	-2.149126	2.773421	-2.288579
	1	-1.290488	1.665359	-2.149126	1.773421	$\boxed{-1.288579 = b_4}$
-1.290488		-1.290488	3.330719	-6.447378	11.093685	
	1	-2.580796	4.996078	-8.596504	$\boxed{12.867106 = c_3}$	

$$P_2 = -1.290488 + \frac{1.288579}{12.867106}$$

$$\boxed{P_2 = -1.190343}$$

The root is -1.190343

iv) $x^6 - x^4 - x^3 - 1 = 0$, $\quad P = 1.5$

<u>Soln:</u> Here $n = 6$, $P_0 = 1.5$

1.5	1	0	-1	-1	0	0	1
		1.5	2.25	1.875	1.3125	1.96875	2.953125
	1	1.5	1.25	0.875	1.3125	1.96875	$\boxed{1.953125 = b_4}$
1.5		1.5	4.5	8.625	14.25	23.34375	
	1	3	5.75	9.5	15.5625	$\boxed{25.3125 = c_3}$	

$$P_1 = 1.5 - \frac{1.953125}{25.3125} = 1.42284.$$

1.42284	1	0	-1	-1	0	0	-1
		1.42284	2.024473	1.457661	0.651179	0.926523	1.318294
	1	1.42284	1.024473	0.457661	0.651179	0.926523	$\boxed{0.318294}$ $= b_4$
1.42284		1.42284	4.048947	7.218665	10.922184	16.467044	
	1	2.84568	5.07342	7.676326	11.573369	$\boxed{17.393567}$ $= c_3$	

$$P_2 = 1.42284 - \frac{0.318294}{17.393567} = 1.4045405$$

$$\boxed{P_2 = 1.4045405}$$

The root is $\boxed{1.4045405}$.

The Bairstow method

The Bairstow method estimate a quadratic factors of the form x^2+px+q from the polynomial equ^n $P_n(x)$. If we divide a polynomial equation $P_n(x)$ by the quadratic factor x^2+px+q, then we obtain a quotient polynomial $Q_{n-2}(x)$ if degree $n-2$ and a remainder term which is a polynomial of degree one, that is $Rx+S$.

i.e) $P_n(x) = (x^2+px+q)\,Q_{n-2}(x) + Rx + S$

using Newton - Raphson's iterative product

$$p_{k+1} = p_k + \Delta p \quad \& \quad q_{k+1} = q_k + \Delta q.$$

Where

$$\Delta p = -\frac{b_n c_{n-3} - b_{n-1} c_{n-2}}{c_{n-2}^2 - c_{n-3}(c_{n-1} - b_{n-1})}$$

$$\Delta q = -\frac{b_{n-1}(c_{n-1} - b_{n-1}) - b_n c_{n-2}}{c_{n-2}^2 - c_{n-3}(c_{n-1} - b_{n-1})}.$$

For computing b_n's and c_n's we use the following scheme

$$\begin{array}{c|cccccc}
 & 1 & a_1 & a_2 \;\ldots & a_{n-2} & a_{n-1} & a_n \\
-p & & -p & -pb_1 \;\ldots & -pb_{n-3} & -pb_{n-2} & -pb_{n-1} \\
-q & & & -q \;\ldots & -qb_{n-4} & -qb_{n-3} & -qb_{n-2} \\
\hline
 & 1 & b_1 & b_2 \;\ldots & b_{n-2} & b_{n-1} & \boxed{b_n} \\
-p & & -p & -pc_1 \;\ldots & bc_{n-3} & -pc_{n-2} & \\
-q & & & -q & -qc_{n-4} & -qc_{n-3} & \\
\hline
 & 1 & c_1 & c_2 \;\ldots & c_{n-2} & \boxed{c_{n-1}} &
\end{array}$$

When p and q have been obtained to the desired accuracy, then we can calculate p and q such that x^2+px+q is a factor of $P_n(x)=0$.

Problem:

Perform one iteration of the Bairstow method to extract a quadratic factor x^2+px+q from the polynomial

$x^4+x^3+x^2+x+1=0$, Let $p_0=0.5=q_0$

Soln: Here $n=4$

	1	1	2	1	1
-0.5		-0.5	-0.25	-0.625	-0.0625
-0.5			-0.5	-0.25	-0.625
	1	$+0.5$ b_1	1.25 b_2	0.125 b_3	$\boxed{0.3125 = b_4}$
-0.5		-0.5	0	-0.375	
-0.5			-0.5	0	
	1	0 c_1	0.75 c_2	$\boxed{-0.25 = c_3}$	

$$\Delta q = -\frac{b_4c_1 - b_3c_2}{c_2^2 - c_1(c_3-b_3)} = 0.1667$$

$$\Delta q = -\frac{b_3(c_3-b_3) - b_4c_2}{c_2^2 - c_1(c_3-b_3)} = 0.5$$

Therefore

$$P_1 = P_0 + \Delta P = 0.6667$$

$$q_1 = q_0 + \Delta q = 1$$

<u>Home work</u>

using Bairstow's method obtain the quadratic factors of the following equation (perform two iterations)

1) $x^4 - 3x^3 + 20x^2 + 44x + 54 = 0 \quad P = 2 = q$

soln:

Here $n = 4$

-2	1	-3	20	44	54
		-2	10	-56	4
-2			-2	10	-56
-2	1	-5	28	-2	$2 = b_4$
		-2	14	-80	
-2			-2	14	
	1	-7	40	$-68 = C_3$	

$$\Delta P = -\left(\frac{b_4 C_1 - b_3 C_2}{C_2^2 - C_1(C_3 - b_3)}\right) = -0.0579965$$

$$\Delta q = -\left(\frac{b_3(c_3-b_3)-b_4c_2}{c_2^2-c_1(c_3-b_3)}\right) = -0.0456942$$

$p_1 = 1.9420035$

$q_1 = 1.9543058.$

	1	-3	20	44	54
-1.942004		-1.942004	9.5973915	-53.6829826	0.0481510
-1.954306			-1.954306	9.6581881	-54.0230479
	1	-4.9420040	27.6430855	-0.0247945	0.0251031 = b_4
-1.942004					
-1.954306		-1.942004	13.3687711	-75.8499195	
			-1.954306	13.4534581	
	1	-6.884008	39.057556	-62.4212559 = c_3	

$$\Delta p = \frac{0.7956025}{1095.954519} = -0.0007259$$

$$\Delta q = \frac{-0.5666234}{1095.954519} = -0.000517$$

$p_2 = 1.9412781$

$q_2 = 1.9537890.$

(2) $x^4 - x^3 + 6x^2 + 5x + 10 = 0$ $\quad p = 1.14,\ q = 1.42$

Soln: Given: $x^4 - x^3 + 6x^2 + 5x + 10 = 0$ $\quad p = 1.14,\ q = 1.42$

Here $n = 4$

	1	-1	6	5	10
-1.14		-1.14	2.4396	-8.002344	-0.0415598
-1.42			-1.42	3.0388	-9.967832
	1	-2.14	7.0196	0.036456	-0.01093918
-1.14		-1.14	3.7392	-10.646232	
-1.42			-1.42	4.6576	
	1	-3.28	9.3388	-5.952176	

$$\Delta p = \frac{0.4600310}{94.8201614} = 0.0045826$$

$$\Delta q = \frac{-16.399432}{94.8201614} = -0.0172953$$

$p_1 = 1.1445826$

$q_1 = 1.42193300$

	1	-1	6	5	10
-1.144583	0	-1.144583	2.454653	-8.049532	0.000090
-1.421933	0	0	-1.421933	3.049453	-10.000057
	1	-2.144583	7.032720	-0.000079	$\boxed{0.000033 = b_4}$
-1.144583	0	-1.144583	3.764723	-10.731050	
-1.421933	0	0	-1.421933	4.676974	
	1	-3.289166	9.375510	$\boxed{-6.054155 = c_3}$	

$\Delta p = -0.000009$

$\Delta q = 0.022737$

$p_2 = 1.144594$

$q_2 = 1.444670.$

(3) $x^3 - 3.7x^2 + 6.25x - 4.09 = 0 \quad p = -2.5,\ q = 3$

Here $\underline{\underline{n=3}}$

Soln:

2.5	1	-3.7	6.25	-4.069
		2.5	-3	0.625
-3			-3	3.6
2.5	1	-1.2	0.25	$0.156 = b_3$
-3		2.5	3.25	
			-3	
	1	1.3	0.50	

$$\Delta p = -\left(\frac{b_3 c_0 - b_2 c_1}{c_1^2 - c_0(c_2 - b_2)}\right) = 0.1173611$$

$$\Delta q = -\left(\frac{b_2(c_2 - b_2) - b_3 c_1}{c_1^2 - c_0(c_2 - b_2)}\right) = 0.0974306$$

$P_1 = -2.3826389$

$q_1 = 3.0974306$

2.3826389	1	3.7	6.25	-4.069
-3.0974306		2.3826389	-3.1387958	0.0328175
			-3.0974306	4.0804346
	1	-1.3173611	0.0137736	0.0442521
2.3826389		2.3826389	2.5381723	
-3.0974306			-3.0974306	
	1	1.0652778	-0.5454847	

$\Delta p = -0.0174568$

$\Delta q = 0.0323739$

$P_2 = -2.4000957$

$q_2 = 3.1298045$

To Find the rate of Convergence of Secant method:

The iterative formula for Secant or Chord method to find the root of $f(x)=0$ is

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k \longrightarrow (1)$$

Let ξ be the single root of $f(x)=0$ and to find the rate of $f(x)=0$ is convergence Let us ~~x_{k+1} =~~ substitute $x_k = \xi + \epsilon_k$ in (1).

We get

$$\xi + \epsilon_{k+1} = \xi + \epsilon_k - \frac{(\xi+\epsilon_k) - (\xi + \epsilon_{k-1})}{f(\xi+\epsilon_k) - f(\xi+\epsilon_{k-1})} \times \left(f(\xi+\epsilon_k)\right)$$

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) f(\xi + \epsilon_k)}{f(\xi+\epsilon_k) - f(\xi+\epsilon_{k-1})}$$

Using Taylor's Series expansion of $f(\xi)=0$, we get,

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left[f^0(\xi) + \frac{\epsilon_k}{1} f'(\xi) + \frac{\epsilon_k^2}{2} f''(\xi) + \cdots\right]}{\left[f^0(\xi) + \frac{\epsilon_k}{1!} f'(\xi) + \frac{\epsilon^2}{2!} f''(\xi) + \cdots\right] - \left[f^0(\xi) + \frac{\epsilon_{k-1}}{1} f'(\xi) + \frac{\epsilon_{k-1}^2}{2} f''(\xi) + \cdots\right]}$$

$$= \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left[\epsilon_k f'(\xi) + \frac{\epsilon_k^2}{2} f''(\xi) + \cdots\right]}{(\epsilon_k - \epsilon_{k-1}) f'(\xi) + \frac{(\epsilon_k^2 - \epsilon_{k-1}^2)}{2} f''(\xi) + \cdots}$$

$$= \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) \cdot f'(\xi)\left[\epsilon_k + \frac{\epsilon_k^2}{2}\frac{f''(\xi)}{f'(\xi)} + \cdots\right]}{(\epsilon_k - \epsilon_{k-1}) f'(\xi)\left[1 + \frac{(\epsilon_k + \epsilon_{k-1})}{2}\frac{f''(\xi)}{f'(\xi)} + \cdots\right]}$$

$$\epsilon_{k+1} = \epsilon_k - \left[\epsilon_k + \frac{\epsilon_k^2}{2}\frac{f''(\xi)}{f'(\xi)} + \cdots\right]\left[1 + \frac{(\epsilon_k + \epsilon_{k-1})}{2}\frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \epsilon_k \epsilon_{k-1} + O(\epsilon_k^2)$$

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \epsilon_k \epsilon_{k-1} + O(\epsilon_k^2)$$

i.e.) $\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \epsilon_k \epsilon_{k-1}$ [omitting ϵ_k^2 & higher powers]

$\epsilon_{k+1} = C \epsilon_k \epsilon_{k-1}$ where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$

This is called error equation. We can write this as $\epsilon_{k+1} = A \epsilon_k^p$, where A and P are to be determined.

ie) $\epsilon_k = A\, \epsilon_{k-1}^{p}$

$$\epsilon_{k-1} = A^{-1} |\epsilon_k|^{1/p}$$

The error equation becomes,

$$A\,\epsilon_k^{p} = C\, A\, \epsilon_{k-1}^{p} \cdot A^{-1} (\epsilon_k)^{1/p}$$

$$= C\, \frac{\epsilon_k}{A} \cdot (\epsilon_k)^{1/p}$$

$$A\,\epsilon^{p} = C A^{-1} (\epsilon_k)^{1+1/p}$$

Comparing the powers of ϵ_k on both sides,

$$p = 1 + \frac{1}{p} \quad \text{(or)} \quad p^2 - p - 1 = 0$$

$$p = \frac{1 \pm \sqrt{5}}{2}$$

Neglecting the minus sign, we find that the rate of convergence of secant method is

$$\boxed{p = 1.618.}$$

X

Aitken Δ^2 method :-

The linear convergence of the iterative method is imporved with the help of the Aitken Δ^2 method.

The iterative formula of Aitken Δ^2 method is

Let $f(x)=0$ be the given equation the linear iterative Process to find sequence $\{x_k\}$ is

$$x_{k+1} = x_k + \frac{1}{2} f(x_k) \text{ where } k=0,1,2\ldots$$

Let x_0 be the initial approximation to the root and $x_0 \in [a,b]$, where $[a,b]$ be the interval in which the root lie. Using Aitken Δ^2 method,

$$x^*_{3k+2} = x_{3k} - \frac{(x_{3k+1}-x_{3k})^2}{x_{3k+2} - 2x_{3k+1} + x_{3k}}$$

1. Solve $f(x) = \cos x - xe^x = 0$ using Aitken Δ^2 method

Soln:

$f(x) = \cos x - xe^x = 0$

$f(0) = 1$ (+ve)

$f(1) = -2.17797523$

One root is lies between 0 and 1

Let $I = [a,b] = [0,1]$

Let $x_0 = 0 \in [0,1]$ be the initial approximation

Let the linear iterative Process be

$$x_{k+1} = x_k + \frac{1}{2}\left[\cos x_k - x_k e^{x_k}\right], \quad k=0,1,2\ldots$$

$$x^*_{3k+2} = x_{3k} - \frac{(x_{3k+1} - x_{3k})^2}{x_{3k+2} - 2x_{3k+1} + x_{3k}} \quad , \; k = 0,1,2.$$

k	x_k	x_{k+1}
0	0	0.5
1	0.5	0.52661096
2	0.52661096	0.52810686
3	0.52810686	0.51222771
4	0.51222771	0.52059979
5	0.520599979	0.51770956
6	0.51770956	0.51778227
7	0.51778227	0.51774438
8	0.51774438	0.51775736

GRAEFFE's ROOT SQUARING METHOD

This is a direct method and it is used to find the roots of a polynomial with real co-efficients. The roots may be real and distinct, real and equal of complex roots.

Let $P_n(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1}x + a_n = 0$ $\longrightarrow$ (1).

be a Polynomial equation of degree n. Then the Graffe's root squaring method table is

1	a_1	a_2	a_3	$\dots$	a_{n-1}	a_n
1	a_1^2	a_2^2	a_3^2	$\dots$	a_{n-1}^2	a_n^2
	$-2a_2$	$-2a_1a_3$	$-2a_2a_4$	$\dots$	$-2a_{n-2}a_n$	
		$+2a_4$	$+2a_1a_3$		$\vdots$	
		$\vdots$	$\vdots$			
1	b_1	b_2	b_3	$\dots$		b_n

This Procedure is repeated on n times, we obtain

$x^n + B_1 x^{n-1} + B_2 x^{n-2} + \cdots + B_{n-1} x + B_n = 0$ and

Let whose roots are $R_1, R_2 \ldots R_n$ are the 2^m th power of the roots of the equation (1) where

$$R_i = -\xi_i^{2m}, \quad i = 1, 2 \ldots n$$

Let us assume that

$$|\xi_1| > |\xi_2| > \cdots > |\xi_n|, \text{ then}$$

$$|R_1| > |R_2| > \cdots > |R_n|$$

then $\xi_i = |R_i|^{1/2m}, \quad i = 1, 2 \ldots n$

ie) $\xi_i = |R_i|^{1/2m} = \left| \dfrac{B_i}{B_{i-1}} \right|^{1/2m}, \quad i = 1, 2 \ldots n$

Problem

(1) Find all the roots of the polynomial $x^3 - 6x^2 + 11x - 6 = 0$, using the Graeffe's root squaring method.

Soln: Let the polynomial equation be

$$x^3 - 6x^2 + 11x - 6 = 0 \longrightarrow (1)$$

The co-efficients of the successive root squarings are given in the following table.

m	2^m				
0	1	1	-6	11	-6
		1	36	121	36
			-22	-72	
1	2	1	14	49	36
		1	196	2401	1296
			-98	-1008	
2	4	1	98	1393	1296
		1	9604	1940449	1679616
			-2786	-254016	
3	8	1	6818	1686433	1679616
		1	46485124	2.8440562(12)	2.821109907(12)
			-3372866	-2.2903243(10)	
4	16	1	43112258	2.8211530(12)	2.821109907(12)

Successive approximation to the roots are given in the following table.

m	α_1	α_2	α_3
1	3.7417	1.8708	0.8571
2	3.1465	1.9417	0.9821
3	3.0144	1.9914	0.9995
4	3.0003	1.9998	1.0000

the exact root is 3, 2, 1.

(2) Find all the roots of the Polynomial $x^3 - 4x^2 + 5x - 2 = 0$ using Graeffe's root squaring method.

Soln: The co-efficient in the successive root squarings are tabulated in the following table

m	2^m				
0	1	1	-4	5	-2
		1	16	25	4
			-10	-16	
1	2	1	6	9	4

		1	36	81	16
			−18	−48	
2	4	1	18	33	16
		1	324	1089	256
			−66	−576	
3	8	1	258	573	256
		1	66564	263169	65536
			−1026	132096	
4	16	1	65538	131073	65536
		1	0.4295(10)	1.7180(10)	0.4295(10)
			− 0.2622(6)	−0.859(10)	
5	32	1	0.4294(10)	0.859(10)	0.4295(10)

Successive approximations to the roots are given in the following table.

m	α_1	α_2	α_3
1	2.4495	1.2247	0.6666
2	2.0598	1.1636	0.9135
3	2.0019	1.0897	0.9168
4	2.0008	1.0443	0.9576
5	1.9999	1.0219	0.9786

The exact root is 2, 1, 1. It has a double root.

(1) Find all the roots of the polynomial
$x^3 - x^2 - x - 2 = 0$ (complex roots)

Soln: Using the Graeffe's root squaring method, the coefficients in the successive roots squarings are tabulated in the following table.

m	2^m				
		1	−1	−1	−2
		1	1	1	4
			2	−4	
1	2	1	3	−3	4
		1	9	9	16
			6	−24	
2	4	1	15	−15	16
		1	225	225	256
			30	−480	

3	8	1	255	−255	256
		1	65025	65025	65536
			510	−130560	
4	16	1	65535	−65535	65535
		1	4294836225	4294836225	4294967296
			131070	−8589803520	
5	32	1	4294967295	−4294967295	4294967296

We observe from above that the magnitude of the coefficient B_1 has become constant (upto four decimal places), whereas the magnitude of the coefficient B_2 oscillates. This indicates that ξ_1 is the real root and ξ_2 and ξ_3 are a pair of complex roots.
The real root is given by

$$|\alpha_1|^{32} = (4294967295)$$

$$|\alpha_1| = 2.0000$$

It can be verified that this root is positive.
The magnitude of the complex root is obtained as

$$\beta_2^{64} = \left| \frac{B_3}{B_1} \right| = 1.0000$$

Which gives $\beta_2 = 1.0000$

If we write

$$\xi_2 = p + iq \quad \text{and} \quad \xi_3 = p - iq$$

then using the given equation, we get

$$\xi_1 + \xi_2 + \xi_3 = -a_1$$

$$\xi_1 + 2p = 1 \quad \text{or} \quad p = -0.5$$

we Find $q = \sqrt{(\beta_2^2 - p^2)} = 0.8660$

Thus, the required roots of the given equation are

$$\xi_1 = 2.0000$$

$$\xi_2 = -0.5000 + 0.8660\,i$$

$$\xi_3 = -0.5000 - 0.8660\,i$$

The exact roots of the equation are $2.0, (-1.0 \pm i\sqrt{3}/2)$

(2) Find all the roots of the polynomial

$$x^4 - x^3 + 3x^2 + x - 4 = 0$$

using the Graeffe's root squaring method.

Soln: The coefficients in the successive root squarings are given the table.

m	2^m					
0	1	1	-1	3	1	-4
		1	1	9	1	16
			-6	2	24	
				-8		
1	2	1	-5	3	25	16
		1	25	9	625	256
			-6	250	-96	

2	4	1	19	291	529	256
		1	361	84681	279841	65536
			-582	-20102	-148992	
				512		
3	8	1	-221	65091	130849	65536

Since B_1 alternates in sign, we have a pair of complex roots, which can be obtained using B_0, B_1 and B_2.

First real root:

$$|\alpha_1|^8 = \left|\frac{B_4}{B_3}\right|$$

which gives $|\xi_1| = 0.9172$

On substituting in the given polynomial, this root is found to be negative

Second real root:

$$|\alpha_2|^8 = \left|\frac{B_3}{B_2}\right|$$

which gives $|\xi_2| = 1.0912$ and this root is found to be positive

To obtain the pair of complex roots $p \pm iq$, we have

$$\sqrt{p^2+q^2} = \beta$$

where $$|\beta|^{16} = \left|\frac{B_2}{B_0}\right|$$

which gives $\beta = 1.9991$

We also have

$$2p + \xi_1 + \xi_2 = 1$$

and $p^2 + q^2 = \beta^2$

From the above two relations, we obtain

$p = 0.4130$

$q = 1.9560$

Hence the roots of the given polynomial are

$$-0.9172,\ 1.0912,\ 0.4130 \pm 1.9560\, i$$

Methods For Multiple Roots

If ξ is a multiple root of multiplicity m of the equation then we have from definition

$$f(\xi) = f'(\xi) = \cdots = f^{(m-1)}(\xi) = 0 \text{ and}$$

$$f^{(m)}(\xi) \neq 0.$$

$$\boxed{x_{k+1} = x_k - \frac{f_k}{f'_k}} \longrightarrow (1)$$

This method is called the Newton - Raphson method

$$\boxed{x_{k+1} = x_k - \frac{f_k f'_k}{f_k'^2 - f_k f''_k}} \longrightarrow (2)$$

It can be easily verified that this method has second order convergence.

The secant method can also be generalized to find a multiple root. We can apply the secant method on $g = f/f'$, which has a simple root. In this case the secant method becomes

$$x_{k+1} = \frac{x_{k-1} g_k - x_k g_{k-1}}{g_k - g_{k-1}}$$

$$\boxed{x_{k+1} = \frac{x_{k-1} f_k f'_{k-1} - x_k f_{k-1} f_k}{f_k f'_{k-1} - f_{k-1} f'_k}} \longrightarrow (3)$$

However, this method can be made derivative free, if we use the approximation

$$g(x) \simeq G(x) = - \frac{f^2(x)}{f(x - f(x)) - f(x)}$$

When $G(x)$ is substituted in the secant method, we have

$$\boxed{x_2 = x_1 - (x_0 - x_1) \frac{G_1}{G_0 - G_1}} \longrightarrow (4)$$

Problem 1 Find the multiple root of the equation $f(x) = 27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1 = 0$

Soln: Given:

$$f(x) = 27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1$$

$f(-1) = -16$ and $f(0) = 1$

The roots lies in $(-1, 0)$

For the newton raphson method and the initial eqn ①, ②

Approximation taken as $-1, 0$.

Let $x_0 = -1 \Rightarrow f'(x) = 135x^4 + 108x^3 + 108x^2 + 56x + 9$

$$f''(x) = 540x^3 + 324x^2 + 216x + 56$$

$$x_{k+1} = x_k - \frac{f_k}{f'_k} \quad \text{— ①}$$

$k = 0, 1, 2 \ldots$

$$x_{k+1} = x_k - \frac{f_k f'_k}{f'^2_k - f_k f''_k} \quad \text{— ②}$$

①

K	x_{k+1}	f_{k+1}
1	-0.818182	$-0.514(1)$
2	-0.678667	$-0.162(1)$
3	-0.574679	$-0.504(0)$
4	-0.499459	$-0.155(0)$
5	-0.446432	$-0.468(-1)$

2

K	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	$x_{k+1} = x_k - \frac{f_k f'_k}{f'^2_k - f_k f''_k}$	f_{k+1}
1	-1	-16	88	-376	-0.18518518	0.090802(-1)
2	-0.1851851	0.090802187	1.806232873	23.68178585	-0.33266153	0.909(-8)
3.	-0.33266153	0.909(-8)	4.0597(-5)	0.120827279	-0.333333214	0.139(-15)
4.	-0.333333214				-0.33333320	-0.208(-15)
5.	-0.				-0.33333313	-0.153(-15)

$\underline{k=1}$

$$x_{k+1} = \frac{x_{k-1} f_k f'_{k-1} - x_k f_{k-1} f'_k}{f_k f'_{k-1} - f_{k-1} f'_k}$$

$$x_2 = \frac{x_0 f_1 f_0' - x_1 f_0 f_1'}{f_1 f_0' - f_0 f_1'}$$

$x_1 = -1$
$x_0 = 0$

$$x_2 = \frac{0 - (-1)(1)(88)}{(-16)(9) - (1)(88)} = \frac{88}{-144-88} = -0.37931034$$

$f(x_2) = -0.300(-2)$

K=2

$$x_3 = \frac{x_1 f_2 f_1' - x_2 f_1 f_2'}{f_2 f_1' - f_1 f_2'} \qquad x_2 = -0.300(-2), \quad x_1 = -1$$

$$= \frac{(-1)(-0.300\times10^{-2})(88) - (-0.37931034)(-16)(0.19785063)}{(-0.300\times10^{-2})(88) - (-16)(0.19785063)}$$

$$x_3 = \frac{0.264 - 1.20074848}{-0.264 + 3.16561008} = -0.322837535$$

$$f(x_3) = 0.3447207358\times10^{-4}$$

K=3

$$x_4 = \frac{x_2 f_3 f_2' - x_3 f_2 f_3'}{f_3 f_2' - f_2 f_3'} \qquad x_2 = -0.37931034, \quad x_3 = -0.322837535$$

$$x_4 = \frac{(-0.379310344)\times(0.34472\times10^{-4})(0.19785063) - (-0.32283753\times -0.300\times10^{-2})\times 9.83295909\times10^{-3})}{(0.3447\times10^{-4}\times0.19785) - (-0.3000\times10^{-2}\times 9.83295\times10^{-3})}$$

$$= \frac{-1.211020753\times10^{-5}}{3.631878841\times10^{-5}}$$

$$x_4 = -0.333441947$$

$$f(x_4) = -0.3844\times10^{-10}$$

$\underline{\underline{n=4}}$

$$x_5 = \frac{x_3 f_4 f'_3 - x_4 f_3 f'_4}{f_4 f'_3 - f_3 f_4'}$$

$x_3 = -0.322837535 \qquad x_4 = -0.333441947$

$$x_5 = \frac{(-0.322837535 \times -0.3844\times10^{-10} \times 9.83295909\times10^{-3}) - (-0.333441947 \times 0.3447\times10^{-4}) \times (1.06167893\times10^{-6})}{(-0.3844\times10^{-10}) \times 9.83295909\times10^{-3}) - (-0.333441947 \times 0.3447\times10^{-4} \times 1.06167893\times10^{-6})}$$

$$= \frac{1.232469153\times10^{-4}}{-3.697405167\times10^{-11}}$$

$x_5 = -0.33333354$

$f(x_5) = 0.416(-16)$

$\underline{\underline{n=5}}$

$$x_6 = \frac{x_4 f_5 f'_4 - x_5 f_4 f'_5}{f_5 f'_4 - f_4 f_5'}$$

$x_4 = -0.333441947 \qquad x_5 = -0.33333354.$

$$x_6 = \frac{(-0.333441947 \times 0.416\times10^{-16} \times 1.06167893\times10^{-6}) - (-0.33333354 \times -0.3844\times10^{-10} \times 3.76\times10^{-12})}{(0.416\times10^{-16} \times 1.0616\times10^{-6}) - (-0.3844\times10^{-10}) \times 3.76\times10^{-12}}$$

$$= \frac{-6.290490773\times10^{-23}}{1.887002435\times10^{-22}}$$

$x_6 = -0.333358911$

$f(x_6) = -0.394(-12)$

K	③ x_{k+1}	f_{k+1}	④ x_{k+1}	f_{k+1}
1	-0.37931034	-0.300(-2)	-0.13673044	0.209(0)
2	-0.32280294	0.348(-4)	-0.34455964	-0.427(-4)
3.	-0.33343798	-0.344(-10)	-0.34089774	-0.130(-4)
4	-0.33333355	0.416(-16)	-0.33336414	-0.877(-12)
5	-0.33335693	-0.394(-12)	-0.33336408	-0.872(-12)

UNIT - II

Iteration Methods :- System of Linear Algebraic equations & Eigen value Problems

A general linear iterative method for the solution of the system of equations may be defined in the form

$$x^{(k+1)} = Hx^{(k)} + c, \quad k = 0,1,2\ldots \longrightarrow (1)$$

Where $x^{(k+1)}$ and $x^{(k)}$ are the approximations for x at the $(k+1)^{th}$ and k^{th} iterations, respectively. H is called the iteration matrix depending on A and c is a column vector. In the limiting case when $k \to \infty$, $x^{(k)}$ converges to the exact solution.

$$x = A^{-1}b \longrightarrow (2)$$

and the iteration equation (1) becomes, by sub from (2)

$$A^{-1}b = HA^{-1}b + c \longrightarrow (3)$$

From (3), the column vector c is given by

$$c = (I - H)A^{-1}b \longrightarrow (4)$$

We now determine the iteration matrix H and the column vector c for a few well known iteration method.

Jacobi Iteration Method :-

We assume that the quantities a_{ii} in (3.2) are pivot elements. The equations (3.2) may be written as

$$\left.\begin{aligned}
a_{11}x_1 &= -(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n) + b_1 \\
a_{22}x_2 &= -(a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n) + b_2 \\
&\vdots \\
a_{nn}x_n &= -(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1}) + b_n
\end{aligned}\right\} \longrightarrow (5)$$

The Jacobi iteration method may now be defined as

$$\left.\begin{aligned}
x_1^{(k+1)} &= -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \cdots + a_{1n}x_n^{(k)} - b_1\right) \\
x_2^{(k+1)} &= -\frac{1}{a_{22}}\left(a_{21}x_1^{(k)} + a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)} - b_2\right) \\
&\vdots \\
x_n^{(k+1)} &= -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{n,n-1}x_{n-1}^{(k)} - b_n\right)
\end{aligned}\right\}$$

$$k = 0, 1, 2 \ldots \longrightarrow (6)$$

Which in matrix notation becomes

$$\boxed{x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b}$$

$$= Hx^{(k)} + c, \quad k = 0, 1, 2 \longrightarrow (7)$$

Where

$$H = -D^{-1}(L+U), \quad c = D^{-1}b$$

and L and U are respectively lower and upper triangular matrices with zero diagonal entries, D is the diagonal matrix such that

$$A = L + D + U$$

Equation (7) can alternatively be written as

$$x^{(k+1)} = x^{(k)} - [I + D^{-1}(L+U)]\, x^{(k)} + D^{-1}b$$

$$= x^{(k)} - D^{-1}[D + L + U]\, x^{(k)} + D^{-1}b$$

$$= x^{(k)} + D^{-1}[b - Ax^{(k)}]$$

or $\quad v^{(k)} = D^{-1} r^{(k)}$

where $v^{(k)} = x^{(k+1)} - x^{(k)}$, is the error in the approximation and $r^{(k)} = b - Ax^{(k)}$ is the residual. We may rewrite the above equation as

$$Dv^{(k)} = r^{(k)}$$

We solve for $v^{(k)}$ and find $x^{(k+1)} = x^{(k)} + v^{(k)}$.

Gauss - Seidal Iteration Method :-

We now use on the right hand side (6), all the available values from the present iteration. We write the Gauss-Seidal method as

$$\left\{\begin{aligned} x_1^{(k+1)} &= -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \dots + a_{1n}x_n^{(k)}\right) + \frac{b_1}{a_{11}} \\ x_2^{(k+1)} &= -\frac{1}{a_{22}}\left(a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \dots + a_{2n}x_n^{(k)}\right) + \frac{b_2}{a_{22}} \\ &\vdots \\ x_n^{(k+1)} &= -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \dots + a_{n,n-1}x_{n-1}^{(k+1)}\right) + \frac{b_n}{a_{nn}} \end{aligned}\right. \longrightarrow (8)$$

Which may be rearranged in the form

$$\left.\begin{aligned} a_{11}x_1^{(k+1)} &= -\sum_{i=2}^{n} a_{1i}x_i^{(k)} + b_1 \\ a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} &= -\sum_{i=3}^{n} a_{2i}x_i^{(k)} + b_2 \\ &\vdots \\ a_{n1}(x_1)^{k+1} + \dots + a_{nn}x_n^{(k+1)} &= b_n \end{aligned}\right\} \longrightarrow (9)$$

In matrix notation (9) becomes

$$(D+L)x^{(k+1)} = -Ux^{(k)} + b$$

(or)

$$\boxed{x^{(k+1)} = -(D+L)^{-1}Ux^{(k)} + (D+L)^{-1}b} \longrightarrow (10)$$

$$= Hx^{(k)} + c, \quad k = 0, 1, 2 \dots$$

Where

$$H = -(D+L)^{-1}U \text{ and } c = (D+L)^{-1}b$$

Equation (10) can be alternatively be written as

$$x^{(k+1)} = x^{(k)} - \left[I + (D+L)^{-1} U \right] x^{(k)} + (D+L)^{-1} b$$

$$= x^{(k)} - (D+L)^{-1} A x^{(k)} + (D+L)^{-1} b$$

$$= x^{(k)} + (D+L)^{-1} (b - A x^{(k)})$$

we write

$$v^{(k)} = (D+L)^{-1} r^{(k)}$$

where $v^{(k)} = x^{(k+1)} - x^{(k)}$ and $r^{(k)} = b - Ax^{(k)}$ is the residual. Alternatively, we may write.

$$(D+L)\, v^{(k)} = r^{(k)}$$

and solve $v^{(k)}$ by forward substitution. The solution is then found from

$$x^{(k+1)} = x^{(k)} + v^{(k)}.$$

Example

Consider the system of equations

$$2x_1 - x_2 + 0x_3 = 7$$

$$-x_1 + 2x_2 - x_3 = 1$$

$$0x_1 - x_2 + 2x_3 = 1$$

Use the Gauss Seidal iterative method and Perform three iterations.

Soln:

The given system of equations can be written as $Ax = b$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad D+L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A = L + D + U$$

By Gauss-Seidal method

$$x^{(K+1)} = -(D+L)^{-1} U x^{(K)} + (D+L)^{-1} b.$$

Let
$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

K=0

$$x^1 = -(D+L)^{-1} U (x)^0 + (D+L)^{-1} b.$$

$$-(D+L)^{-1} U = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0.25 & 0.5 \\ 0 & 0.125 & 0.25 \end{bmatrix}$$

$$(D+L)^{-1} b = \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

<u>k=1</u>

$$x^{(2)} = -(D+L)^{-1} U x^{(1)} + (D+L)^{-1} b$$

$$x^{(2)} = \begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix}$$

<u>k=2</u>

$$x^{(3)} = -(D+L)^{-1} U x^{(2)} + (D+L)^{-1} b$$

$$x^{(3)} = \begin{bmatrix} 5.3125 \\ 4.3125 \\ 2.6563 \end{bmatrix}$$

$$x^{(4)} = \begin{bmatrix} 5.6563 \\ 4.6563 \\ 2.8281 \end{bmatrix}$$

$$x^{(5)} = \begin{bmatrix} 5.8282 \\ 4.8281 \\ 2.9141 \end{bmatrix}$$

$$x^{(6)} = \begin{bmatrix} 5.9141 \\ 4.9141 \\ 2.9570 \end{bmatrix}$$

$$x^{(7)} = \begin{bmatrix} 5.9571 \\ 4.9570 \\ 2.9785 \end{bmatrix}$$

The exact solution is $x = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$

Use the Jacobi Iteration method.

Soln: Given: $2x_1 - x_2 + 0x_3 = 7$

$-x_1 + 2x_2 - x_3 = 1$

$0x_1 - x_2 + 2x_3 = 1$

The Given system of equations can be written as

$$Ax = b$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad L+U = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$A = L + D + U$

By Jacobi - Method

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$$

Let $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\underline{k=0}$

$$x^{1} = -D^{-1}(L+U)x^{(0)} + D^{-1}b$$

$$-D^{-1}(L+U) = \begin{pmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{pmatrix} \quad D^{-1}b = \begin{pmatrix} 3.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} 3.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

$\underline{k=1}$

$$x^{(2)} = -D^{-1}(L+U)\,x^{(1)} + D^{-1}b$$

$$x^{(2)} = \begin{pmatrix} 3.75 \\ 2.5 \\ 0.75 \end{pmatrix}$$

$\underline{k=2}$

$$x^{(3)} = -D^{-1}(L+U)\,x^{(2)} + D^{-1}b$$

$$x^{(3)} = \begin{pmatrix} 4.75 \\ 2.75 \\ 1.75 \end{pmatrix}.$$

$$x^{(4)} = \begin{pmatrix} 4.875 \\ 3.75 \\ 1.875 \end{pmatrix} \qquad x^{(5)} = \begin{pmatrix} 5.375 \\ 3.875 \\ 2.375 \end{pmatrix}$$

$$x^{(6)} = \begin{pmatrix} 5.475 \\ 4.375 \\ 2.4375 \end{pmatrix} \qquad x^{(7)} = \begin{pmatrix} 5.6875 \\ 4.4563 \\ 2.6875 \end{pmatrix}$$

$$x^{(8)} = \begin{pmatrix} 5.7282 \\ 4.6875 \\ 2.7282 \end{pmatrix} \qquad x^{(9)} = \begin{pmatrix} 5.8438 \\ 4.7282 \\ 2.8438 \end{pmatrix}$$

$$x^{(10)} = \begin{pmatrix} 5.8641 \\ 4.8438 \\ 2.8641 \end{pmatrix} \qquad x^{(11)} = \begin{pmatrix} 5.9219 \\ 4.8641 \\ 2.9219 \end{pmatrix}$$

$$x^{(12)} = \begin{pmatrix} 5.9321 \\ 4.9219 \\ 2.9321 \end{pmatrix} \qquad x^{(13)} = \begin{pmatrix} 5.9610 \\ 4.9321 \\ 2.9610 \end{pmatrix}$$

$$x^{(14)} = \begin{pmatrix} 5.9661 \\ 4.9610 \\ 2.9661 \end{pmatrix} \qquad x^{(15)} = \begin{pmatrix} 5.9805 \\ 4.9661 \\ 2.9805 \end{pmatrix}$$

$$x^{(16)} = \begin{pmatrix} 5.9831 \\ 4.9805 \\ 2.9831 \end{pmatrix} \quad x^{(17)} = \begin{pmatrix} 5.9903 \\ 4.9831 \\ 2.9903 \end{pmatrix}$$

The exact solution is $x = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$

Home work

Consider the system of equation

$$4x + y + 2z = 4$$
$$3x + 5y + z = 7$$
$$x + y + 3z = 3$$

Use the Gauss Seidal & Jacobi iteration method.

Soln: The Given system of equation can be written as

$$Ax = b$$

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 3 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}. \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix};$$

$$U = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad D + L = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 5 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

$$A = L + D + U$$

By Gauss-Seidal method.

$$x^{[k+1]} = -(D+L)^{-1} U x^{(k)} + (D+L)^{-1} b$$

Let

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{k=0}$

$$x^{(1)} = -(D+L)^{-1}U x^{(0)} + (D+L)^{-1} b$$

$$-(D+L)^{-1}U = \begin{bmatrix} 0 & -0.25 & -0.5 \\ 0 & 0.15 & 0.10 \\ 0 & 0.0333 & 0.1333 \end{bmatrix}$$

$$(D+L)^{-1} b = \begin{bmatrix} 1 \\ 0.8 \\ 0.4 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 1 \\ 0.8 \\ 0.4 \end{bmatrix}$$

$$x^{(2)} = -(D+L)^{-1}U x^{(1)} + (D+L)^{-1} b$$

$$x^{(2)} = \begin{bmatrix} 0.6 \\ 0.96 \\ 0.48 \end{bmatrix} \qquad x^{(6)} = \begin{bmatrix} 0.5002 \\ 0.9999 \\ 0.4999 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} 0.52 \\ 0.992 \\ 0.496 \end{bmatrix}$$

$$x^{(4)} = \begin{bmatrix} 0.504 \\ 0.9984 \\ 0.4992 \end{bmatrix}$$

$$x^{(5)} = \begin{bmatrix} 0.5 \\ 0.9997 \\ 0.4998 \end{bmatrix}$$

The exact root is $\begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$

By Jacobi method

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$$

$$L+U = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Let $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\underline{\underline{k=0}}$

$$x^{(1)} = -D^{-1}(L+U)x^{(0)} + D^{-1}b$$

$$-D^{-1}(L+U) = \begin{pmatrix} 0 & -0.25 & -0.5 \\ -0.6 & 0 & -0.2 \\ -0.3333 & -0.3333 & 0 \end{pmatrix}$$

$$D^{-1}b = \begin{pmatrix} 1 \\ 1.4 \\ 1 \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} 1 \\ 1.4 \\ 1 \end{pmatrix}$$

$\underline{\underline{k=1}}$

$$x^{(2)} = -D^{-1}(L+U)x^{(1)} + D^{-1}b$$

$$x^{(2)} = \begin{pmatrix} 0.15 \\ 0.6 \\ 0.2001 \end{pmatrix}$$

$\underline{\underline{k=2}}$

$$x^{(3)} = -D^{-1}(L+U)x^{(2)} + D^{-1}b$$

$$x^{(3)} = \begin{pmatrix} 0.75 \\ 1.27 \\ 0.75 \end{pmatrix}$$

$$x^{(4)} = \begin{pmatrix} 0.3075 \\ 0.800 \\ 0.3267 \end{pmatrix}$$

$$x^{(5)} = \begin{pmatrix} 0.6367 \\ 1.1592 \\ 0.6309 \end{pmatrix}$$

$$x^{(6)} = \begin{pmatrix} 0.3970 \\ 0.8918 \\ 0.4044 \end{pmatrix}$$

$$x^{(7)} = \begin{pmatrix} 0.5749 \\ 1.0809 \\ 0.5704 \end{pmatrix}$$

$$x^{(8)} = \begin{pmatrix} 0.4446 \\ 0.9410 \\ 0.4481 \end{pmatrix}$$

$$x^{(9)} = \begin{pmatrix} 0.5407 \\ 1.0436 \\ 0.5382 \end{pmatrix}$$

EIGENVALUES AND EIGENVECTORS :-

The eigenvalues of a matrix A are given by the roots of the characteristic equation

$$\det(A - \lambda I) = 0 \longrightarrow (1)$$

which when simplified gives the polynomial equation

$$P(\lambda) = (-1)^n \lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0 \longrightarrow (2)$$

where the sign $(-1)^n$ is used to given the terms of the polynomial the same sign that they would have if the polynomial were generated by expanding the determinant. The coefficients of the polynomial (2) can be determined with the help of the Faddeev - Leverrier method

We have

$$\left.\begin{aligned}
B_1 &= A \text{ and } a_1 = \operatorname{tr} B_1 \\
B_2 &= A(B_1 - a_1 I) \text{ and } a_2 = \frac{1}{2}\operatorname{tr} B_2 \\
B_3 &= A(B_2 - a_2 I) \text{ and } a_3 = \frac{1}{3}\operatorname{tr} B_3 \\
&\vdots \\
B_K &= A(B_{K-1} - a_{K-1} I) \text{ and } a_K = \frac{1}{K}\operatorname{tr} B_K \\
&\vdots \\
B_n &= A(B_{n-1} - a_{n-1} I) \text{ and } a_n = \frac{1}{n}\operatorname{tr} B_n
\end{aligned}\right\} \longrightarrow (3)$$

Where $\operatorname{tr} A = a_{11} + a_{22} + \dots + a_{nn}$.

The roots of the polynomial equation (2) may be determined by the methods discussed in Chapter 2.

A nonzero vector X_i such that

$$AX_i = \lambda_i X_i \longrightarrow (4)$$

is called the eigenvector or characteristic vector corresponding to λ_i. Multiplying (4) by an arbitrary constant c and putting $y_i = cx_i$, we get

$$Ay_i = \lambda_i y_i \longrightarrow (5)$$

Which shows that an eigenvector is determined only within an arbitrary multiplicative constant.

Premultiplying (3.4) m times by A we may obtain

$$A^m x = \lambda^m x \longrightarrow (6)$$

which shows that λ^m is an eigenvalues of A^m and x is the corresponding eigenvector. Substituting (6) into (2) we get

$$P(A) = 0 \longrightarrow (7)$$

which gives the result that a square matrix A satisfies its own characteristic equation. The result is known as the (Cayley-Hamilton) theorem. Replacing the matrix A in (7) by the transpose matrix A^T we find.

$$\det(A^T - \lambda I) = \det(A - \lambda I) = 0 \longrightarrow (8)$$

Thus A & A^T have the same eigenvalues. For distinct eigenvalues, if $u_1, u_2 \dots u_n$, are the eigenvectors of A and $v_1, v_2 \dots v_n$ are the eigenvectors of A^T then we have

$$A u_i = \lambda_i u_i \longrightarrow (9)$$

$$A^T v_j = \lambda_j v_j \longrightarrow (10)$$

We obtain

$$v_j^T A u_i = \lambda_i v_j^T u_i \longrightarrow (11)$$

System of Linear Algebraic equations and Eigenvalues Problems.

Taking transpose of (10) and post-multiplying by u_i, we get

$$v_j^T A u_i = \lambda_j v_j^T u_i \longrightarrow (12)$$

Subtracting (12) from (11) we get

$$(\lambda_i - \lambda_j) v_j^T u_i = 0 \longrightarrow (12)$$

If $i \neq j$, then $\lambda_i \neq \lambda_j$ and we have

$$v_j^T u_i = 0 \longrightarrow (13)$$

If $i = j$, $v_i^T u_i \neq 0$ and since the length of eigenvectors is arbitrary we normalize them such that

$$v_i^T u_i = 1 \longrightarrow (14)$$

Thus, we have

$$v_j^T u_i = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \longrightarrow (15)$$

We conclude that the eigenvectors corresponding to different eigenvalues of a matrix and of its transpose are orthogonal. Such sets of mutually orthogonal vectors are called biorthogonal sets. When A is a real symmetric matrix, $A = A^T$, $u_i = v_i$ and the eigenvectors corresponding to different eigenvalues are orthogonal. Premultiplying (9) by u_i^T, we get

$$\lambda_i = \frac{u_i^T A u_i}{u_i^T u_i} \longrightarrow (16)$$

Which gives an expression for the eigenvalues in terms of the eigenvectors. For arbitrary u, (16) is called the Rayleigh quotient.

Let A and B be two square matrices of same order. If a non-singular matrix S can be determined such that

$$B = S^{-1}AS \longrightarrow (17)$$

then the matrices A & B are said to be similar and (17) is called a similary transformation. The matrix S is called the similarity matrix.

From (17) we may write

$$A = SBS^{-1} \longrightarrow (18)$$

If λ_i is an eigenvalues of A and u_i is the corresponding eigenvector then

$$Au_i = \lambda_i u_i \longrightarrow (19)$$

or

$$S^{-1}Au_i = \lambda S^{-1}u_i$$

substituting $u_i = Sv_i$ in (19) and using (17) we get

$$Bv_i = \lambda_i v_i$$

Thus $S^{-1}AS$ has the same eigenvalues as A and its eigenvectors v_i are obtained from the relation $v_i = S^{-1}u_i$.

Suppose that the matrix A has eigenvalues λ_i with eigenvectors u_i and that A has an inverse A^{-1}. Then

$$Au_i = \lambda_i u_i$$

which may be written as

$$A^{-1}u_i = \frac{1}{\lambda_i} u_i \longrightarrow (20)$$

The inverse matrix A^{-1} has the same eigenvectors as A but has the eigenvalues $1/\lambda_i$. A similarly transformation, where S is the matrix of eigenvector, reduces a matrix A to its diagonal form. The eigenvalues of A are located on the leading diagonal of this diagonal matrix.

Jacobi method for Symmetric Matrices:

Let A be the given real symmetric matrix. The eigenvalues of A are real orthogonal matrix S such that $S^{-1}AS$ is a diagonal matrix D. The diagonalization is done by applying a series of orthogonal transformations $S_1, S_2 \dots S_n \dots$ as follows

Among the off-diagonal elements, Let $|a_{ik}|$ be the numerically largest element. Then the elements $a_{ii}, a_{ik}, a_{ki}, a_{kk}$ form a 2×2 submatrix A_1 which can be transformed to a diagonal form, we choose

$$S_1^* = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \longrightarrow (1)$$

and find θ such that the 2×2 submatrix A_1 is diagonalized, we have

$$S_1^{*-1} A_1 S^* = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ik} \\ a_{ki} & a_{kk} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{ii}\cos^2\theta + a_{ik}\sin 2\theta + a_{kk}\sin^2\theta & (a_{kk}-a_{ii})\sin\theta\cos\theta + a_{ik}\cos 2\theta \\ (a_{kk}-a_{ii})\sin\theta\cos\theta + a_{ik}\cos 2\theta & a_{ii}\sin^2\theta - a_{ik}\sin 2\theta + a_{kk}\cos^2\theta \end{bmatrix} \longrightarrow (2)$$

We now choose θ such that this matrix reduces to a diagonal form. That is we put

$$\frac{1}{2}(a_{kk} - a_{11})\sin 2\theta + a_{1k}\cos 2\theta = 0 \longrightarrow (3)$$

(or) $$\tan 2\theta = \frac{2a_{1k}}{a_{11} - a_{kk}} \longrightarrow (4)$$

This equation produces four values of θ and in order that we may get smallest rotation we require $-\pi/4 \le \theta \le \pi/4$ form (4) we get

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{1k}}{a_{11} - a_{kk}}\right) \longrightarrow (5)$$

if $a_{11} \neq a_{kk}$. If $a_{11} = a_{kk}$, then

$$\theta = \begin{cases} \pi/4, & a_{1k} > 0 \\ -\pi/4, & a_{1k} < 0 \end{cases} \longrightarrow (6)$$

with the value of θ given in (5) the off-diagonal elements in (3) vanish and the diagonal elements are simplified. The first step is now completed by performing the rotation $S_1^{-1}AS_1$. In the next step the largest off-diagonal element in magnitude in the new rotated matrix is found and the procedure is repeated. we now perform a series of such two dimensional rotations. After finding θ at each step, the rotation is performed with the corresponding orthogonal matrix. For example. if $|a_{1k}|$ is the largest off-diagonal element then we write S_1 as

$$S_1 = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \cos\theta & \cdots & -\sin\theta & \\ & & & \vdots & 1 & \vdots & \\ & & & & \ddots & & \\ & & & \sin\theta & \cdots 1 & \cos\theta & \\ & & & & & & \ddots \; 1 \end{bmatrix}$$

where $\cos\theta$, $-\sin\theta$, $\sin\theta$, $\cos\theta$ are located in (i,i) (i,k) (k,i) & (k,k) positions respectively. After making r transformations, we get

$$B_r = S_r^{-1} S_{r-1}^{-1} \cdots S_1^{-1} A S_1 \cdots S_{r-1} S_r$$

$$= (S_1 S_2 \cdots S_r)^{-1} A (S_1 \cdots S_{r-1} S_r)$$

$$= S^{-1} A S \longrightarrow (7)$$

where

$$S = S_1 S_2 \cdots S_r$$

As $r \to \infty$, B_r approaches a diagonal matrix with the eigenvalues on the leading diagonal. We then have the eigenvectors as the corresponding columns of S. The minimum number of rotations required to bring A into a diagonal form may be $(n-1)n/2$. This ~~Prude~~ procedure, described to reduce the symmetric matrix A to the diagonal matrix D is called the Jacobi method. This method suffers from the following disadvantage. The elements annihilated by a plane rotation may not necessarily remain zero during subsequent transformation. The value of θ must be checked for its accuracy by checking whether $|\sin^2\theta + \cos^2\theta - 1|$, is sufficiently small. The convergence to the eigenvalues takes places even if the Pivots are not selected on the basis of their magnitudes, but are selected in the "typewritter fashion". That is annihilate $a_{21}, a_{31}, a_{32}, a_{41}, a_{42}, a_{43}$ etc. This modification is called the special cyclic Jacobi method. In this method there is no search for the Pivots.

Example: Find all the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Soln:

Given:

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Here $a_{ij} = a_{ji}$ for all $i \neq j$

A is a symmetric matrix. Numerically largest off diagonal is 2.

$$a_{13} = a_{31} = 2$$

$$\tan 2\theta = \frac{2a_{ik}}{a_{ii} - a_{kk}} = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2(2)}{1-1} = \infty$$

$$\tan 2\theta = \infty$$

$$2\theta = \tan^{-1}\infty$$

$$2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

Let $S_1 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta & 1 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$\underline{\theta = \pi/4}$

$$S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$B_1 = S_1^{-1} A S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$S_1^{-1} A S_1 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = B_1$$

Again the largest off-diagonal element is 2

$$a_{12} = a_{21} = 2$$

$$\tan 2\theta = \frac{2a_{ik}}{a_{ii} - a_{kk}} = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2(2)}{3-3} = \infty$$

$$\tan 2\theta = \infty$$

$$2\theta = \tan^{-1}\infty$$

$$2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

$$S_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{\theta = \pi/4}$

$$S_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_2 = S_2^{-1} B_1 S_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$

$$B_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The Eigen values of A $(5, 1, -1)$

<u>To find the Eigen vectors</u>:

$$\underline{E} = S_1 S_2$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}$$

The eigen vector of 5 is $\begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$

The eigen vector of 1 is $\begin{pmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{pmatrix}$

The eigen vector of -1 is $\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.

Home work

Find all the eigen values and eigenvectors of the matrix $\begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 1 & 1/4 \\ 1/2 & 1/4 & 2 \end{bmatrix}$

Soln: Given $A = \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 1 & 1/4 \\ 1/2 & 1/4 & 2 \end{bmatrix}$

Here $a_{ij} = a_{ji}$ for all $i \& j$

A is a symmetric matrix.

Numerically largest off diagonal is 1

$$a_{12} = a_{21} = 1$$

$$\tan 2\theta = \frac{2a_{ik}}{a_{ii} - a_{kk}} = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2(1)}{1-1} = \frac{2}{1-1} = \infty$$

$$\tan 2\theta = \infty$$

$$2\theta = \tan^{-1}\infty$$

$$2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

Let $S_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$= \begin{bmatrix} \cancel{\cos\theta} & \cancel{0} & \cancel{-\sin\theta} \\ \cancel{0} & \cancel{1} & \cancel{0} \\ \cancel{\sin\theta} & \cancel{0} & \cancel{\cos\theta} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{\theta = \pi/4}$ $S_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$B_1 = S_1^{-1} A S = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 1 & 1/4 \\ 1/2 & 1/4 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 2 & 0 & 0.5303 \\ 0 & 0 & -0.1768 \\ 0.5303 & -0.1768 & 2 \end{bmatrix}$$

Again the largest off-diagonal element is 0.5303

$$a_{13} = a_{31} = 0.5303$$

$$\tan 2\theta = \frac{2(0.5303)}{2-2} = \infty$$

$$\tan 2\theta = \infty$$

$$2\theta = \tan^{-1}\infty \Rightarrow 2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

$$S_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$\underline{\theta = \pi/4}$ $S_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

$$B_2 = S_2^{-1} B_1 S_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 0.5303 \\ 0 & 0 & -0.1768 \\ 0.5303 & -0.1768 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 2.5303 & -0.125 & 0 \\ -0.125 & 0 & -0.125 \\ 0 & -0.125 & 1.4697 \end{bmatrix}$$

The largest off diagonal $= 0 \Rightarrow a_{13} = a_{31}$

$$\tan 2\theta = \frac{0}{2.5303 - 1.4697} = 0$$

$\tan 2\theta = 0$

$2\theta = \tan^{-1}(0)$

$\theta = 1/2\,(0)$

$\boxed{\theta = 0}$

$$S_3 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$\underline{\underline{\theta = 0}}$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_3 = S_3^{-1} B_2 S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2.5303 & -0.125 & 0 \\ -0.125 & 0 & -0.125 \\ 0 & -0.125 & 1.4697 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5303 & -0.125 & 0 \\ -0.125 & 0 & -0.125 \\ 0 & -0.125 & 1.4697 \end{bmatrix}$$

The Eigen values of A

$(2.5303, 0, 1.4697)$

To find the Eigen vectors

$$E = S_1 \times S_2 \times S_3 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ 1/2 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

The eigen vector of 2.5303 is $\begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$

The eigen vector of 0 is $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

The eigen vector of 14.697 is $\begin{pmatrix} -1/2 \\ -1/2 \\ 1/\sqrt{2} \end{pmatrix}$

Power Method

Find the largest eigenvalue in modulus and the corresponding eigenvector of the matrix

$$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$$ using the power method.

There are four choices $\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$

Soln: Let initially $V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$AV_0 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \\ 18 \end{bmatrix} = 18 \begin{bmatrix} -0.4444 \\ 0.2222 \\ 1 \end{bmatrix} = 18V_1$$

$$AV_1 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -0.4444 \\ 0.2222 \\ 1 \end{bmatrix} = \begin{bmatrix} 10.5548 \\ -1.1104 \\ -7.7768 \end{bmatrix} = 10.5548 \begin{bmatrix} 1 \\ -0.1052 \\ -0.7368 \end{bmatrix}$$

$$= 10.5548\, V_2$$

$$AV_2 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.1052 \\ -0.7368 \end{bmatrix} = \begin{bmatrix} -17.6312 \\ 6.8416 \\ 18.9472 \end{bmatrix} = 18.9472 \begin{bmatrix} -0.9305 \\ 0.3611 \\ 1 \end{bmatrix}$$

$$= 18.9472\, V_3$$

$$AV_3 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -0.9305 \\ 0.3611 \\ 1 \end{bmatrix} = \begin{bmatrix} 18.4019 \\ -7.6382 \\ -18.0544 \end{bmatrix} = 18.4019 \begin{bmatrix} 1 \\ -0.4151 \\ -0.9811 \end{bmatrix}$$

$$= 18.4019\, V_4$$

$$AV_4 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4151 \\ -0.9811 \end{bmatrix} = \begin{bmatrix} -19.6037 \\ 9.0946 \\ 19.6982 \end{bmatrix} = 19.6982 \begin{bmatrix} -0.9952 \\ 0.4617 \\ 1 \end{bmatrix}$$

$$= 19.6982\, V_5$$

$$AV_5 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -0.9952 \\ 0.4617 \\ 1 \end{bmatrix} = \begin{bmatrix} 19.7748 \\ -9.4924 \\ -19.7508 \end{bmatrix} = 19.7748 \begin{bmatrix} 1 \\ -0.48 \\ -0.9988 \end{bmatrix}$$

$$= 19.7748 V_6$$

$$AV_6 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.48 \\ -0.9988 \end{bmatrix} = \begin{bmatrix} -19.9164 \\ 9.7672 \\ 19.9224 \end{bmatrix} = 19.9224 \begin{bmatrix} -0.9997 \\ 0.4903 \\ 1 \end{bmatrix}$$

$$= 19.9224 V_7$$

$$AV_7 = \begin{bmatrix} 19.9567 \\ -9.8806 \\ -19.9552 \end{bmatrix} = 19.9567 \begin{bmatrix} 1 \\ -0.4951 \\ -0.9999 \end{bmatrix} = \cancel{AV_8}\, 19.9567 V_8$$

$$AV_8 = \begin{bmatrix} -19.9801 \\ 9.9418 \\ 19.9806 \end{bmatrix} = 19.9806 \begin{bmatrix} -1 \\ 0.4976 \\ 1 \end{bmatrix} = 19.9806 V_9$$

$$AV_9 = \begin{bmatrix} 19.9904 \\ -9.9712 \\ -19.9904 \end{bmatrix} = 19.9904 \begin{bmatrix} 1 \\ -0.4988 \\ -1 \end{bmatrix} = 19.9904 V_{10}$$

$$AV_{10} = \begin{bmatrix} -19.9952 \\ 9.9856 \\ 19.9952 \end{bmatrix} = 19.9952 \begin{bmatrix} -1 \\ 0.4994 \\ 1 \end{bmatrix} = 19.9952 V_{11}$$

$$AV_{11} = \begin{bmatrix} 19.9976 \\ -9.9928 \\ -19.9976 \end{bmatrix} = 19.9976 \begin{bmatrix} 1 \\ -0.4997 \\ -1 \end{bmatrix} = 19.9976 V_{12}$$

$$AV_{12} = \begin{bmatrix} -19.9988 \\ 9.9964 \\ 19.9988 \end{bmatrix} = 19.9988 \begin{bmatrix} -1 \\ -0.4998 \\ 1 \end{bmatrix} = 19.9988 V_{13}$$

Numerically the largest eigen value of A

$19.9988 \simeq 20$

The corresponding eigen vector is $\begin{bmatrix} -1 \\ -0.4998 \\ 1 \end{bmatrix}$

Interpolation and Approximation

Hermite Interpolation:

The Hermite interpolating polynomial interpolates not only the function $f(x)$ but also its (certain order) derivatives at a given set of tabular points.

$$P(x_i) = f(x_i)$$
$$P'(x_i) = f'(x_i) \quad , \quad i = 0,1,2\dots n$$

Since there are $2n+2$ conditions to be satisfied, $P(x)$ must be a polynomial of degree $\leq 2n+1$

The required polynomial may be written as

$$\boxed{P(x) = \sum_{i=0}^{n} A_i(x) f(x_i) + \sum_{i=0}^{n} B_i(x) f'(x_i)} \longrightarrow (1)$$

Where $A_i(x)$ and $B_i(x)$ are polynomials of degree $2n+1$ and satisfy

$$\left.\begin{aligned}
&(i) \quad A_i(x_j) = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases} \\
&(ii) \quad A_i'(x_j) = 0 \quad \text{for all } i, j \\
&(iii) \quad B_i(x_j) = 0 \quad \text{for all } i, j \\
&(iv) \quad B_i'(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}
\end{aligned}\right\} \longrightarrow (2)$$

Using the Lagrange fundamental polynomial $l_i(x)$ we write

$$\left.\begin{aligned} A_i(x) &= \gamma_i(x)\, l_i^2(x) \\ B_i(x) &= \delta_i(x)\, l_i^2(x) \end{aligned}\right\} \longrightarrow (3)$$

Since $l_i^2(x)$ is a polynomial of degree $2n$, $\gamma_i(x)$ and $\delta_i(x)$ must be linear polynomials,

Let

$$\left.\begin{aligned} \gamma_i(x) &= a_i x + b_i \\ \delta_i(x) &= c_i x + d_i \end{aligned}\right\} \longrightarrow (4)$$

Substituting (4) into (3)

$$\left.\begin{aligned} A_i(x) &= (a_i x + b_i)\, l_i^2(x) \\ B_i(x) &= (c_i x + d_i)\, l_i^2(x) \end{aligned}\right\} \longrightarrow (5)$$

Diff^n (5) w.r.to x

$$A_i'(x) = a_i\, l_i^2(x) + (a_i x + b_i)\, 2\, l_i(x)\, l_i'(x)$$

$$= a_i l_i^2(x) + a_i x\, 2\, l_i(x)\, l_i'(x) + b_i 2\, l_i(x)\, l_i'(x)$$

$$A_i'(x) = \left[l_i^2(x) + 2x\, l_i(x)\, l_i'(x)\right] a_i + 2 l_i(x) l_i'(x)\, b_i \longrightarrow (6)$$

$$B_i'(x) = c_i l_i^2(x) + (c_i x + d_i)\, 2 l_i(x)\, l_i'(x)$$

$$B_i'(x) = c_i\left(l_i^2(x) + x\, 2 l_i(x) l_i'(x)\right) + 2 l_i(x) l_i'(x) d_i \longrightarrow (7)$$

Using the conditions (2) we get

$$(5) \Rightarrow \quad a_i x_i + b_i = 1 \longrightarrow (8a)$$

$$c_i x_i + d_i = 0 \longrightarrow (8b)$$

$$(6) \Rightarrow \left(2x_i l_i'(x_i) + 1\right) a_i + 2 l_i'(x_i) b_i = 0 \longrightarrow (9a)$$

$$(7) \Rightarrow \left(2x_i l_i'(x_i) + 1\right) c_i + 2 l_i'(x_i) d_i = 1 \longrightarrow (9b)$$

Using (8a) and (9a) we get

$$(8a) \times 2 l_i'(x_i) \Rightarrow a_i x_i \left(2 \cancel{l_i'(x_i)}\right) + b_i\, 2 l_i'(x_i) = 2 l_i'(x_i)$$

$$(9a) \times 1 \Rightarrow \overset{(-)}{a_i 2 \cancel{x_i} l_i'(x_i)} \overset{(-)}{+} a_i + 2 l_i'(x_i) b_i \overset{(-)}{=} 0$$

$$\boxed{a_i = 2 l_i'(x_i)}$$

From (8a) $\Rightarrow \quad 2 l_i'(x_i) + b_i = 1$

$$\boxed{b_i = 1 + 2 l_i'(x_i) x_i}$$

Using (8b) and (9b) in a similar way we get

$$(8b) \times 2l_i'(x_i) \Rightarrow c_i x_i \, 2l_i'(x_i) + 2l_i'(x_i) d_i = 0$$

$$(9b) \times 1 \Rightarrow \overset{(-)}{2l_i'(x_i) c_i} + \overset{(-)}{c_i} + 2l_i'(x_i) \overset{(-)}{d_i} = 1$$

$$\boxed{c_i = 1}$$

From (8b) $\Rightarrow$ $(1) x_i + d_i = 0$

$$\boxed{d_i = -x_i}$$

$$\left.\begin{aligned} a_i &= -2l_i(x_i) \\ b_i &= 1 + 2l_i'(x_i)(x_i) \\ c_i &= 1 \\ d_i &= -x_i \end{aligned}\right\} \longrightarrow (10)$$

Substituting (10) into (5), the equ^n (1) becomes

$$P(x) = \sum_{i=0}^{n} \left[1 - 2(x - x_i) l_i'(x_i)\right] l_i^2(x) f(x_i) + \sum_{i=0}^{n} (x - x_i) l_i^2(x) f'(x_i)$$

$$\longrightarrow (11)$$

Which is called the Hermite interpolating Polynomial.

The error in the formula is given by

$$R(x) = \frac{f^{2n+2}(\xi)}{(2n+2)!} \left(\prod (x-x_i)\right)^2$$

Formula :-

$$P(x) = \sum_{i=0}^{n} \left[A_i(x) f'(x_i) + B_i(x) f'(x_i) \right]$$

Where

$$A_i(x) = \left[1 - 2(x-x_i) l_i'(x_i) \right] l_i'(x)$$

$$B_i(x) = (x - x_i) l_i^2(x)$$

and $l_i(x)$ is lagrange Polynomial.

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Problem:

1. Given the following values of $f(x)$ and $f'(x)$

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	1
1	3	7

estimate the values of $f(-0.5)$ and $f(0.5)$ using Hermite interpolation.

Soln:

$$\boxed{l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}$$

$$= \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{(-1)(-2)}$$

$$l_0(x) = \frac{x^2-x}{2}$$

Diff^n w.r.to x

$$l_0'(x) = \frac{1}{2}(2x-1)$$

Put $x = x_0$

$$l_0'(x_0) = \frac{2(-1)-1}{2} = -3/2$$

$$\boxed{l_0'(x_0) = -3/2}$$

$$\boxed{l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}}$$

$$= \frac{(x+1)(x-1)}{(0+1)(0-1)} = \frac{(x+1)(x-1)}{-1}$$

$$l_1(x) = -(x^2-1) = 1-x^2$$

Diff^n w.r.to x

$$l_1'(x) = -2x$$

Put $x=x_1$

$$\boxed{l_1'(x_1) = 0}$$

$$\boxed{l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}$$

$$= \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{(x+1)(x)}{2(1)}$$

$$l_2(x) = \frac{x^2+x}{2}$$

Diff^n w.r.to x

$$l_2'(x) = \frac{2x+1}{2}$$

Put $x=x_2$

$$l_2'(x_2) = \frac{2(1)+1}{2} = 3/2$$

$$\boxed{l_2'(x_2) = 3/2}$$

Put $i=0$ in equ^n (1)

$$A_i(x) = [1 - 2(x - x_i) l_i'(x_i)]\ l_i^2(x)$$

$$A_0(x) = [1 - 2(x - x_0) l_0'(x_0)]\ l_0^2(x)$$

$$= [1 - 2(x+1)(-3/2)] \left[\frac{x^2 - x}{2}\right]^2$$

$$= \frac{1}{4}[1 + 3x + 3]\ [x^4 + x^2 - 2x^3]$$

$$= \frac{1}{4}[x^4 + 3x^5 + 3x^4 + x^2 + 3x^3 + 3x^2 - 2x^3 - 6x^4 - 6x^3]$$

$$\boxed{A_0(x) = \frac{1}{4}[3x^5 - 2x^4 - 5x^3 + 4x^2]}$$

Put $i=1$ in equ^n (1)

$$A_1(x) = [1 - 2(x - x_1) l_1'(x_1)]\ l_1^2(x)$$

$$= [1 - 2(x - 0)(0)]\ (1 - x^2)^2$$

$$= (1)\ (1 + x^4 - 2x^2)$$

$$\boxed{A_1(x) = x^4 - 2x^2 + 1}$$

Put $i=2$ in equ^n (1)

$$A_2(x) = [1 - 2(x - x_2)\ l_2'(x_2)]\ l_2^2(x)$$

$$= [1 - 2(x-1)(3/2)]\ \left[\frac{x^2 + x}{2}\right]^2$$

$$= \left[1-(x-1)3\right]\left[\frac{x^4+x^2+2x^3}{4}\right]$$

$$= \frac{1}{4}\left[1-3x+3\right]\left[x^4+x^2+2x^3\right]$$

$$= \frac{1}{4}\left[x^4-3x^5+3x^4+x^2-3x^3+3x^2+2x^3-6x^4+6x^3\right]$$

$$\boxed{A_2(x) = \frac{1}{4}\left[-3x^5-2x^4+5x^3+4x^2\right]}$$

Put $i=0$ in equ^n (2)

$$B_i(x) = (x-x_i)\, l_i^2(x)$$

$$B_0(x) = (x-x_0)\, l_0^2(x)$$

$$= (x+1)\left(\frac{x^2-x}{2}\right)^2$$

$$= \frac{1}{4}\left[(x+1)(x^4+x^2-2x^3)\right]$$

$$= \frac{1}{4}\left[x^5+x^4+x^3+x^2-2x^4-2x^3\right]$$

$$\boxed{B_0(x) = \frac{1}{4}\left[x^5-x^4-x^3+x^2\right]}$$

Put $i=1$ in equ^n (2).

$$B_1(x) = (x-x_1)(l_1^2(x))$$

$$= (x-0)(1-x^2)^2 = x\,(1+x^4-2x^2)$$

$$= x+x^5-2x^3$$

$$\boxed{B_1(x) = x^5-2x^3+x}$$

Put $i=2$ in eqn (2)

$$B_2(x) = (x-x_2)\left(l_2^{\,2}(x)\right)$$

$$= (x-1)\left(\frac{x^2+x}{2}\right)^2$$

$$= \frac{1}{4}\left[(x-1)(x^4+x^2+2x^3)\right]$$

$$B_2(x) = \frac{1}{4}\left[x^5 - x^4 + x^3 - x^2 + 2x^4 - 2x^3\right]$$

$$\boxed{B_2(x) = \frac{1}{4}\left[x^5 + x^4 - x^3 - x^2\right]}$$

$$P(x) = A_0(x)f(x_0) + A_1(x)f(x_1) + A_2(x)f(x_2) +$$
$$B_0(x)f'(x_0) + B_1(x)f'(x_1) + B_2(x)f'(x_2).$$

$$= \left[\frac{1}{4}\left(3x^5 - 2x^4 - 5x^3 + 4x^2\right)\right](1) + (x^4 - 2x^2 + 1)(1)$$
$$+ \left[\frac{1}{4}\left(-3x^5 - 2x^4 - 5x^3 + 4x^2\right)\right](3) + \left[\frac{1}{4}\left(x^5 - x^4 - x^3 + x^2\right)\right](-5)$$
$$+ (x^5 - 2x^3 + x)(1) + \left[\frac{1}{4}\left(x^5 + x^4 - x^3 - x^2\right)\right](7).$$

$$= \frac{1}{4}\Big[3x^5 - 2x^4 - 5x^3 + 4x^2 + 4x^4 - 8x^2 + 4$$
$$-9x^5 - 6x^4 + 15x^3 + 12x^2 - 5x^5 + 5x^4 + 5x^3 - 5x^2$$
$$+ 4x^5 - 8x^3 + 4x + 7x^5 + 7x^4 - 7x^3 - 7x^2\Big]$$

$$= \frac{1}{4}\left[0x^5 + 8x^4 + 0x^3 - 4x^2 + 4x + 4\right]$$

$$P(x) = \frac{1}{4}\left(8x^4 - 4x^2 + 4x + 4 \right)$$

$$= \frac{4}{4}\left(2x^4 - x^2 + x + 1 \right)$$

$$\boxed{P(x) = 2x^4 - x^2 + x + 1}$$

$$f(-0.5) = P(-0.5) = 0.375$$

$$f(0.5) = P(0.5) = 1.375$$

Given the following values of $f(x)$ and $f'(x)$

x	$f(x)$	$f'(x)$
0	4	-5
1	-6	-14
2	-22	-17

Interpolates $f(x)$ at $x = 0.5$ and 1.5.

Piece wise Interpolation.

Piecewise Linear Interpolation :

The interpolating polynomial is the piecewise linear polynomial. Using the Lagrange interpolation. We have for $x \in [x_{i-1}, x_i]$.

The piecewise linear interpolation,

$$P_{i,1}(x) = \frac{x - x_1}{x_{i-1} - x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i),$$

$$i = 1, 2, \ldots n.$$

Piece wise Cubic Interpolation :

Piece wise Cubic Interpolation using Hermite type data.

Let the Hermite type of data be given on each Sub interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots n$.

$$P_{i,3}(x_{i-1}) = f_{i-1} \quad ; \quad P_{i,3}(x_i) = f_i$$

$$P'_{i,3}(x_{i-1}) = f'_{i-1} \quad ; \quad P'_{i,3}(x_i) = f'_i$$

Then we construct a cubic polynomial $P_{i,3}(x)$ on each of the Sub interval the polynomial thus obtained is Called piecewise Cubic Hermite interpolating polynomial.

$$\boxed{P_{i,3}(x) = A_{i-1}(x) f_{i-1} + A_i(x) f_i + B_{i-1}(x) f'_{i-1} + B_i(x) f'_i \cdot \rightarrow (1)}$$

Where $x_{i-1} \le x \le x_i$ and,

$$A_{i-1}(x) = \frac{(x - x_i)^2}{(x_{i-1} - x_i)^2}\left[1 + \frac{2(x_{i-1} - x)}{(x_{i-1} - x_i)}\right]$$

$$A_i(x) = \frac{(x - x_{i-1})^2}{(x_{i-1} - x_i)^2}\left[1 + \frac{2(x - x_{i-1})}{x_{i-1} - x_i}\right]$$

$$B_{i-1}(x) = \frac{(x - x_{i-1})(x - x_i)^2}{(x_{i-1} - x_i)^2}$$

$$B_i(x) = \frac{(x - x_i)(x - x_{i-1})^2}{(x_{i-1} - x_i)^2}$$

of the interval $[x_i, x_{i+1}]$, we have.

$$\boxed{P_{i+1,3} = A^*_i(x) f_i + A^*_{i+1} f_{i+1} + B^*_i(x) f'_i + B^*_{i+1}(x) f'_{i+1} \cdot \rightarrow (2)}$$

where,

$$A^*_i(x) = \frac{(x - x_{i+1})^2}{(x_i - x_{i+1})^2}\left[1 + \frac{2(x_i - x)}{x_i - x_{i+1}}\right]$$

$$A^*_{i+1}(x) = \frac{(x-x_i)^2}{(x_i-x_{i+1})^2}\left[1+\frac{2(x-x_{i+1})}{x_i-x_{i+1}}\right]$$

$$B^*_{i+1}(x) = \frac{(x-x_{i+1})(x-x_i)^2}{(x_i-x_{i+1})^2}$$

$$B^*_i(x) = \frac{(x-x_i)(x-x_{i+1})^2}{(x_i-x_{i+1})^2}.$$

1. Using the following values of $f(x)$ and $f'(x)$.

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	1
1	3	7

Estimate the values of $f(-0.5)$ and $f(0.5)$ using piecewise cubic Hermite interpolation.

Soln:

Here,

$x_{i-1} = -1$; $x_i = 0$; $x_{i+1} = 1$

$f_{i-1} = 1$; $f_i = 1$; $f_{i+1} = 3$

$f'_{i-1} = -5$; $f'_i = 1$; $f'_{i+1} = 7$

Since, $x = -0.5 \in [x_{i-1}, x_i]$, piecewise Cubic

Hermite interpolation becomes,

$$\boxed{P_{i,3}(x) = A_{i-1}(x) f_{i-1} + A_i(x) f_i + B_{i-1}(x) f'_{i-1} + B_i(x) f'_i .}$$

$$A_{i-1}(x) = \frac{(x-0)^2}{(-1-0)^2}\left[1 + \frac{2(-1-x)}{(-1-0)}\right]$$

$$= x^2\left[1 + \frac{2(-1-x)}{-1}\right]$$

$$= x^2\left[1 + 2(1+x)\right]$$

$$\boxed{A_{i-1}(x) f_{i-1} = x^2\left[1+2(1+x)\right](1)}$$

$$A_i(x) = \frac{(x+1)^2}{(-1)^2}\left[1 + \frac{2(x-0)}{-1}\right]$$

$$= (x+1)^2\left[\frac{-(1-2x)}{-1}\right]$$

$$= (x+1)^2(1-2x)$$

$$\boxed{A_i(x) f_i = (x+1)^2(1-2x)(1)}$$

$$B_{i-1}(x) = \frac{(x+1)(x-0)^2}{(-1-0)^2}$$

$$= (x+1)(x)^2$$

$$\boxed{B_{i-1}(x)\ f'_{i-1} = (x+1)(x)^2(-5)}$$

$$B_i(x) = \frac{(x-x_i)(x-x_{i-1})^2}{(x_{i-1}-x_i)^2}$$

$$= \frac{(x-0)(x+1)^2}{(-1-0)^2}$$

$$= \frac{(x)(x+1)^2}{1}$$

$$\boxed{B_i(x) f'_i = (x)(x+1)^2(1)}$$

$$P_{i,3}(x) = x^2\left[1+2(1+x)\right] + (x+1)^2\left[1-2x\right]$$

$$+ (x+1)(x^2)(-5) + (x)(x+1)^2.$$

$$= x^2 + 2x^2 + 2x^3 + x^2 + 1 + 2x - 2x^3$$

$$-2x - 4x^2 - 5x^3 - 5x^2 + x^3 + x + 2x^2.$$

$$= -4x^3 - 3x^2 + x + 1$$

we get,

$$f(-0.5) \approx 0.25.$$

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Since, $x = 0.5 \in [x_i, x_{i+1}]$

Piecewise Cubic Hermite interpolation becomes

$$\boxed{P_{i+1,3}(x) = A_i^*(x) f_i + A_{i+1}^* f_{i+1} + B_i^*(x) f_i' + B_{i+1}(x) f_{i+1}'}$$

$$A_i^*(x) = \frac{(x - x_{i+1})^2}{(x_i - x_{i+1})^2}\left[1 + \frac{2(x_i - x)}{x_i - x_{i+1}}\right]$$

$$= \frac{(x-1)^2}{(-1)^2}\left[1 + \frac{2(0-x)}{-1}\right]$$

$$= (x-1)^2 (1-2x).$$

$$\boxed{A_i^*(x) f_i = (x-1)^2 (1-2x)(1)}$$

$$A_{i+1}^*(x) = \frac{(x - x_i)^2}{(x_i - x_{i+1})^2}\left[1 + \frac{2(x - x_{i+1})}{x_i - x_{i+1}}\right]$$

$$= \frac{(x-0)^2}{(-1)^2}\left[1 + \frac{2(x-1)}{-1}\right]$$

$$= x^2\left[1 - 2(1-x)\right]$$

$$\boxed{A_{i+1}^*(x) f_{i+1} = x^2\left[1 - 2(1-x)\right](3)}$$

$$B^*_{i+1}(x) = \frac{(x - x_{i+1})(x - x_i)^2}{(x_i - x_{i+1})^2}$$

$$= \frac{(x-1)(x-0)^2}{(0-1)^2}$$

$$= (x-1)(x)^2$$

$$\boxed{B^*_{i+1}(x)\, f_i = (x-1)(x)^2(7)}$$

$$B^*_i(x) = \frac{(x - x_i)(x - x_{i+1})^2}{(x_i - x_{i+1})^2}$$

$$= \frac{(x-0)(x-1)}{(0-1)^2}$$

$$\boxed{B^*_i(x)\, f'_{i+1} = x(x-1)^2\ (1)}$$

$$P_{i+1,3}(x) = (x-1)^2[1-2x] + x^2\left[1 - 2(1-x)\right](8)$$

$$+ (x-1)(x)^2(7) + x(x-1)^2(1)$$

$$= x^2 + 1 - 2x + 2x^3 + 2x - 4x^2 + 3x^2$$

$$- 6x^3 + 6x^2 + x^5 + x - 2x^2 - 7x^3 - 7x^2$$

$$= 4x^3 - 3x^2 + x + 1$$

Hence,

$$f(0.5) \approx P_3(0.5) \Rightarrow 1.25.$$

Spline interpolation

The second order derivative of $P_3(x)$ exists but may not be continuous at the knots. It is possible to determine the $m_0, m_1, \ldots m_n$ in such a way that the resulting piecewise Cublic interpolation is twice continuously differentiable. such an interpolation is called cubic spline interpolation.

A cubic spline statistifies the following conditions :

(i) $f(x)$ is linear polynomial outside the interval $[x_0, x_n]$.

(ii) on each subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$, $f(x)$ is a third degree polynomial.

(iii) $f(x)$, $f'(x)$ and $f''(x)$ are continuous on (x_0, x_n) cubic spline interpolation formula is given by

$$f(x) = \frac{(x_{i+1}-x)^3}{6h} M_i + \frac{(x-x_i)^3}{6h} M_{i+1}$$

$$+ \frac{x_{i+1}-x}{h}\left(y_i - \frac{h^2}{6} M_i\right) + \frac{(x-x_i)}{h}\left[y_{i+1} - \frac{h^2}{6} M_{i+1}\right]$$

where, M_i and M_{i+1} are found out by solving the relations.

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}),$$

$i = 1, 2, 3, \ldots (n-1)$ and $M_0 = M_n = 0$.

problem obtain the cubic spline approximation for the function given in the tabular form

x	0	1	2	3
$y\ (f(x))$	1	2	33	244

Soln

Here, $h = (x_i - x_{i-1})$ $i = 0, 1, 2, 3$

$\therefore h = 1$

We have to find M_i values,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}),$$

$i = 1, 2$

$i = 1 \Rightarrow$

$$M_0 + 4M_1 + M_2 = \frac{6}{(1)^2}(y_0 - 2y_1 + y_2)$$

Given table,

$$x_0 = 0 \; ; \quad x_1 = 1 \; ; \quad x_2 = 2 \; ; \quad x_3 = 3$$

$$y_0 = 1 \; ; \quad y_1 = 2 \; ; \quad y_2 = 33 \; ; \quad y_3 = 244$$

$$M_0 + 4M_1 + M_2 = 6\,(1 - 2(2) + 33)$$

$$= 6(1 - 4 + 33)$$

$$M_0 + 4M_1 + M_2 = 180 \quad \text{———} \; (1)$$

$i = 2 \Rightarrow$

$$M_1 + 4M_2 + M_3 = \frac{6}{(1)^2}(y_1 - 2y_2 + y_3)$$

$$M_1 + 4M_2 + M_3 = 6\,(2 - 2\times 33 + 244)$$

$$= 6(2 - 66 + 244)$$

$$M_1 + 4M_2 + M_3 = 1080 \quad \text{———} \; (2)$$

equation (1) and (2) solving, if

$$M_0 = M_3 = 0$$

$$4M_1 + M_2 = 180$$

$$M_1 + 4M_2 = 1080$$

$$4M_1 + M_2 = 1080$$

$$\underset{(-)}{4M_1} \underset{(-)}{+ 16M_2} = \underset{(-)}{4320}$$

$$-15M_2 = -4140$$

$$\therefore \quad \boxed{M_2 = 276}$$

M_2 values substitude in (1)

$$0 + 4M_1 + 276 = 180$$

$$4M_1 = 180 - 276$$

$$4M_1 = -96$$

$$\boxed{M_1 = -24}$$

Now, cubic spline in $[x_i, x_{i+1}]$ is

$$f(x) = \frac{(x_{i+1} - x)^3}{6h} M_i + \frac{(x - x_i)^3}{6h} M_{i+1} +$$

$$\frac{x_{i+1} - x}{h} \left(y_i - \frac{h^2}{6} M_i\right) + \frac{(x - x_i)}{h} \left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right) \quad (3)$$

Taking $i = 0, 1, 2$, in eqn. (3) and using $M_0 = 0$, $M_1 = -24$, $M_2 = 276$, $M_3 = 0$.

If $i = 0$ in (3)

$$f(x) = \frac{(x_1 - x)^3}{6h} M_0 + \frac{(x - x_0)^3}{6h} M_1 +$$

$$\frac{(x_1 - x)}{1}\left(y_0 - \frac{1^2}{6} M_0\right) + \frac{(x - x_0)}{1}\left(y_1 - \frac{1}{6} M_1\right)$$

$$= \frac{(1-x)^3}{6}(0) + \frac{(x-0)^3}{6}(-24) + (1-x)\left(1-\frac{1}{6}(0)\right) + (x-0)\left(2-\frac{(-24)}{6}\right)$$

$$= \cancel{x^3} - x^3(-4) + 1 - x + (x \times 6)$$

$$= -4x^3 + 1 - x + 6x$$

$$f(x) = -4x^3 + 5x + 1 \quad ; \quad x \in [0,1]$$

If $i = 1$ in (3),

$$f(x) = \frac{(x_2-x)^3}{6} M_1 + \frac{(x-x_1)^3}{6} M_2 + \frac{(x_2-x)}{1}\left(y_1 - \frac{M_1}{6}\right) + \frac{(x-x_1)}{1}\left(y_2 - \frac{M_2}{6}\right) \quad \text{—(3)}$$

$$= \frac{(2-x)^3}{6} \cdot (-24) + \frac{(x-1)^3}{6}(276) + (2-x)\left(2 - \frac{(-24)}{6}\right) + (x-1)\left(2 - \frac{276}{6}\right)$$

$$= (8 - x^3 - 12x + 6x^2)(-4) + (x^3 - 1 - 3x^2 + 3x)(46) + 12 - 6x - 44x + 44$$

$$= (8 - x^3 - 12x + 6x^2)(-4) + 46x^3 - 46 - 138x^2 + 138x + 12 - 6x - 44x + 44$$

$$f(x) = 50x^3 - 162x^2 + 167x - 53 \; ; x \in [1,2]$$

if $i = 2$ in (3)

$$f(x) = \frac{(x_3 - x)^3}{6} \cdot M_2 + \frac{(x - x_2)^3}{6} M_3 +$$

$$\frac{(x_3 - x)}{1}\left(y_2 - \frac{M_2}{6}\right) + \frac{(x - x_2)}{1}\left(y_3 - \frac{M_3}{6}\right)$$

$$= \frac{(3-x)^3}{6} \times 276 + \frac{(x-2)^3}{6}(0) +$$

$$(3-x)\left(33 - \left(\frac{-276}{6}\right)\right) + (x-2)(244-0)$$

$$= (27 - x^3 - 27x + 9x^2)(46) - 39x +$$

$$13x + 244x - 488$$

$$= 1242 - 46x^3 - 1242x + 414x^2 + 13x$$

$$- 39x + 244x - 488$$

$$f(x) = -46x^3 + 414x^2 - 985x + 715$$

$$\therefore f(x) = \cancel{50x^3 - 162x^2}$$

$$-46x^3 + 414x^2 - 985x + 715$$

$$x \in [2,3]$$

Thus, the cubic splines in the corresponding intervals become,

Interval	Cubic spline
$[0, 1]$	$-4x^3+5x+1$
$[1, 2]$	$50x^3-162x^2+167x-53$
$[2, 3]$	$-46x^3+414x^2-985x+715$

BIVARIATE INTERPOLATION

Lagrange Bivariate Interpolation:

let $f(x,y)$ be defined at $(m+1)(n+1)$ distinct points (x_i, y_j), $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$. and denote $f(x_i, y_j)$ by $f_{i,j}$. We want to obtain a polynomial $p(x,y)$ of degree at most m in x and n in y, such that

$$p(x_i, y_j) = f_{i,j} \qquad \begin{matrix} i = 0,1,\ldots,m \\ j = 0,1,\ldots,n \end{matrix} \longrightarrow (1)$$

Using the Lagrange fundamental polynomials of a single Variable, we define

$$X_{m,i}(x) = \frac{w(x)}{(x-x_i)\, w'(x_i)}, \quad i = 0,1,\ldots,m$$

$$Y_{n,j}(y) = \frac{w^*(y)}{(y-y_j)\, w^{*\prime}(y_j)}, \quad j = 0,1,\ldots,n$$

where

$$w(x) = (x-x_0)(x-x_1)\cdots(x-x_m)$$

$$w^*(y) = (y-y_0)(y-y_1)\cdots(y-y_n).$$

Thus, the Lagrange Bivariate interpolation is

$$\boxed{P_{m,n}(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} X_{m,i}(x)\, Y_{n,j}(y)\, f_{i,j}}$$

Ex

The following data for a function $f(x,y)$ is given

$y \backslash x$	0	1	2
0	1	3	5
1	2	5	10

Find $f(1.5, 0.75)$.

Soln

$f(0,0) = 1$, $f(0,1) = 2$

$f(1,0) = 3$, $f(1,1) = 5$. $m = 2$,

$f(2,0) = 5$, $f(2,1) = 10$ $n = 1$

$$X_{m,i}(x) = \frac{w(x)}{(x-x_i)\,w'(x_i)}, \quad i = 0, 1, 2$$

where,

$$w(x) = (x-x_0)(x-x_1)\dots(x-x_m)$$

$$w(x) = (x-x_0)(x-x_1)(x-x_2)$$

$$= (x-0)(x-1)(x-2)$$

$$= x(x-1)(x-2)$$

$$w(x) = x^3 - 3x^2 + 2x$$

$$w'(x_i) = 3x^2 - 6x + 2$$

$$w'(x_0) = 3(0)^2 - 6(0) + 2 = 2$$

$$w'(x_1) = 3(1)^2 - 6(1) + 2 = -1$$

$$w'(x_2) = 3(2)^2 - 6(2) + 2$$

$$= 3(4) - 12 + 2 = 2$$

$i = 0$, $m = 2$

$$X_{2,0}(x) = \frac{w(x)}{(x-x_0)\,w'(x_0)}$$

$$= \frac{x^3 - 3x^2 + 2x}{(x-0)\cdot(2)}$$

$$= \frac{x^3 - 3x^2 + 2x}{2x}$$

$$= \frac{1}{2}\left[\frac{\cancel{x}(x^2 - 3x + 2)}{\cancel{x}}\right]$$

$$= \frac{x^2 - 3x + 2}{2}$$

$i=1,\ m=2$

$$X_{2,1}(x) = \frac{w(x)}{(x-x_1)\,w'(x_1)}$$

$$= \frac{x^3-3x^2+2x}{(x-1)(-1)}$$

$$= -\frac{x(x^2-3x+2)}{(x-1)}$$

$$\begin{array}{r|l} & 2 \\ -1 & -2 \\ \hline & -3 \end{array}$$

$$= -\frac{x\cancel{(x-1)}(x-2)}{\cancel{(x-1)}}$$

$$= -x^2+2x$$

$i=2,\ m=2$

$$X_{2,2}(x) = \frac{w(x)}{(x-x_2)\,w'(x_2)}$$

$$= \frac{x^3-3x^2+2x}{(x-2)(2)}$$

$$= \frac{x(x^2-3x+2)}{2(x-2)}$$

$$= \frac{x(x-1)\cancel{(x-2)}}{2\cancel{(x-2)}}$$

$$= \frac{x^2-x}{2}$$

$$y_{n,j}(y) = \frac{w^*(y)}{(y-y_j)\, w^{*\prime}(y_j)}, \quad j = 0, 1$$

where

$$w^*(y) = (y-y_0)(y-y_1)\cdots(y-y_n)$$

$$w^*(y) = (y-y_0)(y-y_1)$$

$$= (y-0)(y-1)$$

$$= y(y-1)$$

$$w^*(y) = y^2 - y$$

$$w^{*\prime}(y_j) = 2y - 1$$

$$w^{*\prime}(y_0) = 2(0) - 1 = -1$$

$$w^{*\prime}(y_1) = 2(1) - 1 = 1$$

$j = 0,\ n = 1$

$$y_{1,0}(y) = \frac{w^*(y)}{(y-y_0)\, w^{*\prime}(y_0)}$$

$$= \frac{y^2 - y}{(y-0)(-1)}$$

$$= \frac{\cancel{y}(y-1)}{-\cancel{y}}$$

$$= 1 - y$$

$j=1,\ n=1$

$$y_{1,1}(y) = \frac{w^*(y)}{(y-y_1)\,w^{*'}(y_1)}$$

$$= \frac{y^2-y}{(y-1)(1)}$$

$$= \frac{y\cancel{(y-1)}}{\cancel{(y-1)}}$$

$$= y$$

$$P(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} X_{m,i}(x)\, Y_{n,j}(y)\, f_{i,j}$$

$$= X_{2,0}(x)\left[y_{1,0}(y)\, f(0,0) + y_{1,1}(y)\, f(0,1)\right] +$$
$$X_{2,1}(x)\left[Y_{1,0}(y)\, f(1,0) + Y_{1,1}(y)\, f(1,1)\right] +$$
$$X_{2,2}(x)\left[Y_{1,0}(y)\, f(2,0) + Y_{1,1}(y)\, f(2,1)\right]$$

$$= \frac{(x^2-3x+2)}{2}\left[(1-y)(1) + y(2)\right] + (-x^2+2x)$$
$$\left[(1-y)(3) + y(5)\right] + \frac{x^2-x}{2}\left[(1-y)(5) + y(10)\right]$$

$$= \frac{x^2-3x+2}{2}\left[1-y+2y\right] + (-x^2+2x)\left[3-3y+5y\right] +$$
$$\frac{x^2-x}{2}\left[5-5y+10y\right]$$

$$= \frac{(x^2-3x+2)}{2}[1+y] + (-x^2+2x)[3+2y] + \frac{x^2-x}{2}[5+5y]$$

$$= \frac{1}{2}\left[x^2-3x+2+x^2y-3xy+2y\right] + \left[-3x^2+6x-2x^2y+4xy\right]$$

$$+ \frac{1}{2}\left[5x^2-5x+5x^2y-5xy\right]$$

$$= \frac{1}{2}\left[x^2-3x+2+x^2y-3xy+2y+5x^2-5x+5x^2y-5xy\right]$$
$$+ \left[-3x^2+6x-2x^2y+4xy\right]$$

$$= \frac{1}{2}\left[6x^2y+6x^2-8xy-8x+2y+2\right] + \left[-3x^2+6x-2x^2y+4xy\right]$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}\left[3x^2y+3x^2-4xy-4x+y+1\right] + \left[-3x^2+6x-2x^2y+4xy\right]$$

$$= 3x^2y + \cancel{3x^2} - \cancel{4xy} - 4x + y + 1 - \cancel{3x^2} + 6x - 2x^2y + \cancel{4xy}$$

$$= x^2y + 2x + y + 1$$

$$f(1.5, 0.75) = (1.5)^2(0.75) + 2(1.5) + 0.75 + 1$$

$$= 6.4375.$$

① The following data for a function $f(x,y)$ is given

$y \backslash x$	0	1	3
0	1	2	10
1	2	4	14
3	10	14	28

construct the Bivariate interpolating polynomial and find $f(0.5, 0.5)$

Soln

Given, $f(0,0) = 1$, $f(0,1) = 2$, $f(0,2) = 10$

$f(1,0) = 2$, $f(1,1) = 4$, $f(1,2) = 14$

$f(2,0) = 10$, $f(2,1) = 14$, $f(2,2) = 28$

$x_0 = 0$, $x_1 = 1$, $x_2 = 3$ $\quad m = 2$

$y_0 = 0$, $y_1 = 1$, $y_2 = 3$ $\quad h = 2$

$$X_{m,i}(x) = \frac{w(x)}{(x-x_i)\,w'(x_i)}$$

where

$w(x) = (x-x_0)(x-x_1)\cdots(x-x_m)$

$w(x) = (x-x_0)(x-x_1)(x-x_2)$

$= (x-0)(x-1)(x-3)$

$= x(x-1)(x-3)$

$w(x) = x^3 - 4x^2 + 3x$

$w'(x_i) = 3x^2 - 8x + 3$

$w'(x_0) = 3(0)^2 - 8(0) + 3 = 3$

$w'(x_1) = 3(1)^2 - 8(1) + 3 = -2$

$w'(x_2) = 3(3)^2 - 8(3) + 3 = 6$

Put

$i = 0, m = 2$

$$X_{2,0}(x) = \frac{w(x)}{(x - x_0)\, w'(x_0)}$$

$$= \frac{x^3 - 4x^2 + 3x}{(x - 0)(3)}$$

$$= \frac{\cancel{x}(x^2 - 4x + 3)}{3\cancel{x}}$$

$$= \frac{x^2 - 4x + 3}{3}$$

put $i = 1, m = 2$

$$X_{2,1}(x) = \frac{w(x)}{(x - x_1)\, w'(x_1)}$$

$$= \frac{x^3 - 4x^2 + 3x}{(x - 1)(-2)}$$

$$= \frac{x(x^2 - 4x + 3)}{-2(x - 1)}$$

$$= \frac{x\cancel{(x-1)}(x - 3)}{-2\cancel{(x-1)}}$$

$$= \frac{-x^2 + 3x}{2}$$

3

-1 | -3

-4

put $i=2, m=2$

$$X_{2,2}(x) = \frac{w(x)}{(x-x_2)\,w'(x_2)}$$

$$= \frac{x^3-4x^2+3x}{(x-3)(6)}$$

$$= \frac{x(x^2-4x+3)}{6(x-3)}$$

$$= \frac{x(x-1)\cancel{(x-3)}}{6\cancel{(x-3)}}$$

$$= \frac{x^2-x}{6}$$

$$Y_{n,j}(y) = \frac{w^*(y)}{(y-y_j)\,w^{*\prime}(y_j)}$$

where

$$w^*(y) = (y-y_0)(y-y_1)\cdots(y-y_n)$$

$$w^*(y) = (y-y_0)(y-y_1)(y-y_2)$$

$$=(y-0)(y-1)(y-3)$$

$$=y(y-1)(y-3)$$

$$w^*(y_j) = y^3-4y^2+3y$$

$$w^{*\prime}(y_j) = 3y^2-8y+3$$

$$w^{*\prime}(y_0) = 3(0)^2-8(0)+3 = 3$$

$$w^{*\prime}(y_1) = 3(1)^2-8(1)+3 = -2$$

$$w^{*\prime}(y_2) = 3(3)^2-8(3)+3 = 6$$

put $j=0$, $n=2$

$$Y_{2,0}(y) = \frac{w^*(y)}{(y-y_0)\,w^{*\prime}(y_0)}$$

$$= \frac{y^3-4y^2+3y}{(y-0)(3)}$$

$$= \frac{\cancel{y}(y^2-4y+3)}{3\cancel{y}}$$

$$= \frac{y^2-4y+3}{3}$$

put $j=1$, $n=2$

$$Y_{2,1}(y) = \frac{w^*(y)}{(y-y_1)\,w^{*\prime}(y_1)}$$

$$= \frac{y^3-4y^2+3y}{(y-1)(-2)}$$

$$= \frac{y(y^2-4y+3)}{-2(y-1)}$$

$$= \frac{y\,\cancel{(y-1)}(y-3)}{-2\cancel{(y-1)}}$$

$$= \frac{-y^2+3y}{2}$$

put $j=2$, $n=2$

$$Y_{2,2}(y) = \frac{w^*(y)}{(y-y_2)\,w^{*\prime}(y_2)}$$

$$= \frac{y^3-4y^2+3y}{(y-3)(6)} \Rightarrow \frac{y(y^2-4y+3)}{6(y-3)}$$

$$= \frac{y(y-1)\cancel{(y-3)}}{6\cancel{(y-3)}} = \frac{y^2-y}{6}$$

$$p(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} X_{m,i}(x)\, Y_{n,j}(y)\, f_{i,j}$$

$$= \left[X_{2,0}(x) + X_{2,1}(x) + X_{2,2}(x)\right]\left[Y_{2,0}(y) + Y_{2,1}(y) + Y_{2,2}(y)\right] \times f_{ij}$$

$$= X_{2,0}(x)\left[Y_{2,0}(y) f(0,0) + Y_{2,1}(y) f(0,1) + Y_{2,2}(y) f(0,2)\right] +$$

$$X_{2,1}(x)\left[Y_{2,0}(y) f(1,0) + Y_{2,1}(y) f(1,1) + Y_{2,2}(y) f(1,2)\right] +$$

$$X_{2,2}(x)\left[Y_{2,0}(y) f(2,0) + Y_{2,1}(y) f(2,1) + Y_{2,2}(y) f(2,2)\right]$$

$$= \frac{x^2-4x+3}{3}\left[\frac{y^2-4y+3}{3}(1) + \left(\frac{-y^2+3y}{\cancel{2}}\right)(\cancel{2}) + \frac{y^2-y}{\cancel{6}_3}(\cancel{10}^5)\right] +$$

$$\left(\frac{-x^2+3x}{2}\right)\left[\frac{y^2-4y+3}{3}(2) + \left(\frac{-y^2+3y}{\cancel{2}}\right)(\cancel{4}^2) + \frac{y^2-y}{\cancel{6}_3}(\cancel{14}^7)\right] +$$

$$\frac{x^2-x}{6}\left[\frac{y^2-4y+3}{3}(10) + \left(\frac{-y^2+3y}{\cancel{2}}\right)(\cancel{14}^7) + \frac{y^2-y}{\cancel{6}_3}(\cancel{28}^{14})\right]$$

$$= \frac{x^2-4x+3}{3}\left[\frac{y^2-4y+3}{3} - y^2+3y + \frac{5y^2-5y}{3}\right] +$$

$$\left(\frac{-x^2+3x}{2}\right)\left[\frac{2y^2-8y+6}{3} - 2y^2+6y + \frac{7y^2-7y}{3}\right] +$$

$$\frac{x^2-x}{6}\left[\frac{10y^2-40y+30}{3} - 7y^2+21y + \frac{14y^2-14y}{3}\right]$$

$$= \frac{x^2-4x+3}{3}\left[\frac{y^2-4y+3-3y^2+9y+5y^2-5y}{3}\right]+$$

$$\left(\frac{-x^2+3x}{2}\right)\left[\frac{2y^2-8y+6-6y^2+18y+7y^2-7y}{3}\right]+$$

$$\frac{x^2-x}{6}\left[\frac{10y^2-40y+30-21y^2+63y+14y^2-14y}{3}\right]$$

$$= \frac{x^2-4x+3}{3}\left[\frac{3y^2+3}{3}\right]+\left(\frac{-x^2+3x}{2}\right)\left[\frac{3y^2+3y+6}{3}\right]+$$

$$\frac{x^2-x}{6}\left[\frac{3y^2+9y+30}{3}\right]$$

$$= \frac{x^2-4x+3}{3}\left[\frac{\cancel{3}(y^2+1)}{\cancel{3}}\right]+\left(\frac{-x^2+3x}{2}\right)\left[\frac{\cancel{3}(y^2+y+2)}{\cancel{3}}\right]+$$

$$\frac{x^2-x}{6}\left[\frac{\cancel{3}(y^2+3y+10)}{\cancel{3}}\right]$$

$$= \frac{x^2-4x+3}{3}(y^2+1)+\left(\frac{-x^2+3x}{2}\right)(y^2+y+2)+$$

$$\frac{x^2-x}{6}(y^2+3y+10)$$

$$= \frac{x^2y^2-4xy^2+3y^2+x^2-4x+3}{3}+\left[\frac{-x^2y^2-x^2y-2x^2+3xy^2+3xy+6x}{2}\right.$$

$$+\frac{x^2y^2+3x^2y+10x^2-xy^2-3xy-10x}{6}$$

$$= \frac{\begin{array}{l}2x^2y^2-8xy^2+6y^2+2x^2-8x+6-3x^2y^2-\cancel{3x^2y}-6x^2\\ +9xy^2+9xy+18x+x^2y^2+\cancel{3x^2y}+10x^2-xy^2-3xy\\ -10x\end{array}}{6}$$

$$= \frac{6y^2+6x^2+6xy+6}{6}$$

$$= \frac{\cancel{6}(y^2+x^2+xy+1)}{\cancel{6}}$$

$$\boxed{p(x, y) = y^2 + x^2 + xy + 1}$$

$f(0.5, 0.5)$

$$= (0.5)^2 + (0.5)^2 + (0.5)(0.5) + 1$$

$f(0.5, 0.5) = 1.75$

Newton's Bivariate Interpolation for Equispaced points :-

With equispaced points, with spacing h in x and k in y, we define

$$\Delta x\, f(x,y) = f(x+h, y) - f(x,y)$$
$$= (E_x - 1)\, f(x,y)$$

$$\Delta y\, f(x,y) = f(x, y+k) - f(x,y)$$
$$= (E_y - 1)\, f(x,y)$$

$$\Delta xx\, f(x,y) = \Delta x\, f(x+h, y) - \Delta x\, f(x,y)$$
$$= (E_x - 1)^2\, f(x,y)$$

$$\Delta yy\, f(x,y) = \Delta y\, f(x, y+k) - \Delta y\, f(x,y)$$
$$= (E_y - 1)^2\, f(x,y)$$

$$\Delta xy\, f(x,y) = \Delta x\, f(x, y+k) - \Delta x\, f(x,y)$$
$$= \Delta y\, f(x+h, y) - \Delta y\, f(x,y)$$
$$= (E_x - 1)(E_y - 1)\, f(x,y) \quad \text{and so on.}$$

Now, $f(x_0 + mh, y_0 + nk) = E_x^m E_y^n\, f(x_0, y_0)$

$$= (1 + \Delta x)^m (1 + \Delta y)^n f(x_0, y_0)$$

$$= \left[1 + \binom{m}{1}\Delta x + \binom{m}{2}\Delta xx + \cdots\right]$$

$$\left[1 + \binom{n}{1}\Delta y + \binom{n}{2}\Delta yy + \cdots\right] f(x_0, y_0)$$

$$= \left[1 + \binom{m}{1}\Delta x + \binom{n}{1}\Delta y + \binom{m}{2}\Delta xx + \cdots + \binom{m}{1}\binom{n}{1}\Delta xx + \binom{n}{2}\Delta yy + \cdots\right] f(x_0, y_0)$$

This gives the interpolating polynomial

$$p(x,y) = f(x_0, y_0) + \left[\frac{1}{h}(x-x_0)\Delta x + \frac{1}{k}(y-y_0)\Delta y\right] f(x_0,y_0)$$
$$+ \frac{1}{2!}\left[\frac{1}{h^2}(x-x_0)(x-x_1)\Delta xx + \frac{2}{hk}(x-x_0)(y-y_0)\Delta xy + \frac{1}{k^2}(y-y_0)(y-y_1)\Delta yy\right] f(x_0,y_0) + \cdots$$

and is called the Newton's bivariate interpolating polynomial, for equispaced points.

Example :-

The following data for a function $f(x,y)$ is given:

$y \backslash x$	0	1
0	1	1.414214
1	1.732051	2

find $f(0.25, 0.75)$, using linear interpolation.

Soln:-

$f(0,0) = 1$, $f(0,1) = 1.732051$

$f(1,0) = 1.414214$, $f(1,1) = 2$

$h = k = 1$ and $m = 1$, $n = 1$

The linear interpolation is given by

$$p(x,y) = f(x_0,y_0) + \frac{1}{h}(x-x_0)\,\Delta x\, f(x_0,y_0) + \frac{1}{k}(y-y_0)\,\Delta y\, f(x_0,y_0)$$

$$\Delta x\, f(x_0,y_0) = f(x_0+h, y_0) - f(x_0,y_0)$$
$$= f(0+1, 0) - f(0,0)$$
$$= f(1,0) - f(0,0)$$
$$= 1.414214 - 1 \Rightarrow 0.414214$$

$$\Delta y\, f(x_0,y_0) = f(x_0, y_0+k) - f(x_0,y_0)$$
$$\cancel{= f(0+1,0) - f(0,0)}$$
$$\cancel{= f(1,0) - f(0,0)}$$
$$= f(0, 0+1) - f(0,0)$$
$$= f(0,1) - f(0,0)$$
$$= 1.732051 - 1$$
$$= 0.732051$$

$x = 0.25, \quad y = 0.75.$

$$p(x,y) = f(x_0,y_0) + \frac{1}{h}(x-x_0)\Delta x\, f(x_0,y_0) + \frac{1}{k}(y+y_0)\,\Delta y\, f(x_0,y_0)$$

$$p(0.25, 0.75) = f(0,0) + \frac{1}{1}(x-0)(0.414214) + \frac{1}{1}(y-0)(0.732051)$$
$$= 1 + 0.414214\,x + 0.732051\,y$$
$$= 1 + 0.414214(0.25) + 0.732051(0.75)$$
$$= 1 + 0.1035535 + 0.54908825$$

$$\boxed{p(0.25, 0.75) = 1.65259175}$$

Ex:-

The function $f(x, y)$ is given table

$y \backslash x$	0	1	2
0	-1	4	2
1	2	0	-2
2	3	4	3

for the values construct the newton's bivariate polynomial and find the approximate values of $f(1.25, 0.75)$.

$$\boxed{\begin{aligned} h &= x_1 - x_0 \\ k &= y_j - y_0 \end{aligned}}$$

Approximation :

THEOREM: (WEIERSTRSS).

If the function $f(z)$ is Contenuous on a finite interval $[a,b]$ then given any $e>0$, their exist an $= n(e)$ and a polynomial $p(x)$ of degree n such that $|f(x) - p(x)| < e \;\forall\; x \in [a,b]$.

Pf.

To determine an approximation to $f(x)$ we assume an expression of the form,

$$f(x) \simeq p(x, c_0, c_1, \ldots, c_n).$$

$$= c_0 \phi_0(x) + c_1 \phi_1(x) + \cdots + c_n \phi_n(x) \quad \text{--(1)}.$$

where $\phi_i(x)$, $i = 0, 1, \ldots, n$ are n appropriately, chosen linearly independent function and $c_i(i=0,1,2,..n)$ are parameters to be determind. The function $\phi_i(x)$ are also called co-ordinate function be are usually taken as $\phi_i(x) = x^i$, $i = 0, 1, \ldots, n$. for polynomial approximation. The error of approximation is defined as

$E(f:c) = \| f(x) - (c_0 \phi_0(x) + \dots + c_n \phi_n(x)) \|$.

where $\|.\|$ is a well defined norm. This error is as small as possible in some sense. By using different norms.

we obtain different types of approximation. once a criterian [or a particular norm] is fixed. The function (out of a class of given functions). which makes this error smallest according to this criterian is called the best approximation.

Thus the minimization of error solves the problem of best approximation. The most commonly used norms are.

Discrete data:

l^p norm:

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad p \geq 1. \quad -①$$

where $x = \{x_i\}$ is a sequence of real (or) Complex number.

Euclidean norm:

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \quad -(2)$$

For $p=2$ and is also called sequence norm and written $\|x\|_2$.

Uniform norm:

$$\|x\| = \max_{1 \le j \le n} |x_j| \quad -(3)$$

which is a particular case for $p=\infty$.

Continuous Data:

If the function $f(x)$ is continuous on $[a,b]$ and $|f(x)|^p$ is integrable on $[a,b]$ then,

$$\|f\| = \left(\int_a^b w(x) \, |f(x)|^p \, dx \right)^{1/p} \quad p \ge 1 \quad -(4)$$

where $w(x) > 0$ is the weight function is called the L^p norm for $p=2$.

r

we have the Euclidean (or) Square norm:

$$\|f\| = \left(\int_a^b W(x)\, f^2(x)\, dx \right)^{1/2} \quad - (5).$$

For $p = \infty$, we have uniform norm.

$$\|f\| = \max_{a \le x \le b} |f(x)| \quad - (6).$$

When we use the euclidean norm. we obtain the least square approximation.

LEAST SQUARE APPROXIMATION:

Least square approximation are most commonly used approximations for approximating a function $f(x)$ which may be given to tabular form (or) known explicity over a given Interval.

The best approximation in the least square sense is defined as that for which the constants c_i, $i = 0, 1, \ldots, n$ are determined so that the aggregate of $W(x)\, E^2$ over a given domain D is as small as possible, where $W(x) > 0$ is the weight function. For functions whose values are given at $N+1$ points $x_0, x_1, \ldots, x_N$, we have,

$$I(c_0, c_1, \ldots, c_n) = \sum_{k=0}^{n} W(x_k)\left[f(x_k) - \sum_{i=0}^{n} c_i \phi_i(x_k)\right]^2$$

$$= \text{minimum} \qquad (1).$$

For functions which are continuous on $[a,b]$ and are given explicitly.

we have.

$$I(c_0, c_1, \ldots, c_n) = \int_a^b W(x)\left[f(x) - \sum_{i=0}^{n} c_i \phi_i(x)\right]^2 dx.$$

$$= \text{minimum}. \qquad -(2).$$

The co-ordinate functions $\phi_i(x)$ are usually chosen as,

$$\phi_i(x) = x^i \quad , \quad i = 0, 1, \ldots, n.$$

and $W(x) = 1$. The necessary conditions (1) or (2) to have a minimum value is that,

$$\frac{\partial I}{\partial c_i} = 0, \quad i = 0, 1, \ldots, n.$$

This gives a system of $n+1$ linear eqs. in $n+1$ unknowns $c_0, c_1, \ldots, c_n$. These equations are called normal equations. The normal equations for ① & ② become, respectively,

$$\sum_{k=0}^{N} W(x_k)\left[f(x_k) - \sum_{i=0}^{n} c_i \phi_i(x_k)\right]\phi_j(x_k) = 0 \quad j = 0(1)n \qquad -③$$

and

$$\int_a^b W(x)\left[f(x) - \sum_{i=0}^{n} c_i \phi_i(x)\right]\phi_j(x)\,dx = 0, \quad j = 0(1)n \qquad -④.$$

Example 4.13

Obtain a linear polynomial approximation to the function $f(x) = x^3$ on the interval $[0,1]$ using the least square approximation with $W(x) = 1$.

Consider a linear polynomial

$$P(x) = a_0 x + a_1.$$

where a_0 and a_1 are arbitrary parameters.

$$I(a_0, a_1, \dots, a_n) = \int_a^b w(x) \left[f(x) - \sum_{i=0}^{n} a_i \phi_i(x) \right]^2 dx. \quad -①.$$

$$I(a_0, a_1) = \int_a^b w(x) \left[f(x) - P(x) \right]^2 dx.$$

$$I(a_0, a_1). = \int_a^b \left[x^3 - (a_0 x + a_1) \right]^2 dx.$$

$$= \text{minimum}.$$

$$= \int_0^1 \left[(x^3)^2 + (a_0 x + a_1)^2 - 2\, x^3 (a_0 x + a_1) \right] dx.$$

$$= \int_0^1 \left[x^6 + a_0 x^2 + a_1^2 + 2 a_0 a_1 x - 2(a_0 x^4 + a_1 x^3) \right] dx.$$

$$= \int_0^1 \left(x^6 dx + a_0 x^2 dx + a_1^2 dx + 2 a_0 a_1 x\, dx - 2 (a_0 x^4 dx + a_1 x^3 dx) \right)$$

$$= \left(\frac{x^7}{7} + \frac{a_0^2 x^3}{3} + a_1^2 x + \frac{\cancel{2} a_0 a_1 x^2}{\cancel{2}} - 2\left(\frac{a_0 x^5}{5} + \frac{a_1 x^4}{4} \right) \right)_0^1$$

$$= \left(\frac{1}{7} + \frac{a_0^2}{3} + a_1^2 + a_0 a_1 - 2\left(\frac{a_0}{5} + \frac{a_1}{4} \right) \right) - (0).$$

$$I(a_0, a_1) = \frac{1}{7} - 2\left(\frac{a_0}{5} + \frac{a_1}{4} \right) + \frac{a_0^2}{3} + a_0 a_1 + a_1^2.$$

$$= \text{minimum}.$$

The necessary condition for $I(a_0, a_1)$ to be minimum are given by,

$$I(a_0, a_1) = \frac{1}{7} - 2\left(\frac{a_0}{5} + \frac{a_1}{4}\right) + \frac{a_0^2}{3} + a_0 a_1 + a_1^2.$$

$$\frac{\partial I}{\partial a_0} = -\frac{2}{5} + \frac{2}{3} a_0 + a_1. \quad -②.$$

$$I(a_0, a_1) = \frac{1}{7} - 2\left(\frac{a_0}{5} + \frac{a_1}{4}\right) + \frac{a_0^2}{3} + a_0 a_1 + a_1^2.$$

$$\frac{\partial I}{\partial a_1} = -\frac{2}{4} + a_0 + 2a_1 = 0$$

$$\frac{\partial I}{\partial a_0.} = -\frac{1}{2} + a_0 + 2a_1. = 0. \quad -③.$$

which gives,

$$\frac{2}{3} a_0 + a_1 = \frac{2}{5} \quad -②.$$

$$a_0 + 2a_1 = \frac{1}{2} \quad -④.$$

From eqn ② we get, (Substitution method).

$$\frac{2}{3} a_0 + a_1 = \frac{2}{5}.$$

$$\boxed{a_1 = \frac{2}{5} - \frac{2}{3} a_0} \quad -⑤.$$

eqn ⑤ Subs in ④.

$$a_0 + 2\left(\frac{2}{5} - \frac{2}{3}a_0\right) = \frac{1}{2}.$$

$$a_0 + \frac{4}{5} - \frac{4}{3} a_0 = \frac{1}{2}.$$

$$\frac{4}{5} + a_0 - \frac{4}{3} a_0 = \frac{1}{2}.$$

$$\frac{4}{5} + \left(1 - \frac{4}{3}\right) a_0 = \frac{1}{2}.$$

$$-\frac{1}{3} a_0 = \frac{1}{2} - \frac{4}{5}. \quad \Rightarrow -\frac{1}{3} a_0 = \frac{5-8}{10}$$

$$-\frac{1}{3} a_0 = -\frac{3}{10}.$$

$$a_0 = \frac{3}{10} \times 3.$$

$$\boxed{a_0 = \frac{9}{10}}$$

a_0 value Subs in ③.

$$\frac{2}{3}\left(\frac{9}{10}\right) + a_1 = \frac{2}{5}.$$

$$\frac{18}{30} + a_1 = \frac{2}{5}.$$

$$a_1 = \frac{2}{5} - \frac{18}{30}. \Rightarrow \frac{60-90}{150}.$$

$$a_1 = -\frac{30}{150}$$

$$\boxed{a_1 = -\frac{1}{5}}$$

$a_0 = \frac{9}{10}$ and $a_1 = -\frac{1}{5}$. Thus the first degree least Square Approxn. x^3 on $[0,1]$

$$P(x) = a_0 x + a_1.$$

$$= \left(\frac{9}{10}\right) x + \left(-\frac{1}{5}\right).$$

$$= \left(\frac{9}{10}\right) x - \frac{1 \times 2}{5 \times 2}.$$

$$\boxed{P(x) = \frac{9x-2}{10}}$$

Example 4.14:

Obtain the least Square polynomial Approx. of degree one and two for $f(x) = x^{1/2}$ on $[0,1]$.

Soln

For $n=1$, we have,

$$I(C_0, \dots, C_n) = \int_a^b w(x)\left[f(x) - \sum_{i=0}^{n} c_i \phi_i(x)\right]^2 dx.$$

$= \text{minimum}$

The normal equations are,

$$\cancel{\frac{\partial I}{\partial C_0}} = I(C_0, C_1) = \int_0^1 \left(x^{1/2} - c_0 - c_1 x\right)^2 dx.$$

$= \text{minimum}.$

$$\frac{\partial I}{\partial C_0} = -2\int_0^1 (x^{1/2} - C_0 - C_1 x)^2\, dx.$$

$$= -2\int_0^1 (x^{1/2}\cancel{dx} - C_0 - C_1 x\, dx)$$

$$= -2\int_0^1 \left(\frac{x^{3/2}}{3/2} - C_0 - \frac{C_1 x^2}{2}\right)$$

$$= -2\cancel{\int_0^1}\left(\frac{2}{3}x^{3/2} - C_0 - \frac{C_1 x^2}{2}\right)_0^1.$$

$$= -2\left(\frac{2}{3} - C_0 - \frac{C_1}{2}\right) = 0.$$

$$\frac{\partial I}{\partial C_1} = -\int_0^1 2(x^{1/2} - C_0 - C_1 x)\, x\, dx.$$

$$= -2\int_0^1 (x^{3/2} - C_0 x - C_1 x^2)\, dx.$$

$$= -2\int_0^1 (x^{3/2} dx - C_0 x\, dx - C_1 x^2 dx)$$

$$= -2\int_0^1 \left(\frac{x^{5/2}}{5/2} - \frac{C_0 x^2}{2} - \frac{C_1 x^3}{3}\right)$$

$$= -2\left(2/_5\, x^{5/2} - \frac{C_0 x^2}{2} - \frac{C_1 x^3}{3}\right)_0^1$$

$$= -2\left(\frac{2}{5} - \frac{C_0}{2} - \frac{C_1}{3}\right) = 0.$$

which gives,

$$\frac{\partial I}{\partial C_0} \Rightarrow \frac{2}{3} - C_0 - \frac{C_1}{2} = 0 \Rightarrow \quad C_0 + \frac{C_1}{2} = \frac{2}{3}. \quad -(1).$$

$$\frac{\partial I}{\partial C_1} = \frac{2}{5} - \frac{C_0}{2} - \frac{C_1}{3} = 0$$

$$= \frac{C_0}{2} + \frac{C_1}{3} = \frac{2}{5} \quad -(2)$$

From 1 & 2 we get

$$2 \Rightarrow \quad \tfrac{1}{2}\cancel{C_0} + \frac{1}{3}C_1 = \frac{2}{5}$$

$$\tfrac{1}{2}\times 1 \Rightarrow \quad \tfrac{1}{2}C_0 + \tfrac{1}{4}C_1 = \frac{2}{6}$$

$$\left(\tfrac{1}{3}-\tfrac{1}{4}\right)C_1 = \left(\frac{2}{5}-\frac{2}{6}\right)$$

$$\left(\frac{4-3}{12}\right)C_1 = \left(\frac{12-10}{30}\right)$$

$$\tfrac{1}{12}C_1 = \frac{2}{30}$$

$$C_1 = \frac{1}{15}\times 12$$

$$\boxed{C_1 = \frac{4}{5}}$$

C_1, Substitute in eqn 1.

$$C_0 + \frac{1}{2}\left(\frac{4}{5}\right) = \frac{2}{3}.$$

$$C_0 + \frac{4}{10} = \frac{2}{3}.$$

$$C_0 = \frac{2}{3} - \frac{4}{10}. \Rightarrow \frac{20-12}{30} \Rightarrow \frac{8\,4}{30\,15}$$

$$\boxed{C_0 = \frac{4}{15}}$$

We obtain $C_0 = 4/15$ and $C_1 = 4/5$. Thus, the first degree least square approximation to $x^{1/2}$ on $[0,1]$ is

$$P(x) = C_0 + C_1 x$$

$$P(x) = \left(\frac{4}{15} + \frac{4}{5}x\right)$$

$$= \left(\frac{4}{15} + \frac{4\times 3}{5\times 3}x\right)$$

$$= \left(\frac{4+12x}{15}\right)$$

$$p(x) = \frac{4(1+3x)}{15}.$$

For $n=2$, we have.

$$I(c_0, c_1, c_2) = \int_0^1 (x^{1/2} - c_0 - c_1 x - c_2 x^2)^2 \, dx.$$

$$= \text{minimum}$$

The normal equations are,

$$\frac{\partial I}{\partial c_0} = -2\int_0^1 (x^{1/2} - c_0 - c_1 x - c_2 x^2)\, dx.$$

$$= -2\int_0^1 (x^{1/2} dx - c_0 dx - c_1 x\, dx - c_2 x^2 dx)$$

$$= -2\int_0^1 \left(\left(\frac{x^{3/2}}{3/2}\right) - c_0 x - \frac{c_1 x^2}{2} - \frac{c_2 x^3}{3}\right)$$

$$= -2\left(\frac{2}{3}x^{3/2} - c_0 x - \frac{c_1 x^2}{2} - \frac{c_2 x^3}{3}\right)_0^1$$

$$= -2\left(\frac{2}{3} - c_0 - \frac{c_1}{2} - \frac{c_2}{3}\right) \quad -(1)$$

$$\frac{\partial I}{\partial c_1} = -2\int_0^1 \left(x^{1/2} - c_0 - c_1 x - c_2 x^2\right) x\,dx$$

$$= -2\int_0^1 \left(x^{3/2} - c_0 x - c_1 x^2 - c_2 x^3\right) dx.$$

$$= -2\int_0^1 x^{3/2}\,dx - c_0 x\,dx - c_1 x^2\,dx - c_2 x^3\,dx$$

$$= -2\int_0^1 \left(\frac{x^{5/2}}{5/2} - \frac{c_0 x^2}{2} - \frac{c_1 x^3}{3} - \frac{c_2 x^4}{4}\right)$$

$$= -2\left(\frac{2}{5} x^{5/2} - \frac{c_0 x^2}{2} - \frac{c_1 x^3}{3} - \frac{c_2 x^4}{4}\right)_0^1$$

$$= -2\left(\frac{2}{5} - \frac{c_0}{2} - \frac{c_1}{3} - \frac{c_2}{4}\right) = 0 \quad -(2)$$

$$\frac{\partial I}{\partial c_2} = -2\int_0^1 \left(x^{1/2} - c_0 - c_1 x - c_2 x^2\right) x^2\,dx.$$

$$\boxed{e^x \cdot e^y = e^{xy}}$$

$$= -2\int_0^1 \left(x^{1/2} \cdot x^2 - c_0 x^2 - c_1 x^3 - c_2 x^4\right) dx.$$

$$\left[\because x^{1/2} \cdot x^2 = x^{1/2+2} = x^{5/2}\right]$$

$$= -2\int_0^1 \left(x^{5/2}\,dx - c_0 x^2\,dx - c_1 x\,dx - c_2 x^4\,dx\right)$$

$$= -2\int_0^1 \left(\frac{x^{7/2}}{7/2} - \frac{c_0 x^3}{3} - \frac{c_1 x^4}{4} - \frac{c_2 x^5}{5}\right)$$

$$= -2\left(\frac{2}{7}x^{7/2} - \frac{c_0 x^3}{3} - \frac{c_1 x^4}{4} - \frac{c_2 x^5}{5}\right)_0^1$$

$$= -2\left(\frac{2}{7} - \frac{c_0}{3} - \frac{c_1}{4} - \frac{c_2}{5}\right) \quad -(3)$$

which gives,

$$c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 = \frac{2}{3} \quad -(4)$$

$$\frac{1}{2}c_0 + \frac{1}{3}c_1 + \frac{1}{4}c_2 = \frac{2}{5} \quad -(5)$$

$$\frac{1}{3}c_0 + \frac{1}{4}c_1 + \frac{1}{5}c_2 = \frac{2}{7}. \quad -(6).$$

From eqn (4),

$$c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 = \frac{2}{3}.$$

$$c_0 = -\frac{1}{2}c_1 - \frac{1}{3}c_2 + \frac{2}{3} \quad -(7).$$

c_0 subs in eqn (5) & (6).

$$\frac{1}{2}\left(-\frac{1}{2}c_1 - \frac{1}{3}c_2 + \frac{2}{3}\right) + \frac{1}{3}c_1 + \frac{1}{4}c_2 = \frac{2}{5}. \quad -(8).$$

$$\frac{1}{3}\left(-\frac{1}{2}c_1 - \frac{1}{3}c_2 + \frac{2}{3}\right) + \frac{1}{4}c_1 + \frac{1}{5}c_2 = \frac{2}{7}. \quad -(9).$$

From eqn (8)

$$-\frac{1}{4}c_1 - \frac{1}{6}c_2 + \frac{1}{3} + \frac{1}{3}c_1 + \frac{1}{4}c_2 = \frac{2}{5}$$

$$\left(\frac{1}{3}-\frac{1}{4}\right)c_1 + \left(\frac{1}{4}-\frac{1}{6}\right)c_2 = \frac{2}{5} - \frac{1}{3}$$

$$\frac{1}{12}c_1 + \frac{1}{12}c_2 = \frac{1}{15} \quad -(10)$$

From eqn (9).

$$-\frac{1}{6}c_1 - \frac{1}{9}c_2 + \frac{2}{9} + \frac{1}{4}c_1 + \frac{1}{5}c_2 = \frac{2}{7}$$

$$\left(\frac{1}{4}-\frac{1}{6}\right)c_1 + \left(\frac{1}{5}-\frac{1}{9}\right)c_2 = \frac{2}{7} - \frac{2}{9}$$

$$\frac{1}{12}c_1 + \frac{4}{45}c_2 = \frac{4}{63} \quad -(11)$$

(11) − (10)

$$\frac{1}{12}\cancel{c_1} + \frac{4}{45}c_2 = \frac{4}{63}$$

$$\frac{1}{12}c_1 + \frac{1}{12}c_2 = \frac{1}{15}$$

$$\left(\frac{4}{45}-\frac{1}{12}\right)c_2 = \left(\frac{4}{63}-\frac{1}{15}\right)$$

$$\frac{48-45}{540}c_2 = \frac{60-63}{945}$$

$$\frac{3}{540}c_2 = -\frac{3}{945}$$

$$\frac{1}{180}c_2 = -\frac{1}{315}$$

$$c_2 = -\frac{180}{315}$$

$$\boxed{c_2 = -\frac{20}{35}}$$

C_2 value Substitute in (10).

$$\frac{1}{12}c_1 + \frac{1}{12}\left(-\frac{4}{7}\right) = \frac{1}{15}.$$

$$\frac{1}{12}c_1 - \frac{\cancel{4}^{1}}{\cancel{84}_{21}} = \frac{1}{15}.$$

$$\frac{1}{12}c_1 = \frac{1}{15} + \frac{1}{21}.$$

$$c_1 = \frac{36}{315} \times 12.$$

$$c_1 = \frac{\cancel{432}^{48}}{\cancel{315}_{35}}.$$

$$\boxed{c_1 = \frac{48}{35}}.$$

C_1 & C_2 values are Subs in eqn. (7).

$$c_0 = -\frac{1}{2}\left(\frac{48}{35}\right) - \frac{1}{3}\left(-\frac{4}{7}\right) + \frac{2}{3}.$$

$$c_0 = -\frac{48}{70} + \frac{4}{21} + \frac{2}{3}.$$

$$c_0 = -\frac{48}{70} + \frac{54}{63}.$$

$$\frac{12+42}{63}$$

$$\frac{54}{63}.$$

$$c_0 = \frac{-3024 + 3780}{4410}.$$

$$C_0 = \frac{\overset{6}{\cancel{54}}\ \overset{126}{\cancel{756}}}{4410.\ 735.\ 245\ 35.}$$

$$\boxed{C_0 = \frac{6}{35}}.$$

Let $C_0 = \frac{6}{35}$

$C_1 = \frac{48}{35}$

$C_2 = -\frac{20}{35}.$

Let the least square approximation $f(x) = x^{1/2}$.

$P(x) = C_0 + C_1 x + C_2 x^2.$

$= \frac{6}{35} + \frac{48}{35} x - \frac{20}{35} x^2$

$$\boxed{P(x) = \frac{1}{35}\left(6 + 48x - 20x^2\right).}$$

Derive the least square straight line and quadratic fits for the discrete data (x_i, f_i) $i=0,\dots N$.

Soln:

Let $p(x) = c_0 + c_1 x$ be a straight line approximation.

$$I(c_0, c_1) = \sum_{i=0}^{N} \left[f(x_i) - (c_0 + c_1 x_i) \right]^2$$

$= \text{minimum}$

The normal equations are.

$$\frac{\partial I}{\partial c_0} = -\sum_{i=0}^{N} 2 [f(x_i) - (c_0 + c_1 x_i)] = 0 \quad -①$$

$$\frac{\partial I}{\partial c_1} = -\sum_{i=0}^{N} 2 [f(x_i) - (c_0 + c_1 x_i)] x_i = 0 \quad -②$$

These equations simplify to

$$\frac{\partial I}{\partial c_0} = -\sum_{i=0}^{N} 2 [f(x_i) - (c_0 + c_1 x_i)]$$

$$-\sum f(x_i) + \sum c_0 + \sum c_1 x_i = 0.$$

$$c_0(N+1) + c_1\sum x_i = \sum f(x_i) \quad -③$$

$$\frac{\partial I}{\partial c_1} = -\sum_{i=0}^{N} 2\left[f(x_i) - (c_0 + c_1 x_i)\right] x_i$$

$$-\sum x_i f(x_i) + \sum c_0 x_i + \sum c_1 x_i^2 = 0.$$

$$c_0\sum x_i + c_1\sum x_i^2 = \sum x_i f(x_i) \quad -④.$$

The second degree least square approximation

$$p(x) = a + bx + cx^2.$$

$$I(a,b,c) = \sum_{i=0}^{N}\left[f(x_i) - (a + bx + cx^2)\right]^2$$

$$= \text{minimum}$$

The normal equations are.

$$\frac{\partial I}{\partial a} = -\sum_{i=0}^{N} 2\left[f(x_i) - (a + bx + cx^2)\right]. \quad -①$$

$$\frac{\partial I}{\partial b} = -\sum_{i=0}^{N} 2\left[f(x_i) - (a + bx + cx^2)\right] x_i \quad -②$$

$$\frac{\partial I}{\partial c} = -\sum_{i=0}^{N} 2\left[f(x_i) - (a + bx + cx^2)\right] x_i^2 \quad -③$$

The eqn simplify to,

$$\frac{\partial I}{\partial a} = - \sum_{i=0}^{N} 2 \left[f(x_i) - (a + b x_i + c x_i^2) \right]$$

$$- \sum f(x_i) + \sum (a + b x_i + c x_i^2) = 0.$$

$$a(N_0+1) + b \sum x_i + c \sum x_i^2 = \sum f(x_i) \quad -④$$

$$\frac{\partial I}{\partial b} = - \sum_{i=0}^{N} 2 \left[f(x_i) - (a + b x_i + c x_i^2) \right] x_i .$$

$$- \sum x_i f(x_i) + \sum (a x_i + b x_i^2 + c x_i^3) = 0.$$

$$a \sum x_i + b \sum x_i^2 + c \sum x_i^3 = \sum x_i f(x_i) \quad -⑤$$

$$\frac{\partial I}{\partial c} = - \sum_{i=0}^{N} 2 \left[f(x_i) - (a + b x_i + c x_i^2) \right] x_i^2 .$$

$$- \sum x_i^2 f(x_i) + \sum (a x_i^2 + b x_i^3 + c x_i^4) = 0.$$

$$a \sum x_i^2 + b \sum x_i^3 + c \sum x_i^4 = \sum x_i^2 f(x_i) \quad -⑥$$

Find the least squares approximation of second degree for the discrete data.

x	-2	-1	0	1	2
$f(x)$	15	1	1	3	19

Soln.

$n = 2,$

The normal equations for fitting a second degree polynomial

$$P_2(x) = C_0 + C_1 x + C_2 x^2.$$

$$I(C_0, C_1, C_2) = \sum_{i=0}^{N} \left[f(x_i) - (a + b x_i + c x_i^2) \right]^2.$$

(From previous ex.). $[\because (N+1) = 5]$.

$(N+1) = 5.$ $$\frac{\partial I}{\partial C_0} = C_0 (N+1) + C_1 \sum x_i + C_2 \sum x_i^2 = \sum f(x_i) \quad \text{—(1)}$$

$$5C_0 + C_1(-2 - 1 + 0 + 1 + 2) + C_2\left((-2)^2 + (-1)^2 + (0)^2 + (1)^2 + (2)^2\right)$$

$$= (15 + 1 + 1 + 3 + 19)$$

$$5C_0 + C_1(0) + C_2(4 + 1 + 0 + 1 + 4) = 39.$$

$$5C_0 + 10C_2 = 39 \quad - ①.$$

$$\frac{\partial I}{\partial c_1} \Rightarrow C_0 \sum x_i + c_1 \sum x_i^2 + c_2 \sum x_i^3 = \sum x_i f(x_i) - (II).$$

$$\Big[c_0(-2-1+0+1+2) + c_1((-2)^2+(-1)^2+(0)^2+(1)^2+(2)^2 +$$

$$c_2(-8-1+0+1+8)\Big] = \Big[(-2)(15)+(-1)(1)+$$

$$0(1)+1(3)+2(19)\Big].$$

$$C_0(0)+10C_1 + C_2(0) = (-30-1+0+3+38).$$

$$10C_1 = 10. \quad - ②.$$

$$\frac{\partial I}{\partial c_2} \Rightarrow C_0 \sum x_i^2 + C_1 \sum x_i^3 + c_2 \sum x_i^4 = \sum x_i^2 f(x_i) - (III)$$

$$\Big[c_0((-2)^2+(-1)^2+(0)^2+(1)^2+(2)^2) + c_1(-8-1+0+1+8) +$$

$$c_2(16+1+0+16+1) = \Big[(-2)^2(15)+(-1)^2(1)+0^2(1)+(1)^2(3)+(2)^2(19)\Big]$$

$$10C_0 + 0C_1 + 34C_2 = [(60+1+0+3+76)].$$

$$10C_1 + 34C_2 = 140 \quad - ③.$$

$$5C_0 + 10C_2 = 39. \quad -(1)$$

$$10C_1 = 10. \quad -(2)$$

$$10C_0 + 34C_2 = 140. \quad -(3).$$

from eqn (2).

$$10C_1 = 10 \quad \Rightarrow \quad \boxed{C_1 = 1}$$

from eqn (3) & (1)

$$10\cancel{C_0} + 34C_2 = 140.$$

(1)×2, $\quad 10C_0 + 20C_2 = 78$

$$\quad - \quad\quad - \quad\quad -$$

$$14C_2 = 62.$$

$$C_2 = \frac{\cancel{62}^{31}}{\cancel{14}_7}$$

$$\boxed{C_2 = \frac{31}{7}}$$

Subs in eqn (1) we get,

$$5C_0 + 10\left(\frac{31}{7}\right) = 39 \quad \Rightarrow 5C_0 + \frac{310}{7} = 39.$$

$$5C_0 = 39 - \frac{310}{7}. \qquad C_0 = \frac{273-310}{7} \times \frac{1}{5}.$$

$$\boxed{C_0 = -\frac{37}{35}}$$

$$C_0 = -\frac{37}{35}, \quad C_1 = 1 \quad \text{and} \quad C_2 = \frac{31}{7}.$$

The required Approximation is.

$$P_2(x) = \left(-\frac{37}{35} + 1x + \frac{31}{7}x^2\right).$$

$$P_2(x) = \left(-\frac{37\times1}{35\times1}\right) + (1 \times 35)x + \left(\frac{31\times5}{7\times5}\right)x^2\Big)$$

$$P_2(x) = \frac{1}{35}(-37 + 35x + 155x^2)$$

Dfn. (orthogonal over)

A set of functions $\{\phi_i(x)\}$ is said to be orthogonal over a set of points $\{x_i\}$ with respect to weight function $w(x)$, if.

$$\sum_{i=0}^{N} w(x_i)\ \phi_j(x_i)\ \phi_K(x_i) = 0, \quad j \neq K$$

Dfn: (orthogonal on an interval)

A set of functions $\{\phi_i(x)\}$ is said to be orthogonal on an interval $[a,b]$ with respect to the weight function $w(x)$, if

$$\int_a^b w(x)\ \phi_i(x)\ \phi_j(x)\ dx = 0 \quad , i \neq J.$$

If the functions $\phi_i(x)$, $i = 0, 1, \ldots, n$ are orthogonal, then we obtain from.

$$\sum_{k=0}^{n} w(x_k)\ \phi_i^2(x_k) c_i = \sum_{k=0}^{n} w(x_k)\ \phi_i(x_k)\ f(x_k).$$

Hence,

$$C_i = \frac{\sum_{k=0}^{N} w(x_k)\, \phi_i(x_k)\, f(x_k)}{\sum_{k=0}^{N} w(x_k)\, \phi_i^2(x_k)} \qquad i = 0, 1, 2, \ldots, n.$$

Thus, the use of orthogonal functions as coordinate functions, in not only avoids the problem of ill-conditioning in normal equations but also determines the constants C_i, $i = 0, 1, \ldots, n$ directly.

It may be noted that not every set of linearly independent polynomials satisfies the condition of orthogonality but, it can be orthogonalized by using following method.

Gram-Schmidt orthogonalizing process:

Defn:

Given the polynomials $\phi_i(x)$ of degree i, the polynomials $\phi_i^*(x)$ of degree i which are orthogonal over $[a,b]$ with respect to the weight function $W(x)$ can be generated recursively from the relation.

$$\phi_i^*(x) = x^i - \sum_{r=0}^{i-n} a_{ir}\, \phi_r^*(x), \qquad i = 1, 2, \ldots n$$

where

$$a_{ir} = \frac{\int_a^b W(x)\, x^i\, \phi_r^*(x)\, dx}{\int_a^b W(x)\, \phi_r^{*2}(x)\, dx}$$

and $\phi_0^*(x) = 1$.

On a discreate set of points, the integral is replaced by summation.

Example: 4.17.

Using the Gram-Schmidt orthogonalization process, compute the first three orthogonal polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$ which are the orthogonal on $[0,1]$ with respect to the weight function $W(x)=1$. Using three polynomials obtain the least square approximation of second degree for $f(x) = x^{1/2}$ on $[0,1]$

Soln

We have

$$P_0(x) = 1 = \phi_0^*(x)$$

$$\phi_i^*(x) = P_i(x)$$

$$\phi_i^*(x) = x^i - \sum_{i=0}^{N} a_{ir} \phi_r^*(x) \longrightarrow 1$$

$i = 1, 2, \ldots n.$

$i = 1$, $r = 0$ sub in 1

$$P_1(x) = x^1 - a_{10} \phi_0^*(x) \longrightarrow 2$$

$$a_{ir} = \frac{\int_a^b W(x)\, x^i\, \phi_r^*(x)\, dx}{\int_a^b W(x)\, \phi_r^{*2}(x)\, dx}$$

$W(x) = 1$,

$$a_{10} = \frac{\int_0^1 (1)\, \phi_0^*(x)\, dx}{\int_0^1 (1)\, \phi_0^{*2}(x)\, dx}$$

$$= \frac{\int_0^1 x\, dx}{\int_0^1 dx} = \frac{\left[\frac{x^2}{2}\right]_0^1}{[x]_0^1} = \frac{\frac{1}{2}}{1}$$

$$a_{10} = 1/2$$

$$P_1(x) = x - 1/2 = \phi_1^*(x).$$

$i = 2$ & $r = 0, \& 1$

$$P_2(x) = x^2 - a_{20}\,\phi_0^*(x) - a_{21}\,\phi_1^*(x).$$

$$a_{20} = \frac{\int_0^1 (1)\, x^2\, \phi_0^*(x)\, dx}{\int_0^1 (1)\, \phi_0^{*2}(x)\, dx}$$

$$= \frac{\int_0^1 x^2\, dx}{\int_0^1 dx} = \frac{\left[\frac{x^3}{3}\right]_0^1}{[x]_0^1} = \frac{1/3}{1}$$

$$a_{20} = \frac{1}{3}$$

$$a_{21} = \frac{\int_0^1 (1)\, x^2\, \phi_1^*(x)\, dx}{\int_0^1 (1)\, \phi_1^{*2}(x)\, dx.}$$

$$= \frac{\int_0^1 x^2 (x - 1/2)\, dx}{\int_0^1 (x - 1/2)^2\, dx.}$$

$$= \frac{\int_0^1 (x^3 - \frac{1}{2}x^2)\, dx}{\int_0^1 (x^2 - \frac{1}{\cancel{2}}\cdot\cancel{2}x + \frac{1}{4})\, dx} = \frac{\left[\frac{x^4}{4} - \frac{x^3}{6}\right]_0^1}{\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1}$$

$$= \frac{\frac{1}{4} - \frac{1}{6}}{1/3 - 1/2 + 1/4}$$

$$= \frac{\frac{2}{24}}{7/12 - 1/2} = \frac{2/24}{2/24} = 1$$

$a_{21} = 1$

$$P_2(x) = x^2 - \frac{1}{3}(1) - 1(x - 1/2)$$

$$= x^2 - 1/3 - x + 1/2$$

$$P_2(x) = x^2 - x + 1/6$$

Using these polynomials, we have for $n=2$.

$$I(c_0, c_1, \ldots c_n) = \int_a^b W(x)\left[f(x) - \sum_{i=0}^{N} c_i\phi_i(x)\right]^2 dx.$$

$$= \text{minimum}.$$

$$\int_a^b W(x)\left[f(x) - \sum_{i=0}^{N} c_i\phi_i(x)\right]\phi_j(x)\,dx = 0.$$

$$I(c_0, c_1, c_2) = \int_0^1 (1)\left[x^{1/2} - c_0P_0(x) - c_1P_1(x) - c_2P_2(x)\right]^2 dx$$

$$= \text{minimum}$$

$$\frac{\partial I}{\partial c_0} = -2\int_0^1 \left[x^{1/2} - c_0P_0 - c_1P_1 - c_2P_2\right]P_0\,dx = 0$$

$$\frac{\partial I}{\partial c_1} = -2\int_0^1 [x^{1/2} - c_0 p_0 - c_1 p_1 - c_2 p_2] p_1 \, dx = 0$$

and

$$\frac{\partial I}{\partial c_2} = -2\int_0^1 [x^{1/2} - c_0 p_0 - c_1 p_1 - c_2 p_2] p_2 \, dx = 0$$

Using the orthogonally conditions.

we obtain,

$$c_i = \frac{\sum_{k=0}^{N} w(x_k)\, \phi_i(x_k)\, f(x_k)}{\sum_{k=0}^{N} w(x_k)\, \phi_i^2(x_k)} \qquad i = 0, 1, \dots n.$$

$$C_0 = \frac{\int_0^1 (1)\, x^{1/2}\, p_0(x)\, dx}{\int_0^1 p_0^2(x)\, dx}.$$

$$= \frac{\int_0^1 x^{1/2} (1)\, dx}{\int_0^1 (1)\, dx}$$

$$= \frac{\left[\frac{x^{3/2}}{3/2}\right]_0^1}{[x]_0^1} = \frac{\left[\frac{2}{3}(x)^{3/2}\right]_0^1}{[x]_0^1}$$

$$= \frac{\frac{2}{3}}{1} = 2/3.$$

$$C_0 = 2/3.$$

$$C_1 = \frac{\int_0^1 x^{1/2}\, p_1(x)\,dx}{\int_0^1 p_1^2(x)\,dx}$$

$$= \frac{\int_0^1 x^{1/2}\,(x - 1/2)\,dx}{\int_0^1 (x - 1/2)^2\,dx}$$

$$= \frac{\int_0^1 \left(x^{3/2} - \frac{1}{2}x^{1/2}\right)dx}{\int_0^1 \left(x^2 - \frac{1}{\cancel{2}}\cdot\cancel{2}x + \frac{1}{4}\right)dx}$$

$$= \frac{\left[\frac{x^{5/2}}{5/2} - \frac{1}{2}\frac{x^{3/2}}{3/2}\right]_0^1}{\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1} = \frac{\left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^{3/2}\right]_0^1}{\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1}$$

$$= \frac{\left(\frac{2}{5} - 1/3\right)}{\left(1/3 + 1/4 - 1/2\right)} = \frac{\frac{6-5}{15}}{1/12} = 1/15 \times \frac{12}{1}$$

$$C_1 = 4/5$$

$$C_2 = \frac{\int_0^1 x^{1/2}\,(x^2 - x + 1/6)\,dx}{\int_0^1 (x^2 - x + 1/6)^2\,dx}$$

$$= \frac{\int_0^1 x^{5/2} - x^{3/2} + \frac{1}{6} x^{1/2} \cdot dx}{\int_0^1 (x^2 - x + 1/6)(x^2 - x + 1/6)\, dx} = -4/7$$

$c_2 = -4/7.$

The required least square approximation is,

$$y(x) = \frac{2}{3} P_0(x) + 4/5\, P_1(x) - 4/7\, P_2(x)$$

$$= \frac{2}{3}(1) + 4/5\,(x - 1/2) - 4/7\,(x^2 - x + 1/6)$$

$$= \frac{2}{3} + \frac{4x}{5} - 2/5 - 4/7\, x^2 + 4/7\, x - 2/21$$

$$= 2/3 + \frac{4x}{5} + \frac{4x}{7} - \frac{2}{5} - \frac{2}{21} - 4/7\, x^2.$$

$$= \frac{2}{3} - \frac{52}{105} - x^2\left(\frac{4}{7}\right) + x\left(\frac{48}{35}\right)$$

$$= \frac{210 - 156}{315} - \frac{4}{7}x^2 + \frac{48}{35}x$$

$$= \frac{54}{315} - \frac{4x^2}{7} + \frac{48}{35}x$$

186 105 35

$$\left(\frac{4 \times 5}{7 \times 5}\right)$$

$$= \frac{6}{35} - \frac{20x^2}{35} + \frac{48x}{35}.$$

$$y(x) = \frac{1}{35}(6 - 20x^2 + 48x)$$

Legendre polynomials:

The legendre polynomials $P_n(x)$ defined on $[-1,1]$ are given by

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!\, x^{n-2m}}{2^n m!\,(n-m)!\,(n-2m)!}$$

where $M = n/2$ (or) $(n-1)/2$ whichever is an integer.

The legendre polynomials satisfy the differential equation.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

The legendre polynomials satisfy the recurrence relation.

$$(n+1)\, P_{n+1}(x) = (2n+1)\, x\, P_n(x) - n P_{n-1}(x).$$

and possess the following properties:

(i) $P_n(x)$ is even polynomial if n is even and an odd polynomial if n is odd.

(ii) $P_n(x)$ are orthogonal polynomial and satisfy,

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \qquad m \neq n$$

$$= \frac{2}{2n+1} \qquad m = n$$

(iii) $P_n(-x) = (-1)^n P_n(x)$.

Chebyshev polynomial :

The chebyeshev polynomials of first kind $T_n(x)$ deffined on $[-1,1]$ are given by

$$T_n(x) = \cos(n\cos^{-1}x) = \cos n\theta$$

where $\theta = \cos^{-1}x$ (or) $x = \cos\theta$.

This polynomials satisfy the differential eqn

$$(1-x^2)y'' - xy' + n^2y = 0$$

This polynomial possess tho following properties:

(i) $T_n(x)$ is a polynomial of degree n. If n is even $T_n(x)$ is an even polynomial and if n is odd. $T_n(x)$ is an odd polynomial.

(ii) $T_n(x)$ has n simple zeros

$$x_k = \cos\left(\frac{2k+1}{2n}, \pi\right), \ k = 1, 2, \ldots n$$ on the interval $[-1,1]$.

(iii) $T_n(x)$ assumes extreme values at $n+1$ points $x_k = \cos(k\pi/n)$, $k = 0, 1, \ldots n$ and the extreme value at x_k is $(-1)^k$.

(iv) $|T_n(x)| \leq 1, \quad x \in [-1, 1]$.

(v) If $P_n(x)$ is any Polynomial of degree n with leading co-efficient unity and $\tilde{T}_n(x) = T_n(x)/2^{n-1}$ is the monic Chebyshev polynomial then

$$\max_{-1 \leq x \leq 1} |\tilde{T}_n(x)| \leq \max_{-1 \leq x \leq 1} |P_n(x)|$$

(vi) $T_n(x)$ are orthogonal with respect to the weight function

$$W(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0. \end{cases}$$

Example 4:18

Using the Chebyshev polynomials, obtain the least square approximation of second degree for $f(x) = x^4$ on $[-1, 1]$.

We write

$$f(x) = c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x).$$

we have

$$I(c_0, c_1, c_2) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[x^4 - (c_0T_0 + c_1T_1 + c_2T_2)\right]^2 dx.$$

= minimum.

$$\frac{\partial I}{\partial c_0} = -2\int_{-1}^{1} \left[x^4 - (c_0T_0 + c_1T_0 + c_2T_2)\right] \frac{T_0\,dx}{\sqrt{1-x^2}} = 0$$

$$\frac{\partial I}{\partial c_1} = -2\int_{-1}^{1} \left[x^4 - (c_0T_0 + c_1T_1 + c_2T_2)\right] \frac{T_1\,dx}{\sqrt{1-x^2}} = 0$$

$$\frac{\partial I}{\partial c_2} = -2\int_{-1}^{1} \left[x^4 - (c_0T_0 + c_1T_1 + c_2T_2)\right] \frac{T_2\,dx}{\sqrt{1-x^2}} = 0$$

We find

$$c_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{x^4 T_0}{\sqrt{1-x^2}}\,dx = \frac{3}{8}.$$

$$c_1 = \frac{2}{\pi}\int_{-1}^{1} \frac{x^4 T_1}{\sqrt{1-x^2}}\,dx.$$

$$= \frac{2}{\pi}\int_{-1}^{1} \frac{x^4\, x}{\sqrt{1^2-x^2}}\,dx$$

$$= \frac{2}{\pi} \int_{-1}^{1} \frac{x^5/\cancel{8}}{1-x} \, dx.$$

$$= \frac{2}{\pi} \left[\frac{x^6/6}{x - \frac{x^2}{2}} \right]_{-1}^{1}$$

$$= \frac{2}{\pi} \left[\frac{1/6 - 1/6}{1 - \frac{1}{2} - 1 - 1/2} \right]$$

$$= 2/\pi \left[\frac{0}{-1} \right]$$

$$c_1 = 0$$

$$c_2 = \frac{2}{\pi} \int_{-1}^{1} \frac{x^4 T_2}{\sqrt{1-x^2}} \, dx.$$

$$c_2 = 1/2 .$$

The required approximation is

$$f(x) = \frac{3}{8} T_0 + (0) T_1 + \frac{1}{2} T_2 .$$

$$f(x) = \frac{3}{8} T_0 + 1/2 \, T_2 .$$

Numerical Analysis

Numerical Differentiation :-

i) methods based on Interpolation :-

Given values of $f(x)$ at set of points $x_0, x_1, \dots, x_n$ the general approach for deriving numerical differentiation methods to obtaining interpolating polynomials $P_n(x)$ and differentiate this polynomial r times $(n \geq r)$ to get $P_n^r(x)$.

The value $P_n^{(r)}(x_k)$ give the approximate value of $f^r(x)$ at point x_k.

$E^r(x) = f^{(r)}(x) - P_n^{(r)}(x) \rightarrow (1)$ is called the error of approximation in r^{th} order derivative at any point x.

a) Linear Interpolation :-

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$P_1(x) = l_0(x) f_0 + l_1(x) f_1$$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1.$$

$$\therefore \boxed{P_1'(x) = \frac{f_1 - f_0}{x_1 - x_0}}, \text{ for all } x \in [x_0, x_1].$$

Also $E_1'(x_0) = \dfrac{x_0 - x_1}{2} \cdot f''(\xi)$

$$E_1'(x_1) = \frac{x_1 - x_0}{2} f''(\xi), \quad x_0 < \xi_j < x_1.$$

b) <u>Quadratic Interpolation</u> :-

If x_i, f_i, $i = 0, 1, 2, \ldots$ are given

$$P_2(x) = l_0(x) f_0 + l_1(x) f_1 + l_2(x) f_2 \rightarrow (1)$$

$$P_2'(x) = l_0'(x) f_0 + l_1'(x) f_1 + l_2'(x) f_2 \rightarrow (2)$$

Eqn (1) $\Rightarrow$ $l_0(x) = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$ $\quad l_0'(x) = \dfrac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)}$

$l_1(x) = \dfrac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$ $\quad l_1'(x) = \dfrac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)}$

$l_2(x) = \dfrac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$ $\quad l_2'(x) = \dfrac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)}$

(2), $\Rightarrow$

$$P_2'(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f_1 + \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} f_2. \rightarrow (3)$$

$$P_2'(x_0) = \frac{2x_0 - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{2x_0 - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f_1 + \frac{2x_0 - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} f_2. \rightarrow (4)$$

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Diff. eqn. ③ with respect to x,

$$P_2''(x) = \frac{2}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{2}{(x_1-x_0)(x_1-x_2)} f_1 +$$

$$\frac{2}{(x_2-x_0)(x_2-x_1)} f_2, \text{ for all } x \in [x_0, x_n].$$

$$E_2'(x_0) = \frac{1}{6}(x_0-x_1)(x_0-x_2) f'''(\xi_1), \quad x_0 < \xi < x_2$$

$$E_2''(x_0) = \frac{1}{3}(2x_0 - x_1 - x_2) f'''(\xi_1) + \frac{1}{24}(x_0-x_1)(x_0-x_2)\left[f^{iv}(\xi_2) + f^{iv}(\xi_3)\right]$$

Problem:-

1) Given the following values of $f(x) = \log x$, find the approximate value of $f'(2.0)$, $f''(2.0)$ using linear and quadratic interpolation methods. Also find upper bound on the error.

i	0	1	2
x_i	2.0	2.2	2.6
f_i	0.69315	0.78846	0.95551

Solution:-

i) To find $f'(2.0)$:

i) linear interpolation :

$$f'(x_0) = \frac{f_1 - f_0}{x_1 - x_0}$$

$$= \frac{0.78846 - 0.69315}{2.2 - 2.0} = \frac{0.09531}{0.2}$$

$$\boxed{f'(x_0) = 0.47655}$$

ii) quadratic interpolation :

$$f'(x_0) = \frac{2x_0 - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f_1 +$$

$$\frac{x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} f_2.$$

$$f'(2.0) = \frac{2(2.0) - 2.2 - 2.6}{(2 - 2.2)\ (2 - 2.6)} (0.69315) + \frac{2 - 2.6}{(2.2 - 2)\ (2.2 - 2.6)} (0.78846)$$

$$+ \frac{2.0 - 2.2}{(2.2 - 2.0)\ (2.6 - 2.2)} (0.95551)$$

$$= \frac{4 - 2.2 - 2.6}{(-0.2)\ (-0.6)} (0.69315) + \frac{-0.6}{(0.2)\ (-0.4)} (0.78846)$$

$$+ \frac{(-0.2)}{(0.2)\ (0.4)} (0.95551)$$

$$= \frac{-0.8}{0.12}(0.69315) + \frac{-0.473076}{-0.08} + \frac{-0.191102}{0.08}$$

$$= -4.62100 + 5.91345 - 0.79626$$

$$\boxed{f'(2.0) = 0.49619}$$

$\therefore$ The exact value is 0.5

ii) To find $f''(2.0)$:

$$f''(x_0) = 2\left[\frac{f_0}{(x_0-x_1)(x_0-x_2)} + \frac{f_1}{(x_1-x_0)(x_1-x_2)} + \frac{f_2}{(x_2-x_0)(x_2-x_1)}\right]$$

$$f''(2.0) = 2\left[\frac{0.69315}{(2.0-2.2)(2.0-2.6)} + \frac{0.78846}{(2.2-2.0)(2.2-2.6)} + \frac{0.95551}{(2.6-2.0)(2.6-2.2)}\right]$$

$$= 2\left[\frac{0.69315}{0.12} + \frac{0.78846}{(-0.08)} + \frac{0.95551}{0.24}\right]$$

$$= 2\left[5.77625 - 9.85575 + 3.98129\right]$$

$$= 2(-0.098210)$$

$$\boxed{f''(2.0) = -0.19642}$$

To find exact solution :-

given $f(x) = \log x$

$f'(x) = \frac{1}{x} \Rightarrow f'(2.0) = \frac{1}{2} = 0.5$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(2.0) = -\frac{1}{(2.0)^2} = -0.25$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(2.0) = \frac{2}{(2.0)^3} = 0.25$

linear error :

$E'(x) = f'(x) - \frac{f_1 - f_0}{h}$; $h = x_1 - x_0 = 2.2 - 2.0$

$h = 0.2$

$= 0.5 - \frac{0.78846 - 0.69315}{0.2}$

$= 0.5 - \frac{0.09531}{0.2} = 0.5 - 0.47655$

$\boxed{E'(2.0) = 0.02345}$

quadratic error :

$E_2'(x) = f'(x) + \frac{d}{dx} \frac{1}{6}\left[(x_0 - x_1)(x_0 - x_2)\right] f'''(x)$

$= \frac{1}{6}\left[(2.0 - 2.2)(2.0 - 2.6)\right](0.25)$

$= \frac{1}{6}(0.03)$

$\boxed{E_2'(x) = 0.005}$

$$E_{2}''(x) = \frac{1}{3}\left\{(2x_0 - x_1 - x_2) f'''(\xi_0) + \frac{1}{24}(x_0 - x_1)(x_1 - x_2)\right\}$$
$$[f^{iv}(\xi_1) + f^{iv}(\xi_2)]$$

$$= \frac{1}{3}\left[(4 - 2.2 - 2.6)(0.25)\right] + \frac{1}{24}\left[(2 - 2.2)(2.2 - 2.6)\right.$$
$$\left.(0.75)\right]$$

$$\boxed{E_{2}''(x) = 0.06917}$$

Exercise :-

1) The following for $f(x) = x^4$ table!

x	0.4	0.6	0.8
$f(x)$	0.0256	0.1296	0.4096

find $f'(0.8)$, $f''(0.8)$ using linear interpolation and quadratic interpolation.

Solution :

i) linear interpolation :-

$$f'(x_0) = \frac{f_1 - f_0}{x_1 - x_0}$$

$$= \frac{0.1296 - 0.0256}{0.6 - 0.4}$$

$$\boxed{f'(x_0) = 0.52}$$

ii) quadratic interpolation :-

$$f'(x_0) = \frac{2x_0 - x_1 - x_2}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{x_0 - x_2}{(x_1-x_0)(x_1-x_2)} f_1 +$$

$$\frac{x_0 - x_1}{(x_2-x_0)(x_2-x_1)} f_2 .$$

$$= \frac{2(0.4) - 0.6 - 0.8}{(0.4-0.6)(0.4-0.8)}(0.0256) + \frac{0.4 - 0.8}{(0.6-0.4)(0.6-0.8)}(0.1296) +$$

$$\frac{0.4 - 0.6}{(0.8-0.4)(0.8-0.6)}(0.4096)$$

$$= \frac{-0.6}{(-0.2)(-0.4)}(0.0256) + \frac{-0.4}{(0.2)(-0.2)}(0.1296) + \frac{-0.2}{0.08}(0.4096)$$

$$= (-7.5 \times 0.0256) + 10(0.1296) - 2.5(0.4096)$$

$$= -0.192 + 1.296 - 1.024$$

$$\boxed{f'(x_0) = 0.080}$$

$$f''(x_0) = 2\left[\frac{f_0}{(x_0-x_1)(x_0-x_2)} + \frac{f_1}{(x_1-x_0)(x_1-x_2)} + \frac{f_2}{(x_2-x_0)(x_2-x_1)}\right]$$

$$= 2\left[\frac{0.0256}{0.08} - \frac{0.1296}{0.04} + \frac{0.4096}{0.08}\right]$$

$$= 2(0.32 - 3.24 + 5.12)$$

$$\boxed{f''(0.8) = 4.4}$$

Optimum choice of step lenght

when $f(x)$ is given in tabular from. these values may not be exact. these values contain round off errors If $f(x_k) = f_k + \epsilon_k$, where $f(x_k)$ is exact value. f_k is tabulated value. and ϵ_k is round-off error.

$$f'(x_k) = \left(\frac{f_{k+1} - f_k}{h}\right) - \frac{h}{2} f''(\alpha),$$

$$x_k < \alpha < x_{k+1}$$

If round-off errors in f_k and f_{k+1} are ϵ_k and ϵ_{k+1} respectively then.

$$f'(x_k) = \frac{1}{h}\left[(f_{k+1} + \epsilon_{k+1}) - (f_k + \epsilon_k)\right] - \frac{h}{2} f''(\alpha)$$

$$= \left(\left[\frac{f_{k+1} - f_k}{h}\right] + \left[\frac{\epsilon_{k+1} - \epsilon_k}{h}\right]\right) - \frac{h}{2} f''(\alpha)$$

$$f'(x_k) = \left(\frac{f_{k+1} - f_k}{h}\right) + RE + TE.$$

where RE is rounded error and. TE is truncation error

Take. $\epsilon = \max(|\epsilon_k|, |\epsilon_{k+1}|)$, $M_2 = \max[|f''(x)|]$

then $|RE| \le \frac{2\epsilon}{h}$, $|TE| \le \frac{h}{2} M_2$

we define. optimum value of h as satisfies either one following condition

i) $|RE| = |TE|$ ii) $|RE| + |TE| = \min$

i) $|RE| = |TE|$

$$\frac{2\epsilon}{h} = \frac{h}{2} M_2.$$

$$\frac{h^2 M_2}{2} = 2\epsilon \Rightarrow h^2 = \frac{4\epsilon}{M_2}$$

$$\boxed{h = 2\sqrt{\epsilon / M_2}}$$

(ii) $|RE| + |TE| = \text{minimum}$.

$$\frac{2\epsilon}{h} + \frac{h}{2} M_2 = \min.$$

Diff. w.r.to h.

$$\frac{2\epsilon}{h^2} + \frac{M_2}{2} = 0$$

$$\frac{M_2}{2} = \frac{2\epsilon}{h^2}$$

$$h^2 = \frac{4\epsilon}{M_2}$$

$$\boxed{h = 2\sqrt{\epsilon / M_2}}$$

5.3 For the method.

$$f'(z_0) = \frac{-3f(z_0) + 4f(z_1) - f(z_2)}{2h} + \frac{h^2}{3} f'''(\varepsilon)$$

$z_0 < \varepsilon < z_2$

determine. the optimal value of h . using. the criteria

(i) $|RE| = |TE|$

(ii) $|RE| + |TE|$ = minimum.

using this method and the. value of h. obtained from the. criterion. $|RE| = |TE|$ determine. an approximate. value. of $f'(2.0)$ from the following table lated value. of $f(x) = \log x$

z	2.0	2.01	2.02	2.06	2.12
f(z)	0.69315	0.69613	0.70310	0.72271	0.75142

given the table. maximum rounded error in function evaluation in 5×10^{-6}.

Soln

If $\epsilon_0, \epsilon_1, \epsilon_2$ are round-off error in the given function f_0, f_1, f_2 respectively, then we have.

$$f'(x_0) = \frac{-3f_0 + 4f_1 - f_2}{2h} + \frac{-3\epsilon_0 + 4\epsilon_1 - \epsilon_2}{2h} + \frac{h^2}{3} f'''(\varepsilon)$$

$$f'(x_0) = -\frac{3f_0 + 4f_1 - f_2}{2h} + RE + TE$$

Let $\epsilon = \max\left(|E_0|, |E_1|, |E_2|\right)$ $m_3 = \max|f'''(x)|$

$$|RE| \leq \frac{4\epsilon}{h}, \quad |TE| \leq \frac{h^3 m_3}{3}$$

(i) $|RE| = |TE|$, $\Rightarrow$ $\frac{4\epsilon}{h} = \frac{h^2 m_3}{3}$

$$h^3 = \frac{12\epsilon}{m_3}$$

$$|RE| = |TE| = \frac{4\epsilon^{2/3} m_3^{1/3}}{(12)^{1/3}}$$

(ii) $|RE| + |TE| = \min$

$$\frac{4\epsilon}{h} + \frac{m_3}{3}h^2 = \min.$$

w.r. to h. $-\frac{4\epsilon}{h^2} + \frac{m_3 2h}{3} = 0$

$$\Rightarrow \frac{4\epsilon}{h^2} = \frac{m_3 2h}{3}$$

$$h_{opt} = \left(\frac{6\epsilon}{m_3}\right)^{1/3}$$

minimum total error $= 6^{2/3} \epsilon^{2/3} m_3^{1/3}$

given $f(x) = \log x$

we have. $m_3 = \max\limits_{2 \leq x \leq 2.12} |f'''(x)|$

$f(x) = \log x$
$f'(x) = 1/x$
$f''(x) = -1/x^2$
$f'''(x) = 2/x^3$

$x = 2.0$

$M_3 = \max \left|\frac{2}{x^3}\right| = -\left|\frac{2}{8}\right| = 0.25$

$$h_{opt} = \left[\frac{12E}{M_3}\right]^{1/3}$$

$$h_{opt} = \left[\frac{12 \times 5 \times 10^{-6}}{1/4}\right]^{1/3} \simeq 0.06.$$

$h = 0.06.$

$$f'(2.0) = \frac{-3(0.69315) + 4(0.72271) - 0.75142}{0.12.}$$

$f'(2.0) = 0.49975.$

If we take. $h = 0.01$

$$f'(2.0) = \frac{-3(0.69315) + 4(0.7221) - 0.75142}{0.2}$$

$= 0.49850.$

Exact value. is $f'(2.0) = 0.5.$

2 determine hopt using the criteria.

(i) $|R| = |TE|$ and.

(ii) $|R| + |TE| =$ minimum.

using this method and the secound criterion. find hopt for $f(x) = \ln x$ and determine the value of $f'(2.03)$ from the following table of values of $f(x)$ if it is given that the maximum round-off error in the function evaluation is 5×10^{-6}

x	:	0.2	2.01	2.02	2.03	2.04	2.66
$f(x)$	:	0.69315	0.69813	0.70310	0.70804	0.71295	0.72271

Soln

If $\epsilon_{-1}, \epsilon_0, \epsilon_1$ are the round-off error in the given function evaluation f_{-1}, f_0, f_1 respectively and if $\epsilon = \max(|\epsilon_{-1}|, |\epsilon_0|, |\epsilon_1|)$ and. ~~If ϵ = max~~ $M_3 = \max |f'''(x)|$ then.

$$|R| \leq \frac{\epsilon}{h} \text{ and } |TE| \leq \frac{h^2}{3} M_3.$$

$$\|R\| = \|TE\|$$

$$\text{hopt } \left(\frac{3\epsilon}{M_3}\right)^{1/3}$$

$$|R| = |TE| = \epsilon^{2/3}\left(\frac{M_3}{3}\right)^{1/3}.$$

If we use $|R| = |TE| = \min$ then.

$f(x) = \log x$

$h_{opt} \left(\frac{3E}{2M_3}\right)^{1/3}$

$$\text{error is} = \frac{M^{1/3}}{3}\left(\frac{3E}{2}\right)^{2/3}$$

$f(x) = \ln x$ and using the secound.

criterion, we get.

$\cancel{h_{opt} = (30 \times 10^{-6})^{1/3} = 0.03}$

$f(x) = \log x$

$M_3(x) = \left|\frac{2}{x^3}\right|$

$$h_{opt} = \left[\frac{3E}{2M_3}\right]^{1/3}$$

$$= \left(\frac{3 \times 5 \times 10^{-6}}{2 \times 0.25}\right)^{1/3} = \left(\frac{15 \times 10^{-6}}{2 \times 0.25}\right)^{1/3}$$

$$= \left(\frac{15 \times 10^{-6}}{0.5}\right)^{1/3} = (30 \times 10^{-6})^{1/3}$$

$\boxed{h = 0.03}$

given to find.

$$f'(2.03) = \frac{f_{k+1} - f_{k-1}}{2h}$$

$$= \frac{0.72271 - 0.69315}{2 \times 0.03} = \frac{0.02956}{0.06}$$

$\boxed{f'(203) = 0.492666}$

5.4 EXTRAPOLATION METHODS

To obtain differentiation methods of high order, we require a large number to tabular points and thus a large number of function evaluations. Consequently there is a possibility that the roundoff errors may increase so much that the numerical results may become meaningless.
However, it is generally possible to obtain highly accurate results by combianing the computed values obtained by using a certain method with different step sizes.

Let $g(h)$ denote the approximate value of g, obtained by using a method of order p, with step length h and $g(qh)$ denote the value of g obtained by using the same method of order p, with step length qh.

We have

$$g(h) = g + ch^p + O(h^{p+1})$$

$$g(qh) = g + cq^p h^p + O(h^{p+1})$$

Eliminating c from the above equation, we get

$$g = \frac{q^p g(h) - g(qh)}{q^p - 1} + O(h^{p+1})$$

Thus we obtain

$$g^{(1)}(h) = \frac{q^p g(h) - g(qh)}{q^p - 1} = g + \underbrace{O(h^{p+1})}_{\to 0}$$

which is of order $p+1$. This technique of combining two computed values obtained obtained by using the same Method with two diffrent step sizes, to obtain a higher order Method is called the <u>Extrapolation Method</u> <u>or</u> <u>Richardson's</u> <u>extropolation</u>.

If the Local trunction error associated with the method is known as a power Series in h, then by repeating the extrapolation procedure. a number of times, we can obtain the Methods of any arbitary order. The application of this procedore becomes simplified when the step lengths from a geometric sequence. For simplicity we generally take $q = 1/2$. To illustrate the procedure we consider the Method.

$$f'(x_0) = \frac{f_1 - f_{-1}}{2h} \rightarrow (2)$$

Wher $f_1 = f(x_0+h)$ and $f_{-1} = f(x_0-h)$.

The Local trucation error associated with rhod (1) is obtained as

$$E'(x_0) = c_1 h^2 + c_2 h^4 + c_3 h^6 + \cdots \rightarrow (3)$$

Wher $c_1, c_2, c_3 \ldots$ are constants independent of h.

Let $g(x) = f'(x_0)$ be the quantity which is to be obtained and $g(h/2^r)$ denote the approximate value of $g(x)$ obtained by using the Method (1) with step length $h/2^r$, $r = 0, 1, 2 \ldots$

Thus we have

$$g(h) = g(x) + c_1 h^2 + c_2 h^4 + c_3 h^6 + \ldots$$

Differentiation and Intergration.

$$g(h/2) = g(x) + \frac{c_1 h^2}{4} + \frac{c_2 h^4}{16} + \frac{c_3 h^6}{64} + \ldots \rightarrow (4)$$

$$g(h/2^2) = g(x) + \frac{c_1 h^2}{16} + \frac{c_2 h^4}{256} + \frac{c_3 h^6}{4096} + \ldots$$

Eliminating c_1 from the above equation we obtain

$$g^{(1)}(h) = \frac{4g(h/2) - g(h)}{3}$$

$$= g(x) - \frac{1}{4} c_2 h^4 - \frac{5}{16} c_3 h^6 \ldots$$

$$g^{(1)}(h) = \frac{4g(h/2^2) - g(h/2)}{3}$$

$$= g(x) - \frac{1}{64} c_2 h^4 - \frac{5}{1024} c_3 h^6 \ldots \rightarrow (5)$$

Thus $g^{(1)}(h), g^{(1)}(h/2) \ldots$ given by (5.49) are $O(h^4)$ approximations to $g(x)$.

Eliminating c_2 from (5.49)

We obtain

$$g^2(h) = \frac{4^2 g^{(1)}(h/2) - g^{(1)}(h)}{4^2 - 1} + \frac{1}{64} c_3 h^6 + \ldots \rightarrow (6)$$

With gives an $O(h^6)$ apponoximation.

Thus the successive higher order result can be obtained from the formula.

$$g^{(m)}(h) = \frac{4^m g^{(m-1)}(h/2) - g^{(m-1)}(h)}{4^m - 1} \rightarrow (7)$$

$m = 1, 2, 3.$

where

$$g^0(h) = g(h)$$

This procedure is called repeated extrapolation to the limit. The sucessive values of $g^{(m)}(h)$ for various values of M can be evaluated as givens in Table.

It may be noted, that Table the successive entries in a particular column give better approximations than the preceding entries. similarly the successive columns give better approximations that the preceding columns. The best results can be obtained from the Lower diagonal terms the extrapolation can be stopped when

$$\left| g^{(k)}(h) - g^{(k-1)}(h/2) \right| < \epsilon \rightarrow (8)$$

for a given error tolerance ϵ.

Table Extrapolation table.

order h	Second	Fourth	sixth	Eights.
h	$g(h)$			
		$g^{(1)}(h)$		
$h/2$	$g(h/2)$		$g^{2}(h)$	
		$g^{(1)}(h/2)$		$g^{(3)}(h)$
			$g^{2}(h/2)$	
$h/2^3$	$g(h/2^2)$			
		$g^{(1)}(1/2^2)$		
$h/2^3$	$g(h/2^3)$			

Example

5.3 The following table of values is given

x	-1	1	2	3	4	5	7
$f(x)$	1	1	16	81	256	625	2401

Using the formula $f'(x_1) = \dfrac{(f(x_2) - f(x_0))}{2h}$ and the Richardson extrapolation, find $f'(3)$.

Soln:

Using the Taylor series expansions. We find,

$$\frac{f(x_2) - f(x_0)}{2h} = f'(x_1) + \frac{h^2}{6} f^{(3)}(x_1) + \frac{h^4}{120} f^{(5)}(x_1) + \cdots$$

Hence, for extrapolation.

We find.

h	$f'(3)$ $O(h^2)$	$O(h^4)$	$O(h^6)$
4	300		
		108	
			108
2	156		
		108	
1	120		

Obviously $f'(3) = 108$

Must be the extract solution as the exact arithmetic is done and

$$g^{(1)}(4) = g^{(1)}(2) = g^{(2)}(4).$$

This is true,

Since the data reperents $f(x) = x^4$ and the second column must produce

The exact solution as the loading error tern is $ch^4 f^{(5)}(\xi)$.

Partial Differentiation :-

we can any of three techniques discussed in the previous sections to obtain numerical partial differentiation methods. we consider only one variable at a time and treat the remaining variables as constants. we consider here a function $f(x,y)$ of two variables only. Let the values of the function $f(x,y)$ be given at a set of points (x_i, y_j) in the (x,y) plane with spacing h and k in x and y directions respectively. we have

$$x_i = x_0 + ih \quad , \quad y_j = y_0 + jk \quad , \quad i,j = 1,2..$$

we can now we write,

$$\left(\frac{\partial f}{\partial x}\right)_{(x_i, y_j)} = \frac{f_{i+1,j} - f_{i,j}}{h} + O(h)$$

$$= \frac{f_{i,j} - f_{i-1,j}}{h} + O(h)$$

$$\left(\frac{\partial f}{\partial x}\right)_{(x_i, y_j)} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} + O(h^2).$$

where $f_{i,j} = f(x_i, y_j)$

III'ly we can write,

$$\left(\frac{\partial f}{\partial y}\right)_{(x_i, y_j)} = \frac{f_{i,j+1} - f_{i,j}}{k} + O(k)$$

$$= \frac{f_{i,j} - f_{i,j-1}}{k} + O(k)$$

$$\left(\frac{\partial f}{\partial y}\right)_{(x_i, y_j)} = \frac{f_{i,j+1} - f_{i,j-1}}{2k} + O(k^2)$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(x_i, y_j)} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} + O(h^2)$$

$$= \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2}.$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(x_i, y_j)} = \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{k^2} + O(k^2)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Since we write $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)_{(x_i, y_j)}$

$$\approx \frac{\partial}{\partial x}\left(\frac{f_{i,j+1} - f_{i,j-1}}{2k}\right)$$

$$\approx \frac{1}{2k}\left[\frac{f_{i+1,j+1} - f_{i-1,j+1}}{2h} - \frac{f_{i+1,j-1} - f_{i-1,j-1}}{2h}\right]$$

$$\approx \frac{f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}}{4hk}.$$

The method (5.57) is of $O(h^2+k^2)$ we can also write.

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(x_i, y_j)} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_{(x_i, y_j)}$$

$$\approx \frac{\partial}{\partial y}\left(\frac{f_{i+1,j} - f_{i-1,j}}{2h}\right)$$

$$\approx \frac{1}{2h}\left[\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2k} - \frac{f_{i-1,j+1} - f_{i-1,j-1}}{2k}\right]$$

$$\approx \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4hk}$$

Example 5.4

Find the Jacobian matrix for the system of equations $f_1(x,y) = x^2+y^2-x=0$

$f_2(x,y) = x^2-y^2-y=0$. at point $(1,1)$ using methods.

$$\left(\frac{\partial f}{\partial x}\right)_{(x_i, y_j)} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}.$$

$$\left(\frac{\partial f}{\partial y}\right)_{(x_i, y_j)} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}.$$

with $h=k=1$

The Jacobian Matrix is given as

$$\vec{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

we have $x_i = 1 \quad y_j = 1$

$$\left(\frac{\partial f_1}{\partial x}\right)_{(1,1)} = \frac{f_1(1+h,1) - f_1(1,1)}{2h}$$

$$= \frac{f_1(1+1,1) - f_1(1,1)}{2(1)}$$

$$= \frac{f_1(2,1) - f_1(1,1)}{2}$$

$$f_1(2,1) = x^2+y^2-x=0$$

$$= 2^2+1^2-2 = 4+1-2 = 3$$

$$f_1(1,1) = 1^2+1^2-1 = 1+1-1$$

$$= 2-1 = 1$$

$$\Rightarrow \frac{f_1(2,1) - f_1(1,1)}{2}$$

$$= \frac{3-1}{2} = \frac{2}{2} = 1$$

$$\left(\frac{\partial f_1}{\partial x}\right)_{(1,1)} = 1.$$

$$\left(\frac{\partial f_1}{\partial y}\right)_{(1,1)} = \frac{f_1(1,1+k) - f_1(1,1)}{2k}$$

$$= \frac{f_1(1,2) - f_1(1,1)}{2}$$

$$\left(\frac{\partial f_1}{\partial y}\right)_{(1,1)} = 2.$$

$$f(1,2) = 1^2 + 2^2 - 1$$
$$= 5 - 1$$
$$= 4$$
$$f(1,1) = 1$$

$$\left(\frac{\partial f_2}{\partial x}\right)_{(1,1)} = \frac{f_2(1+h,1) - f_2(1,1)}{2h}$$

$$= \frac{f_2(2,1) - f_2(1,1)}{2}$$

$$= 2$$

$$\left(\frac{\partial f_2}{\partial y}\right)_{(1,1)} = \frac{f_2(1,1+k) - f_2(1,1)}{2k}$$

$$= \frac{f_2(1,2) - f_2(1,1)}{2}$$

$$= -3$$

Hence we got.

$$J = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$$

UNIT-4

Differentiation & Integration

Methods based on undetermined co-efficients and Gauss methods :

In the integration method, if we have to determine the nodes and weights, then this method is called Gaussian Integration method. In this method, the interval $[a,b]$ is changed into $[-1,1]$, then the linear transformation is

$$x = pt + q$$

$x = a,\ t = -1 \Rightarrow a = -p + q$

$x = b,\ t = 1 \Rightarrow b = p + q$

$$a+b = -p+q+p+q$$

$$a+b = 2q$$

$$q = \frac{a+b}{2}$$

$$p = \frac{b-a}{2}$$

$$\therefore\ x = \frac{b-a}{2}t + \frac{b+a}{2}$$

Then, we consider the integral in the form

$$\int_{-1}^{1} w(x) f(x)\, dx = \sum_{k=0}^{n} \lambda_k f_k \rightarrow (1)$$

Gauss legendre integration method :

For $w(x) = 1$, the method ① reduces

to $$\int_{-1}^{1} f(x)\,dx = \sum_{k=0}^{n} \lambda_k f_k \rightarrow ②$$

is a Gauss legendre method. In this case, all the nodes and weights are unknown.

For $n = 2$, it becomes

$$I = \int_{-1}^{1} f(x)\,dx = \sum_{k=0}^{n} \lambda_k f_k$$

$$\therefore \int_{-1}^{1} f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2) \rightarrow ③$$

There are six unknowns in ③ and it can be made exact for polynomials of degree upto five $(1, x, x^2, x^3, x^4, x^5)$

For $f(x) = x^i$, $i = 0(1)5$, we get

$$\lambda_0 + \lambda_1 + \lambda_2 = 2 \quad (\text{when } f(x) = 1) \rightarrow Ⓐ$$

$$\left[\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} 1\,dx = [x]_{-1}^{1} = 1 + 1 = 2 \right]$$

$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0$ (when $f(x) = x$) $\rightarrow$ (B)

$\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = 2/3$ (when $f(x) = x^2$) $\rightarrow$ (C)

$\lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 = 0$ (when $f(x) = x^3$) $\rightarrow$ (D)

$\lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 = \frac{2}{5}$ (when $f(x) = x^4$) $\rightarrow$ (E)

$\lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_2^5 = 0$ (when $f(x) = x^5$) $\rightarrow$ (F)

Solving the above equations,

Let $x_1 = 0$,

(D) $\Rightarrow \lambda_0 x_0^3 + 0 + \lambda_2 x_2^3 = 0$

(D) $\Rightarrow \lambda_0 x_0^3 = -\lambda_2 x_2^3$

(B) $\Rightarrow \lambda_0 x_0 + 0 + \lambda_2 x_2 = 0$

$\lambda_0 x_0 = -\lambda_2 x_2$

$\frac{(D)}{(B)} \Rightarrow \frac{\lambda_0 x_0^3}{\lambda_0 x_0} = \frac{\lambda_2 x_2^3}{\lambda_2 x_2}$

$x_0^2 = x_2^2$

$x_0^2 - x_2^2 = 0$

But, $x_0 - x_2 \neq 0$,

$\boxed{x_0 = -x_2}$ (by symmetry).

(B) $\Rightarrow \lambda_0 x_0 + 0 + \lambda_2 x_2 = 0$

$\lambda_0 x_0 + \lambda_2(-x_0) = 0$

$\lambda_0 x_0 - \lambda_2 x_0 = 0$

$\lambda_0 x_0 = \lambda_2 x_0$

$\boxed{\lambda_0 = \lambda_2}$

$$\frac{E}{C} \Rightarrow$$

$$E \Rightarrow \lambda_0 x_0^4 + 0 + \lambda_2 x_2^4 = 2/5$$

$$\lambda_0 x_0^4 + \lambda_0(-x_0)^4 = 2/5$$

$$\lambda_0 x_0^4 + \lambda_0 x_0^4 = 2/5$$

$$E \Rightarrow 2\lambda_0 x_0^4 = 2/5$$

$$C \Rightarrow \lambda_0 x_0^2 + 0 + \lambda_2 x_2^2 = 2/3$$

$$\lambda_0 x_0^2 + \lambda_0(-x_0)^2 = 2/3$$

$$\lambda_0 x_0^2 + \lambda_0 x_0^2 = 2/3$$

$$C \Rightarrow 2\lambda_0 x_0^2 = 2/3$$

$$\frac{E}{C} \Rightarrow \frac{2\lambda_0 x_0^4}{2\lambda_0 x_0^2} = \frac{2/5}{2/3}$$

$$x_0^2 = 2/5 \times 3/2$$

$$x_0^2 = 3/5$$

$$x_0 = \pm\sqrt{3/5}$$

$$x_0 = -\sqrt{3/5} \qquad x_2 = \sqrt{3/5}$$

$$C \Rightarrow \lambda_0\left(-\sqrt{3/5}\right)^2 + 0 + \lambda_0\left(\sqrt{3/5}\right) = 2/3$$

$$\lambda_0(3/5) + \lambda_0(3/5) = 2/3$$

$$2\lambda_0(3/5) = 2/3$$

$$2\lambda_0 = \frac{2}{3} \times \frac{5}{3}$$

$$2\lambda_0 = \frac{10}{9}$$

$$\lambda_0 = \frac{10}{18}$$

$$\boxed{\lambda_0 = 5/9}$$

$$\lambda_0 = 5/9 \qquad \lambda_2 = 5/9$$

$$① \Rightarrow 5/9 + \lambda_1 + 5/9 = 2$$

$$10/9 + \lambda_1 = 2$$

$$\boxed{\lambda_1 = 8/9}$$

Substitute all values in ③,

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{3/5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{3/5}\right)$$

$$= 1/9 \left[5f\left(-\sqrt{3/5}\right) + 8f(0) + 5f\left(\sqrt{3/5}\right)\right]$$

With the error term

$$R_5 = \frac{c}{6!} f^{6}(\xi), \quad -1 < \xi < 1 \rightarrow ④$$

where, $c = \int_{-1}^{1} f(x)\,dx - (\lambda_0 x_0^6 + \lambda_1 x_1^6 + \lambda_2 x_2^6)$

$$= \int_{-1}^{1} x^6\,dx - (\lambda_0 x_0^6 + \lambda_1 x_1^6 + \lambda_2 x_2^6)$$

$$= \left[\frac{x^7}{7}\right]_{-1}^{1} - \left[\frac{5}{9}\left(-\sqrt{3/5}\right)^6 + 0 + \frac{5}{9}\left(\sqrt{3/5}\right)^6\right]$$

$$= \frac{2}{7} - \left[\frac{5}{9} \times \frac{27}{125} + \frac{5}{9} \times \frac{27}{125}\right]$$

$$= \frac{2}{7} - \frac{6}{25}$$

$$= \frac{50-42}{175}$$

$$c = \frac{8}{175}$$

$$R_5 = \frac{8/175}{6!} f^6(\xi)$$

$$R_5 = \frac{8}{175(6!)} f^6(\xi) \quad , \quad -1 < \xi < 1$$

<u>Nodes and weights for Gauss-Legendre Integration methods:</u>

n	nodes x_k	weights λ_k
0	0	2
1	$\pm 0.57735 \ (\pm \frac{1}{\sqrt{3}})$	1
2	0 $\pm 0.774596 \ (\pm \sqrt{3/5})$	$0.888889 \ (8/9)$ $\pm 0.555556 \ (5/9)$

One point formula : $\int_{-1}^{1} f(x)\, dx = 2f(0)$

Two point formula : $\int_{-1}^{1} f(x)\, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Problem

1) Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ using Gauss legendre three point formula.

Soln

To use Gauss-legendre formula, we have to change $[0,1]$ to $[-1,1]$.

Then, the transformation is

$$x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$$

$$x = \left(\frac{1-0}{2}\right)t + \left(\frac{1+0}{2}\right)$$

$$x = \frac{1}{2}t + \frac{1}{2}$$

$$dx = \frac{dt}{2}, \quad x = \frac{t+1}{2}$$

$$1+x = 1 + \frac{t+1}{2} = \frac{2+t+1}{2}$$

$$\therefore \quad 1+x = \frac{t+3}{2}$$

$$I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{dt/2}{\frac{t+3}{2}}$$

$$I = \int_{-1}^{1} \frac{dt}{t+3}$$

Gauss legendre three point formula is

$$I = \int_{-1}^{1} \frac{dt}{t+3} = \frac{1}{9}\left[5f\left(-\sqrt{3/5}\right) + 8f(0) + 5f\left(\sqrt{3/5}\right)\right]$$

$$= 1/9\left[\frac{5}{3-\sqrt{3/5}} + \frac{8}{3} + \frac{5}{3+\sqrt{3/5}}\right]$$

$$= \frac{1}{9}\left[\frac{8}{3} + 5\left(\frac{3+\sqrt{3/5}+3-\sqrt{3/5}}{9-3/5}\right)\right]$$

$$= 1/9\left[8/3 + 5\left(\frac{6}{48/5}\right)\right]$$

$$= 1/9\left[\frac{8}{3} + \frac{25}{7}\right]$$

$$= 1/9\left[\frac{56+75}{21}\right]$$

$$= 1/9\left[\frac{131}{21}\right]$$

$$= \frac{131}{189} = 0.693122$$

The exact solution is $I = 0.69314 7$

Lobatto Integration methods:

In this case, $w(x) = 1$ and two end points -1 and 1 are always taken as nodes. The remaining $n-1$ nodes are to be determined. Then, the integration method is

$$\int_{-1}^{1} f(x)\,dx = \lambda_0 f(-1) + \lambda_n f(1) + \sum_{k=1}^{n-1} \lambda_k f_k \quad \to (1)$$

is called Lobatto integration method. Here, there are $2n$ unknowns, this method can be made exact for Polynomial of degree upto $2n-1$.

For $n=2$ in (1), it becomes

$$\int_{-1}^{1} f(x)\,dx = \lambda_0 f(-1) + \lambda_1 f(x_1) + \lambda_2 f(1)$$

It can be made exact for Polynomial of degree upto 3. For $f(x) = x^i$, $i = 0, 1, 2, 3$, we get,

$$\lambda_0 + \lambda_1 + \lambda_2 = 2 \quad \to (A)$$

$$-\lambda_0 + \lambda_1 x_1 + \lambda_2 = 0 \rightarrow (B)$$

$$\lambda_0 + \lambda_1 x_1^2 + \lambda_2 = 2/3 \rightarrow (C)$$

$$-\lambda_0 + \lambda_1 x_1^3 + \lambda_2 = 0 \rightarrow (D)$$

Solve the above equations, we get

Let $x_1 = 0$

$$(B) \Rightarrow -\lambda_0 + 0 + \lambda_2 = 0$$

$$-\lambda_0 = -\lambda_2$$

$$\boxed{\lambda_0 = \lambda_2}$$

$$(C) \Rightarrow \lambda_0 + 0 + \lambda_2 = 2/3 \Rightarrow \lambda_0 + \lambda_0 = 2/3$$

$$2\lambda_0 = 2/3$$

$$\lambda_0 = 1/3$$

$$\lambda_0 = \lambda_2 = 1/3$$

$$(A) \Rightarrow \lambda_0 + \lambda_1 + \lambda_2 = 2$$

$$1/3 + \lambda_1 + 1/3 = 2$$

$$2/3 + \lambda_1 = 2$$

$$\lambda_1 = 2 - 2/3$$

$$\lambda_1 = 4/3$$

Then, it becomes

$$\int_{-1}^{1} f(x)\,dx = 1/3\left[f(-1) + 4f(0) + f(1)\right]$$

with the error term,

$$R_3 = \frac{c}{4!} f^{iv}(\xi) \quad , -1 < \xi < 1$$

Where

$$c = \int_{-1}^{1} f(x)\,dx - (\lambda_0 f(-1) + \lambda_1 f(0) + \lambda_2 f(1))$$

$$= \int_{-1}^{1} x^4\,dx - (\lambda_0 + \lambda_1 x_1^4 + \lambda_2)$$

$$= \left[\frac{x^5}{5}\right]_{-1}^{1} - (1/3 + 0 + 1/3)$$

$$= \frac{2}{5} - \frac{2}{3}$$

$$= \frac{6-10}{15}$$

$$c = -\frac{4}{15}$$

$$R_3 = \frac{-4/15}{4!} f^{iv}(\xi)$$

$$R_3 = \frac{-1}{90} f^{iv}(\xi) \quad , \quad -1 < \xi < 1$$

<u>Nodes and weights for Lobatto Integration methods</u>:

n	nodes x_k	weights λ_k
2	± 1 0	$0.33333\ (1/3)$ $1.33333\ (4/3)$
3	± 1 $\pm 0.4472136\ (\pm\sqrt{1/5})$	$0.166667\ (1/6)$ $0.833333\ (5/6)$

Radau Integration methods :

In this method, $w(x) = 1$ and the lower limit (-1) is fixed mode. The remaining n nodes are to be determined. The integration method of the form,

$$\int_{-1}^{1} f(x)\,dx = \lambda_0 f(-1) + \sum_{k=1}^{n} \lambda_k f_k \longrightarrow (1)$$

is called Radau - Integration method.

Since there are $2n+1$ unknowns (n nodes & $n+1$ weights). It can be made exact for

Polynomial of degree upto $2n$. For $n=2$, it becomes

$$\int_{-1}^{1} f(x)\,dx = \lambda_0 f(-1) + \lambda_1 f(x_1) + \lambda_2 f(x_2) \quad \to (2)$$

There are five unknowns in (2). For $f(x)=x^i$, $i = 0(1)4$, we get

$$\lambda_0 + \lambda_1 + \lambda_2 = 2 \quad \longrightarrow (A)$$

$$-\lambda_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0 \quad \longrightarrow (B)$$

$$\lambda_0 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = 2/3 \quad \longrightarrow (C)$$

$$-\lambda_0 + \lambda_1 x_1^3 + \lambda_2 x_2^3 = 0 \quad \longrightarrow (D)$$

$$\lambda_0 + \lambda_1 x_1^4 + \lambda_2 x_2^4 = 2/5 \quad \longrightarrow (E)$$

Solving the above system of equations, we get –

$$x_1 = \frac{1-\sqrt{6}}{5}$$

$$x_2 = \frac{1+\sqrt{6}}{5}$$

$$\lambda_0 = \frac{2}{9}, \quad \lambda_1 = \frac{16+\sqrt{6}}{18}, \quad \lambda_2 = \frac{16-\sqrt{6}}{18}$$

and it becomes.

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{9} f(-1) + \frac{16+\sqrt{6}}{18} f\left(\frac{1-\sqrt{6}}{5}\right)$$

$$+ \frac{16-\sqrt{6}}{18} f\left(\frac{1+\sqrt{6}}{5}\right) \rightarrow (3)$$

with the error term

$$R_4 = \frac{c}{5!} f^{v}(\xi) \quad , -1 < \xi < 1$$

where

$$c = \int_{-1}^{1} x^5\,dx - \left(-\lambda_0 + \lambda_1 x_1^5 + \lambda_2 x_2^5\right)$$

$$= \int_{-1}^{1} x^5\,dx - \left(-\frac{2}{9} + \left(\frac{16+\sqrt{6}}{18}\right)\left(\frac{1-\sqrt{6}}{5}\right)^5 + \left(\frac{16-\sqrt{6}}{18}\right)\left(\frac{1+\sqrt{6}}{5}\right)^5\right)$$

$$= 0 + 2/9 - \left(\frac{16+\sqrt{6}}{18}\right)\left(\frac{1-\sqrt{6}}{5}\right)^5 + \left(\frac{16-\sqrt{6}}{18}\right)\left(\frac{1+\sqrt{6}}{5}\right)^5$$

$$= \frac{2}{9} - \left(\frac{16+\sqrt{6}}{18}\right)\left(\frac{1-\sqrt{6}}{5}\right)^5 - \left(\frac{16-\sqrt{6}}{18}\right)\left(\frac{1+\sqrt{6}}{5}\right)^5$$

$$c = -\frac{37}{225}$$

Hence $R_4 = \dfrac{-37/225}{5!} f^{v}(\xi)$

$$R_4 = \frac{-37}{225(5!)} f^{v}(\xi), \quad -1 < \xi < 1$$

Nodes and weights for Radau integration method:

n	nodes x_k	weights λ_k
1	-1 0.333333	0.50000 1.50000
2	-1.00000 -0.2898979 0.6898979	0.22222 1.0249717 0.7628061
3	-1.000000 -0.5753189 0.1810663 0.8228241	0.125000 0.6576886 0.7763870 0.4409244

Homework:

1) Evaluate the integral $I = \int_{-1}^{1} (1-x^2)^{3/2} \cos x \, dx$ Using Gauss-legendre three point formula.

Home work :

Evaluate the integral $I = \int_{-1}^{1} (1-x^2)^{3/2} \cos x \, dx$

using Gauss legendre three point formula.

Soln Gauss - legendre three point Formula

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5 f\left(-\sqrt{3/5}\right) + 8 f(0) + 5 f\left(\sqrt{3/5}\right)\right]$$

Here, $f(x) = (1-x^2)^{3/2} \cos x$

$$f\left(-\sqrt{3/5}\right) = \left(1-\frac{3}{5}\right)^{3/2} \cos\sqrt{3/5}$$

$$= \left(\frac{2}{5}\right)^{3/2} \cos\sqrt{3/5}$$

$$f\left(\sqrt{3/5}\right) = \left(1-3/5\right)^{3/2} \cos\left(-\sqrt{3/5}\right)$$

$$= \left(2/5\right)^{3/2} \cos\left(\sqrt{3/5}\right)$$

$f(0) = 1 \quad \cos(0) = 1$

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5 \times 2/5 \cdot \sqrt{2/5} \cos\sqrt{3/5} + 8 + 5 \times \frac{2}{5}\sqrt{2/5} \cos\sqrt{3/5}\right]$$

$$= \frac{2}{9}\left[\sqrt{2/5} \cos\sqrt{3/5} + 4 + \sqrt{2/5} \cos\sqrt{3/5}\right]$$

$$= \frac{4}{9}\left[2 + \sqrt{2/5} \cos\sqrt{3/5}\right]$$

$$= \frac{4}{9}\left[2 + 0.4520179\right]$$

$$= \frac{9.8080716}{9}$$

$$\int_{-1}^{1} f(x)\,dx = 1.08979$$

Gauss chebyshev:

When $w(x) = \dfrac{1}{\sqrt{1-x^2}}$, the method of the form

$$\int_{-1}^{1} \frac{1}{\sqrt{(1-x^2)}} f(x)\,dx = \sum_{k=0}^{n} \lambda_k f_k \longrightarrow ①$$

are called Gauss chebyshev integration method.

The methods are exact for polynomials of degree $2n+1$

The nodes x_k's are found to be the roots of the chebyshev polynomials.

$$T_{n+1}(x) = \cos\left((n+1)\cos^{-1}x\right) = 0.$$

we get

$$x_k = \cos\left(\frac{(2k+1)\pi}{2n+2}\right) \longrightarrow ②, \quad k = 0, 1, \ldots n$$

$\because$ the method is to be exact $f(x) = x^i$, $i = 0(5)$

we get the system of equation.

②

$$\left.\begin{aligned}
\lambda_0 + \lambda_1 + \lambda_2 &= \pi \\
\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 &= 0 \\
\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 &= \pi/2 \\
\lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 &= 0 \\
\lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 &= 3\pi/8 \\
\lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_1^5 &= 0
\end{aligned}\right\} \rightarrow (5)$$

we obtain from (5)

$$x_k = \cos(2k+1)\pi/6, \quad k = 0, 1, 2, \ldots \text{ (or)}$$

$k=0 \Rightarrow x_0 = \cos[2(0)+1]\pi/6$

$= \cos(1)\pi/6 = \cos\pi/6$

$\boxed{x_0 = \sqrt{3}/2}$

$k=1 \Rightarrow x_1 = \cos[2(1)+1]\pi/6$

$= \cos[3]\pi/6 = \cos\pi/2$

$= \cos(\pi/2)$

$\boxed{x_1 = 0}$

$k=2 \Rightarrow x_2 = \cos[2(2)+1]\pi/6$

$= \cos[4+1]\pi/6$

$= \cos[5]\pi/6$

$= \cos 5\pi/6$

$\boxed{x_2 = -\sqrt{3}/2}$

Substituting the values x_0, x_1 and x_2 in ③ we get,

$$\lambda_0+\lambda_1+\lambda_2 = \pi \quad \text{---(a)}$$

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0.$$

$$\lambda_0(\sqrt{3}/2) + \lambda_1(0) + \lambda_2(-\sqrt{3}/2) = 0.$$

$$\Rightarrow \lambda_0(\sqrt{3}/2) - \lambda_2(\sqrt{3}/2) = 0.$$

$$\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = \pi/2$$

$$\lambda_0(\sqrt{3}/2)^2 + \lambda_1(0)^2 + \lambda_2(-\sqrt{3}/2) = \pi/2$$

$$\lambda_0(3/4) + \lambda_2(3/4) = \pi/2$$

$$(\lambda_0+\lambda_2)\, 3/4 = \pi/2$$

$$\lambda_0+\lambda_2 = \pi/\cancel{2}_1 \times \frac{\cancel{4}^2}{3}$$

$$\boxed{\lambda_0+\lambda_2 = 2\pi/3} \rightarrow \text{(i)}$$

$$\text{(i)} \Rightarrow \boxed{\lambda_0 = 2\pi/3 - \lambda_2} \rightarrow \text{(ii)}$$

$$\text{(a)} \Rightarrow \lambda_0+\lambda_2+\lambda_1 = \pi$$

$$2\pi/3 + \lambda_1 = \pi$$

$$\lambda_1 = \pi - \frac{2\pi}{3}$$

$$\boxed{\lambda_1 = \frac{3\pi-2\pi}{3} = \frac{\pi}{3}} \rightarrow \text{(iii)}$$

(ii) & (iii) in ⓐ

$$\Rightarrow \quad \lambda_0 + \lambda_1 + \lambda_2 = \pi$$

$$\pi/3 + \left[\frac{2\pi}{3} - \lambda_2\right] + \lambda_2 = \pi$$

$$\pi/3 + \frac{2\pi}{3}$$

$$\pi/3 + 2\pi/3 = \pi$$

$$3\pi/3 = \pi$$

$$\boxed{\pi = \pi}$$

$$\therefore \quad \boxed{\lambda_1 = \pi/3 = \lambda_2 = \lambda_3}$$

Thus we get the method,

$$\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{(1-x^2)}} = \frac{\pi}{3}\left[f\left(\sqrt{3}/2\right) + f(0) + f\left(-\sqrt{3}/2\right)\right]$$

with the error term

$$R_5 = \frac{c}{6!} f^{vi}(\xi), \quad -1 < \xi < 1.$$

Where

$$c = \int_{-1}^{1} \frac{x^6}{\sqrt{(1-x^2)}}\,dx - (\lambda_0 x_0^6 + \lambda_1 x_1^6 + \lambda_2 x^6)$$

$$= \pi/32$$

It may be verified that eqn ① all the weights λ_k's and are given by $\lambda_k = \frac{\pi}{n+1}$, $k = 0, 1, \ldots n$.

Example: Evaluate the integral

$$I = \int_{-1}^{1} (1-x^2)^{3/2} \cos x \, dx.$$ using Gauss - chebyshev three point formula:

soln: Given, $I = \int_{-1}^{1} (1-x^2)^{3/2} \cos x \, dx$

Gauss - chebyshev three point formula is

$$\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{(1-x^2)}} = \frac{\pi}{3}\left[f(\sqrt{3}/2) + f(0) + f(-\sqrt{3}/2)\right]$$

Let $f(x) = (1-x^2)^2 \cos x$ and

$x = \sqrt{3}/2$, $x=0$, $x = -\sqrt{3}/2$ we get.

$$f(\sqrt{3}/2) = \left[1 - (\sqrt{3}/2)^2\right]^2 \cos x$$

$$= \left[1 - 3/4\right]^2 \cos x \quad = \left(\frac{4-3}{4}\right]^2 \cos x$$

$$= \left[\frac{1}{4}\right]^2 \cos x$$

$$\boxed{f(\sqrt{3}/2) = 1/16 \cos x}$$

$$f(0) = [1-0]^2 \cos x \quad = (1)^2 \cos x$$

$$\boxed{f(0) = \cos x}$$

$$f(-\sqrt{3}/2) = \left[1-(-\sqrt{3}/2)^2\right]^2 \cos x$$

$$= \left[1-3/4\right]^2 \cos x = \left[\frac{4-3}{4}\right]^2 \cos x$$

$$= (1/4)^2 \cos x$$

$$\boxed{f(-\sqrt{3}/2) = 1/16 \cos x}$$

$$\therefore I = \int_{-1}^{1} (1-x^2)^{3/2} \cos x \, dx$$

$$= \pi/3 \left[\cancel{1/16}\, f(\sqrt{3}/2) + f(0) + f(-\sqrt{3}/2) \right]$$

$$= \pi/3 \left[1/16 \cos x + \cos x + 1/16 \cos x \right]$$

$$= \pi/3 \left[1/16 \cos(\sqrt{3}/2) + \cos(1) + 1/16 \cos(-\sqrt{3}/2) \right]$$

$$= \pi/3 \left[1/16 \cos(\sqrt{3}/2) + \cos(1) + 1/16 \cos(\sqrt{3}/2) \right]$$

$$\left[\because \cos(-x) = \cos(x)\right]$$

$$= \pi/3 \left[2/16 \cos(\sqrt{3}/2) + 1 \right]$$

$$= \pi/3 \left[1/8 \cos(\sqrt{3}/2) + 1 \right] = \frac{\pi}{3\times 8}\left[\cos \sqrt{3}/2 + 8\right]$$

$$= \frac{\pi}{24}\left[\cos \sqrt{3}/2 + 8\right]$$

$$\boxed{I = 1.132002}$$ //.

Gauss Laguerre Integration Method:

Here we consider the integral of the form

$$\int_0^\infty e^{-x} f(x)\, dx = \sum_{k=0}^{n} \lambda_k f_k \cdots \longrightarrow (1)$$

The nodes x_k's are found to be the roots,

$$L_{n+1}(x) = (-1)^{n+1} e^{x} \frac{d^{n+1}}{dx^{n+1}} (e^{-x} x^{n+1}) \longleftarrow (2)$$

we have

$$L_0(x) = 1$$

put $n=-1$ in (2)

$$L_0(x) = (-1)^{-1+1} e^{x} \frac{d^{-1+1}}{dx^{-1+1}} (e^{-x} x^{-1+1})$$

$$= (-1)^0 e^{x} \frac{d^0}{dx^0} (e^{-x} x^0) = 1\, e^{x} (e^{-x} x^0)$$

$$= e^{x} \cdot e^{-x} = e^{x-x} = e^0 = 1$$

$$\boxed{L_0(x) = 1}$$

put $n=0$ in (2)

$$L_1(x) = (-1)^1 e^{x} \frac{d^1}{dx^1} (e^{-x} x^1)$$

$$= (-1) e^{x} [-x e^{-x} - (-e^{-x})]$$

$$= (-1) e^{x} \cdot e^{-x} [-x+1] = e^{x-x} (-1)(-x+1)$$

$$= e^0 (-1)(-x+1)$$

$$\boxed{L_1(x) = x-1}$$

put $n=1$ in ②

$$L_2(x) = (-1)^2 e^x \frac{d^2}{dx^2}\left(e^{-x} \cdot x^2\right)$$

$$= e^x \frac{d^1}{dx^1}\left[-\underbrace{e^{-x}x^2}_{uv} - \underbrace{2x e^{-x}}_{uv}\right]$$

$$= e^x \left[\left[-x^2 e^{-x}(-1) - 2x(-e^{-x})\right] - \left[-2e^{-x} - 2e^{-x}x(-1)\right]\right]$$

$$= e^x \left[x^2 e^{-x} - 2x e^{-x} + 2e^{-x} - 2e^{-x}x\right]$$

$$= e^x \cdot e^{-x}\left[x^2 - 2x + 2 - 2x\right]$$

$$= e^{x-x}\left[x^2 - 4x + 2\right]$$

$$= e^0 (x^2 - 4x + 2$$

$$\boxed{L_2(x) = x^2 - 4x + 2}$$

put $n=2$ in ②

$$L_3(x) = (-1)^{2+1} e^x \frac{d^3}{dx^3}\left(e^{-x} x^{2+1}\right)$$

$$= (-1)^3 e^x \frac{d^3}{dx^3}\left(e^{-x} x^3\right) = (-1)^3 e^x \frac{d^3}{dx^3}\left(x^3 e^{-x}\right)$$

$$= \cancel{(-1)^3} \cancel{e^x} \frac{d}{dx}\left[x^2 \cancel{e^{-x}} - \cancel{4}x e^{-x} \cancel{+} 2x \cancel{e^{-x}}\right]$$

$$L_3(x) = (-1)^3 e^x \frac{d^2}{dx^2}\left[x^3 e^{-x}(-1) - e^{-x}(3x^2)\right]$$

$$= (-1)^3 e^x \frac{d^2}{dx^2}\left[-x^3 e^{-x} - e^{-x} 3x^2\right]$$

$$= (-1)^3 e^x \frac{d}{dx}\left[-(3x^2 e^{-x} - x^3 e^{-x}(-1)) - [6x e^{-x} - 3x^2 e^{-x}(-1)]\right]$$

$$= (-1)^3 e^x \frac{d}{dx}\left[-3x^2 e^{-x} + x^3 e^{-x} - 6x e^{-x} - 3x^2 e^{-x}\right]$$

$$= (-1)^3 e^x \Big[-(3x^2 e^{-x}(-1) - e^{-x} 6x) - (x^3 e^{-x}(-1) - e^{-x}(3x)) - (6x e^{-x}(-1) - 6e^{-x}) - (3x^2 e^{-x}(-1) - 3e^{-x}(2x))\Big]$$

$$= (-1)^3 e^x \Big[3x^2 e^{-x} + e^{-x} 6x + x^3 e^{-x} + e^{-x} 3x + 6x e^{-x} + 6e^{-x} + 3x^2 e^{-x} + 3e^{-x} 2x\Big]$$

$$= (-1)^3 e^x \Big[3x^2 e^{-x} + 6x e^{-x} + x^3 e^{-x} + 3x^2 e^{-x} + 6x e^{-x} + 6e^{-x} + 3x^2 e^{-x} + 6x e^{-x}\Big]$$

$$= (-1)^3 e^x \cdot e^{-x}\left[3x^2 + 6x + x^3 + 3x^2 + 6x + 6 + 3x^2 + 6x\right]$$

$$= (-1)^3 e^0 \left[-x^3 + 9x^2 + 18x + 6\right]$$

$$= (-1)(1)\,(-x^3 + 9x^2 - 18x + 6] = x^3 - 9x^2 + 18x - 6$$

$$\boxed{L_3(x) = x^3 - 9x^2 + 18x - 6}$$

The corresponding weights are obtained from the relation

$$\lambda_k = \int_0^\infty \frac{e^{-x} L_{n+1}(x)}{(x-x_k)\, L'_{(n+1)}(x_k)}\, dx \longrightarrow (3)$$

(1) ⇒ producess exact results for polynomials of degree upto $2n+1$.

The nodes and weigth for the (1) for $n=1(1)5$ are given table: $L_n(x)$ are orthogonal w.r.to weight function e^{-x} on $(0,\infty)$

$$\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = 0 \quad , \quad m \neq n.$$

n	nodes x_k	weights λ_k.
1	0.5857864376 3.4142135624	0.8535533906 0.1464466094
2	0.4157745568 2.2942803603 6.2899450829	0.7110930099 0.2785177336 0.0103892565
3 .	0.3225476896 1.7457611012 4.5366202969 9.3950709123	0.6031541043 0.3574186924 0.0388879085 0.0005392947.

4	0.2635603197	0.5217556106
	1.4134030591	0.3986668111
	3.5964257710	0.0759424497
	7.0858100059	0.0036117587
	12.6408008443	0.0000233700
5	0.2228466042	0.4589646740
	1.1889321017	0.4170008308
	2.9927363261	0.1133733821
	5.7751435691	0.0103991975
	9.8374674184	0.0002610172
	15.9828739806	0.0000008985

Gauss Hermite: Integration Methods:

The methods of the form

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = \sum_{k=0}^{n} \lambda_k f_k \longrightarrow (1)$$

are called Gauss-Hermite Integration methods

The nodes x_k's are the roots of the Hermite polynomial

$$H_{n+1}(x) = (-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}} \left(e^{-x^2}\right) \longrightarrow (2)$$

we have

$n = -1$

$$H_0(x) = (-1)^0 e^{x^2} \frac{d^0}{dx^0} (e^{-x^2})$$

$$= 1\, e^{x^2} \cdot (e^{-x^2})$$

$$= e^{-x^2+x^2} = e^0$$

$$\boxed{H_0(x) = 1}$$

$n = 0$

$$H_1(x) = (-1)^1 e^{x^2} \frac{d}{dx} (e^{-x^2})$$

$$= (-1) e^{x^2} \left(-e^{-x^2}(2x)\right)$$

$$= e^{x^2} \cdot e^{-x^2} (2x)$$

$$= 2x\, e^{-x^2+x^2}$$

$$= 2x \cdot e^0$$

$$\boxed{H_1(x) = 2x}$$

$n = 1$

$$H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2} (e^{-x^2})$$

$$= e^{x^2} \, d/dx \left(-e^{-x^2}(2x)\right)$$

$$= e^{x^2} \cdot \left[-e^{-x^2}(2) - 2x\left(-e^{-x^2}(2x)\right)\right]$$

$$= \cancel{e^{x^2} \cdot e^{-x^2}}$$

$$= e^{x^2} \cdot e^{-x^2} \left[-2 + 4x^2\right]$$

$$= e^0 (4x^2 - 2)$$

$$\boxed{H_2(x) = 2(2x^2 - 1)}$$

put $n = 2$

$$H_3(x) = (-1)^3 e^{x^2} \frac{d^3}{dx^3} (e^{-x^2})$$

$$= (-1) e^{x^2} \frac{d^2}{dx^2} \left(e^{-x^2} (-2x) \right)$$

$$= (-1) e^{x^2} \frac{d^2}{dx^2} (-2x e^{-x^2})$$

$$= (-1) e^{x^2} \frac{d}{dx} \left[-2x e^{-x^2} (-2x) - (-2 e^{-x^2} (1)) \right]$$

$$= (-1) e^{x^2} \frac{d}{dx} \left[4x^2 e^{-x^2} + 2 e^{-x^2} \right]$$

$$= (-1) e^{x^2} \left[4x^2 e^{-x^2} (-2) - 4 e^{-x^2} (-2x) + 2 e^{-x^2} (-2x) \right]$$

$$= (-1) e^{x^2} \left[-8x^2 e^{-x^2} + 8x e^{-x^2} + 4x e^{-x^2} \right]$$

$$= (-1) e^{x^2} (-8x^2 e^{-x^2} + 12x e^{-x^2})$$

$$= (-1) . e^{x^2} . e^{-x^2} (-8x^2 + 12x)$$

$$= (-1) (-8x^2 + 12x)$$

$$= 8x^2 - 12x$$

$$\boxed{H_3(x) = 4(2x^2 - 3x)}$$

The corresponding weights λ_k's are obtained from the relation

$$\lambda_k = \int_{-\infty}^{\infty} \frac{e^{-x^2} H_{n+1}(x)}{(x-x_k)\, H'_{n+1}(x_k)}\, dx.$$

the eqn ① produces exact results for polynomials of degree upto $2n+1$.

The nodes and weights for eqn ① for $n = 0(1)5$ given table: The hermite polynomial are orthogonal w.r.to the weight function e^{-x^2} on $(-\infty, \infty)$.

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 0 \quad , \quad m \neq n.$$

n	nodes x_k	weights λ_k.
0	0.0000000000	1.7724538569
1	± 0.7071067812	0.8862289256
2	0.0000000000	1.1816359006
	±1.2247448714	0.2954689752
3	± 0.5246476233	0.8049140900
	± 1.6506801239	0.0812128354
4	0.0000000000	0.9453087265
	± 0.9585724646	0.3936193232
	± 2.0201828705	0.0199532421

5	$\pm$ 0.4360774119	0.7264295952
	$\pm$ 1.3358490740	0.1570673203
	$\pm$ 2.3506049737	0.0045300099

~~Theorem~~:

Consider again the integration formula

$$I = \int_a^b w(x) f(x)\, dx \approx \sum_{k=0}^{n} \lambda_k f_k \longrightarrow ①$$

Theorem: If x_k are selected as zeros of an orthogonal polynomial, orthogonal with respect to the weight function $w(x)$, over (a,b) then the formula ① has precision $2n+1$ (or exact for polynomials of degree $\leq 2n+1$). Further $\lambda_k > 0$.

Soln:

Let $f(x)$ be a polynomial of degree less than (or) equal to $2n+1$.

Let $q_n(x)$ be the Lagrange interpolating polynomial of degree $\leq n$, interpolating the data (x_i, f_i), $i = 0, 1, \ldots n$

$$q_n(x) = \sum_{k=0}^{n} l_k(x)\, f(x_k) \quad \text{with}$$

$$l_k(x) = \frac{\pi(x)}{(x-x_k)\,\pi'(x_k)}.$$

The polynomial $[f(x) - q_n(x)]$ has zeroes at $x_0, x_1, \ldots x_n$.

Hence it can be written as

$$f(x) - q_n(x) = P_{n+1}(x)\, r_n(x) \longrightarrow (5)$$

Where $r_n(x)$ is a polynomial of degree atmost n &

$P_{n+1}(x_i) = 0, \quad i = 0, 1, 2 \ldots n.$

Integrating (2) we get.

$$\int_a^b w(x)\,[f(x) - q_n(x)]\,dx = \int_a^b w(x)\, P_{n+1}(x)\, r_n(x)\, dx.$$

(or)

$$\int_a^b w(x)\, f(x)\, dx = \int_a^b w(x)\, q_n(x)\, dx + \int_a^b w(x)\, P_{n+1}(x)\, r_n(x)\, dx$$

∴ the second integral on the right hand side is zero. If $P_{n+1}(x)$ is an orthogonal polynomial. their degree $\leq n$.

$$\int_a^b w(x)\, p_{n+1}(x)\, r_n(x)\, dx = 0.$$

ie) $$\int_a^b w(x) f(x)\, dx = \int_a^b w(x)\, q_n(x)\, dx + 0$$

$$= \int_a^b w(x)\, q_n(x)\, dx$$

$$= \sum_{k=0}^{n} \lambda_k f(x_k) \rightarrow \text{from } ①.$$

$\longrightarrow$ ③.

where
$$\lambda_k = \int_a^b w(x)\, l_k(x)\, dx.$$

this proves that the formula of ① has precision $2n+1$.

Observe $l^2(x)$ is a polynomial of degree $\leq 2n+1$.

Choosing $f(x) = l_j^2(x)$ & substituting ③ we have.

$$\int_a^b w(x)\, l_j^2(x)\, dx = \sum_{k=0}^{n} \lambda_k\, l_j^2(x_k).$$

$\because$ $l_j(x_k) = \delta_{jk}$ we get,

$$\lambda_j = \int_a^b w(x)\, l_j^2(x)\, dx > 0.$$

$\therefore$ Hence the proved.

unit - V

Define: (Mesh points (or) grid points)

A numerical solution of the differential equation is to partition the interval $[t_0, b]$ on which the solution is desire into a finite number of subintervals by the points,

$$t_0 < t_1 < t_2 \cdots < t_N = b$$

The point are called the mesh points (or) grid points.

Define (mesh spacing (or) step length)

The spacing between the points is given by

$$h_j = t_j - t_{j-1}, \quad j = 1, 2, 3, \ldots N.$$

which is called mesh spacing (or) step length

For simplicity we assume that the points are spaced uniformly, (i.e.) $h_j = h$ = constant $, j = 1, 2, \ldots N$

The mesh points are given by

$$t_j = t_0 + jh, \quad j = 0, 1, 2, \ldots N$$

Define (Numerical Solution)

A numerical methods we determine a number u_j which is an approximation to the value of the solution $u(t)$ at the point t_j. The set of numbers $\{u_j\}$, (i.e) $u_0, u_1, \dots u_N$ is the Numerical solution.

Define :- (Difference equation)

The Numbers $\{u_j\}$ are determined from a set of algebraic equation is called the difference equation.

Euler Method :-

we have,

$$\frac{u_{j+1} - u_j}{h} = f(t_j, u_j) \qquad \text{(f: for Taylor's series)}$$

$$u_{j+1} = u_j + h f(t_j, u_j)$$

$$\boxed{u_{j+1} = u_j + h f_j} \rightarrow (1) \qquad [\text{where } f_j = f(t_j, u_j)]$$

This is called the Euler (or) First order Adams-Bashforth Method.

The method is applied at the mesh points t_j, $j = 0(1)N-1$ to get the numerical solution of the differential equation.

we have,

$$u_1 = u_0 + h f_0$$

$$u_2 = u_1 + h f_1$$

where $f_j = f(t_j, u_j)$

$$\vdots$$

$$u_N = u_{N-1} + h f_{N-1}$$

Thus the Euler method provides a simple procedure for computing approximation u_j to the exact solution $\boxed{u(t_j)}$

Local Truncation Error :-

The error in the approximation is the difference between the exact solution at $t = t_{j+1}$ and the solution value u_{j+1} determined ① using the exact arithmetic, and is called the local truncation error (or) discretization error.

we have,

$$T_{j+1} = u(t_{j+1}) - u_{j+1}, \qquad j = 0, 1, \ldots N-1 \quad \text{(2)}$$

using, (1), the equation (2) becomes.

$$T_{j+1} = u(t_{j+1}) - u(t_j) - h f(t_j, u(t_j))$$

$$T_{j+1} = \frac{h^2}{2} u''(\xi) \qquad \text{wher } t_j < \xi < t_{j+1} \quad \text{(3)}$$

Example: 1

use the Euler method to solve numerically the initial value problem. $u' = -2tu^2$, $u(0) = 1$ with $h = 0.2, 0.1$ and 0.05 on the interval $[0, 1]$. Neglecting the roundoff errors, determine the bound for the error. Apply Richardson's extrapolation to improve the computed value $u(1.0)$

~~Soln.~~

~~we have,~~

Soln:

Gn. $u' = -2tu^2 \qquad u(0) = 1$

with $h = 0.2$, 0.1 , 0.05 on interval $[0, 1]$

we have,

$$u_{j+1} = u_j + h f_j \qquad \text{(for Euler formula)}$$

$$u_{j+1} = u_j - 2h t_j u_j^2 \quad ; \quad j = 0, 1, 2, 3, 4$$

with $h = 0.2$. the initial condition gives $u_0 = 1$

For $j = 0$; $t_0 = 0$, $u_0 = 1$

$$u(0.2) \approx u_1 = u_0 - 2h t_0 u_0^2 = 1 - 0 = 1$$

$$\boxed{u(0.2) \approx u_1 = 1}$$

For $j = 1$; $t_1 = (0.2)$, $u_1 = 0$

$$u(0.4) \approx u_2 = u_1 - 2h t_1 u_1^2$$

$$= 1 - 2(0.2)(0.2)(1)^2$$

$$= 1 - 0.08$$

$$\boxed{u(0.4) \approx u_2 = 0.92}$$

For $j = 2$; $t_2 = 0.4$, $u_2 = 0.92$

$$u(0.6) \approx u_3 = u_2 - 2ht_2u_2^2$$

$$= 0.92 - 2\,(0.2)(0.4)(0.92)^2$$

$$= 0.92 - 0.135424$$

$$\boxed{u(0.6) \approx u_3 = 0.78458}$$

Similarly we get,

$$u(0.8) \approx u_4 = 0.63684$$

$$u(1) \approx u_5 = 0.50706$$

when $h = 0.1$, we get,

For $j = 0$; $t_0 = 0$ $u_0 = 1$

$$u(0.1) \approx u_1 = u_0 - 2ht_0u_0^2 = 1 - 0 = 1$$

$$\boxed{u(0.1) \approx u_1 = 1}$$

For $j = 1$; $t_1 = 0.1$, $u_1 = 1$

$$u(0.2) \approx u_2 = u_1 - 2ht_1u_1^2$$

$$= 1 - 2(0.1)(0.1)(1)^2$$

$$= 1 - 0.02$$

$$\boxed{u(0.2) \approx u_2 = 0.98}$$

For $j = 2$, $t_2 = 0.2$, $u_2 = 0.98$

$$u(0.3) \approx u_3 = u_2 - 2ht_2 u_2^2$$

$$= 0.98 - 2(0.1)(0.2)(0.98)^2$$

$$= 0.98 - 0.038416$$

$$\boxed{u(0.3) \approx u_3 = 0.941584}$$

similarly, we get,

$u(0.4) \approx u_4 = 0.88839$

$u(0.5) \approx u_5 = 0.82525$

$u(0.6) \approx u_6 = 0.75715$

$u(0.7) = u_7 = 0.68835$

$u(0.8) = u_8 = 0.62202$

$u(0.9) = u_9 = 0.56011$

$u(1.0) = u_{10} = 0.50364$

For $h = 0.05$ we get,

$u(0.05) \approx 1.0$

$u(0.1) \approx 0.995$

$u(0.15) \approx 0.9851$

$u(0.2) \approx 0.97054$

$u(0.25) \approx 0.9517$

$u(0.3) \approx 0.92906$

$u(0.35) \approx 0.90316$

$u(0.4) \approx 0.87461$

$u(0.45) \approx 0.84401$

$u(0.50) \approx 0.81195$

$u(0.55) \approx 0.77899$

$u(0.6) \approx 0.74561$

$u(0.65) \approx 0.71225$

$u(0.7) \approx 0.67928$

$u(0.75) \approx 0.64698$

$u(0.8) \approx 0.61559$

$u(0.85) \approx 0.58527$

$u(0.9) \approx 0.55615$

$u(0..95 \approx 0.52831$

$u(1.0) \approx 0.50179$

The Truncation error in the Euler method is given as

$$TE = \frac{h^2}{2} u''(\xi_j)$$

$$|TE| = \frac{h^2}{2} |u''(\xi_j)| \le \frac{h^2}{2} \max_{0 \le t \le 1} |u''(t)|$$

Since the exact solution is $u(t) = 1/_{1+t^2}$

we get,

$$|TE| \le \frac{h^2}{2} \max_{0 \le t \le 1} \left| \frac{2(1-3t^2)}{(1+t^2)^3} \right| \le h^2$$

The error in the Euler method is of the form.

$$u(t_j) - u_j(h) = c_1 h + c_2 h^2 + c_3 h^3 + \dots$$

Richardson's Extrapolation gives.

$$u^{(k)}(h) = \frac{2^k u_j^{(k-1)}(h/2) - u_j^{(k-1)}(h)}{2^k - 1}$$

we have the following extrapolated value for $u(1.0)$

h	$u^{0}(h)$	$u^{(1)}(h)$	$u^{(2)}(h)$	$u^{(3)}(h)$
0.20	0.50766			
		0.50022		
0.10	0.50364		0.49985	0.5
		0.49994		
0.05	0.50179			

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MID POINT METHOD

* The equation $\frac{u_{j+1} - u_{j-1}}{2h} = f(t_j, u_j)$.

written as,

$$u_{j+1} - u_{j-1} \Rightarrow f(t_j, u_j)\, 2h.$$

$$\boxed{u_{j+1} = u_{j-1} + 2h f_j} \longrightarrow (1)$$

∴ This equation is called mid point (or) the second order Nystrom method.

The solution values are given by,

$(1) \Rightarrow \quad j = 1, 2, 3, \dots N$

$j=1 \Rightarrow \quad u_{1+1} = u_{1-1} + 2hf_1$

$$u_2 = u_0 + 2hf_1$$

|||ly

$$u_3 = u_1 + 2hf_2$$

$$\vdots$$

$$u_N = u_{N-2} + 2hf_{N-1}$$

There value is u_0 is known the initial Condition.

The value u_1 is Unknown and must be determined by some other method.

$u_N = u_{N-2} + 2hf_{N-1}$ is used to determine successively.

$\Rightarrow$ $u_2, u_3, \ldots u_N$. The Local Truncation error is given by,

$$T_{j+1} = u(t_{j+1}) - u_{j+1}$$

$$= u(t_{j+1}) - (u_{j-1} + 2hf_j) \quad \text{from } (1).$$

$$= u(t_{j+1}) - u(t_{j-1}) - 2hf(t_j, u(t_j)).$$

$$= \frac{h^3}{3} u'''(\xi), \longrightarrow (2)$$

$$t_{j-1} < \xi < t_{j+1} \qquad [\because M_3 = u'''(\xi).]$$

which may be written as,

$$|T_{j+1}| \le \frac{h^3}{3} M_3 \longrightarrow (3)$$

where $\max\limits_{t \in [t_0, b]} |u'''(\xi)| = M_3. \longrightarrow (4)$

Neglecting the roundoff error equation becomes.

$$e_{j+1} = e_{j-1} + 2h\lambda e_j + \frac{h^3}{3} M_3$$

where $u_j - u(t_j) = e_j$

Let, $|e_j| \le e_j$, $\Rightarrow$ $j = 0, 1, 2, \ldots$

$$e_{j+1} = e_{j-1} + 2h\lambda e_j + \frac{h^3}{3} M_3 \longrightarrow (5)$$

This is the second order differencial equation. with ~~constants and coefficients.~~

The homogoneous diff. equation becomes.

$$e_{j+1} - 2h\lambda e_j - e_{j-1} = 0 \longrightarrow (6)$$

To find the solution of equ: $e_{j+1} - 2h\lambda e_j - e_{j-1} = 0$.

$$\Rightarrow \quad e_j = A\xi^j \longrightarrow (7)$$

where $A \neq 0$ and ξ is a Number to be determined.

we find that non-trivial solution exists if ξ is a root of the polynomial.

$$\xi^2 - 2h\lambda\xi - 1 = 0 \longrightarrow (8)$$

$\therefore$ This equation is called characteristic equation.

It has Two roots,

$$\xi_1 = h\lambda + \sqrt{h^2\lambda^2+1}$$

$$= e^{h\lambda} - \frac{1}{6}\cdot h^3\cdot\lambda^3 + O(h^4) \longrightarrow (9)$$

$$\Rightarrow \quad \xi_2 = h\lambda - \sqrt{h^2\lambda^2+1}$$

$$= -e^{-h\lambda} + \frac{1}{6}h^3\lambda^3 + O(h^4). \longrightarrow (10)$$

The general solution is,

$$e_j = C_1\xi_1^j + C_2\xi_2^j - \frac{h^2M_3}{6\lambda} \longrightarrow (11)$$

where, c_1 and c_2 ~~are the found by solving equati~~ arbitary constants to be determined. we denote,

$$E_j = e_j + \frac{h^2 M_3}{6\lambda}, \quad j = 0, 1.$$

The constants c_1 and c_2 are found by solving equati

$$E_0 = c_1 + c_2$$

$$E_1 = c_1 \xi_1 + c_2 \xi_2.$$

The initial errors e_0, e_1 are constants and equal to e.

Then c_1 and c_2 from substituting from (11)

(11) $\Rightarrow$

$$e_j - \left(e + \frac{h^2 M_3}{6\lambda}\right) \frac{1}{\xi_1 - \xi_2} \left[(1 - \xi_2)\xi_1^j + (\xi_1 - 1)\xi_2^j\right] - \frac{h^2 M_3}{6\lambda} \quad \text{(12)}$$

we find the equn:-

$$\xi_1^j = e^{j\lambda h}\left(1 - \frac{j}{6} h^3 \lambda^3 + O(h^4)\right).$$

$$\xi_2^j = (-1)^j e^{-j\lambda h}\left(1 + \frac{j}{6} h^3 \lambda^3 + O(h^4)\right)$$

$$\xi_1 - \xi_2 = 2 + h^2\lambda^2 + O(h^4)$$

$$1 - \xi_2 = 2 - h\lambda + \frac{h^2\lambda^2}{2} + O(h^4)$$

$$\xi_1 - 1 = h\lambda + \frac{1}{2} h^2 \lambda^2 + O(h^4).$$

The values are substituting equ (12).

$$(12) \Rightarrow \quad e_j = \frac{1}{2+h^2\lambda^2}\left(e + \frac{h^2 M_3}{6\lambda}\right)\Big[\left(2 - h\lambda + \frac{h^2\lambda^2}{2} + o(h^4)\right)$$

$$e^{jh\lambda}\left(1 - \frac{jh^3\lambda^3}{6} + o(h^4)\right) + \left(h\lambda + \tfrac{1}{2}h^2\lambda^2 + o(h^4)\right)$$

$$(-1)^j e^{-jh\lambda}\left(1 + \frac{jh^3\lambda^3}{6} + o(h^4)\right)\Big] - \frac{h^2 M_3}{6\lambda} \rightarrow (13)$$

Put $e = 0$ in (13)

we find $e_j \rightarrow 0$. and $h \rightarrow 0$

if λ is positive $\xi_1 > 1$ and if λ is negative

$\xi_2 < -1$.

Therefore one of the term increases for any fixed $h > 0$.

The second term is small when h and $(t_j - t_0)$ are small, but eventually as $(t_j - t_0)$ increases. with h fixed. it will dominate over the first term,

$\Rightarrow$ for any fixed positive h.

Enamplo:

Solve mid point mothod. follow The Initial. value problem.

$$u' = -2tu^2, \quad u(0) = 1$$

with $h = 0.2$ over the interval $[0, 1]$

The percentage relative error at, $t = 1$

Solution The mid point method gives,

$$u_{j+1} = u_{j-1} + 2hf_j \longrightarrow (1)$$

$\Rightarrow$ equation (1).

$$\Rightarrow u_{j+1} = u_{j-1} - 4ht_j u_j^2,$$

$j = 1, 2, 3, 4$ ....

we calculate u_1 from the exact solution,

$$u(t) = \frac{1}{(1+t^2)}$$

we get,

$$u_1 = u(0.2) = \frac{1}{1.04} = 0.9615385$$

For, $j = 1$

$$u_0 = 1, \quad u_1 = 0.9615385,$$

For, $\underline{\underline{j = 1}}$

$u_0 = 1, \quad u_1 = 0.9615385, \quad t_1 = 0.2.$

$$u(0.4) \approx u_2 = u_0 - 4ht_1 u_1^2$$

$$= 1 - 4(0.2)(0.2)(0.9615385)^2$$

$$= 0.8520710.$$

For, $\underline{\underline{j = 2}}$

$u_1 = 0.09615385, \quad u_2 = 0.8520710, \quad t_2 = 0.4$

$$u(0.6) \approx u_3 = u_1 - 4ht_2 u_2^2$$

$$= 0.9615385 - 4(0.2)(0.4)(0.8520710)^2$$

$$= 0.7292105.$$

III'ly we got,

$$u(0.8) \approx u_4 = 0.5968320.$$

$$u(1.0) \approx u_5 = 0.5012371.$$

The percentage relative error is given by,

$$PRE = \frac{|u - u^*|}{|u|} \times 100. \longrightarrow (I)$$

Where u is the ~~entract~~ solution and

u^* is the approximate solution.

The exact solution at $t = 1$ is (0.5)

The percentage relative error is mid points method.

$\Rightarrow \quad u = 0.5 \quad , \quad u^* = 0.5012371$.

(I) $\Rightarrow$

$$\left| \frac{0.5 - 0.5012371}{0.5} \right| \times 100 .$$

$$\Rightarrow \frac{|0.5 - 0.5012371|}{0.5} \times 100$$

$\Rightarrow \quad 0.25$ %.

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Taylor Series Method :-

We assume that the function $u(t)$ can be expanded in a Taylor series about any point t_j

$$u(t) = u(t_j) + (t-t_j)u'(t_j) + \frac{1}{2!}(t-t_j)^2 u''(t_j) + \cdots$$
$$+ \frac{1}{p!}(t-t_j)^p u^{(p)}(t_j) + \frac{1}{(p+1)!}(t-t_j)^{p+1} u^{(p+1)}(t_j+\theta h) \longrightarrow (1)$$

The expansion holds good for $t \in [t_0, b]$ and $0 < \theta < 1$.

Substituting $\boxed{t = t_{j+1}}$ in (1), we get

$$u(t_j+1) = u(t_j) + hu'(t_j) + \frac{h^2}{2!}u''(t_j) + \cdots + \frac{1}{p!}h^p u^{(p)}(t_j)$$
$$+ \frac{1}{(p+1)!}h^{p+1}u^{(p+1)}(t_j+\theta h)$$

We define,

$$h\phi(t_j, u(t_j), h) = hu'(t_j) + \frac{h^2}{2!}u''(t_j) + \cdots + \frac{h^p}{p!}u^{(p)}(t_j)$$

and $h\phi(t_j, u_j, h)$ is to be obtained from $\phi(t_j, u(t_j), h)$ by using to approximate value u_j in place of the exact value $u(t_j)$. We compute

$$u_{j+1} = u_j + h\phi(t_j, u_j, h). \quad j = 0, 1, \cdots, N-1 \longrightarrow (2)$$

to approximate $u(t_j+1)$. This is called the Taylor series method of order p. Substituting $p=1$ in (2). We get

$$u_{j+1} = u_j + h u'_j$$

Which is the Euler method.

To apply ②, it is necessary to know $u(t_j), u'(t_j), \cdots u^{(p)}(t_j)$. If t_j and $u(t_j)$ are known then the derivatives can be calculated as follows

First the known values t_j and $u(t_j)$ are substituted into the differential equation to give

$$u'(t_j) = f(t_j, u(t_j)).$$

Next, the differential equation ① is differentiated to obtain expressions for the higher order derivatives of $u(t)$.

Thus,

$$u' = f(t, u)$$

$$u'' = f_t + f f_u$$

$$u''' = f_{tt} + 2f f_{tu} + f^2 f_{uu} + f_u (f_t + f f_u)$$

$$\vdots$$

Where $f_t + f_u + \cdots$ represent the partial derivatives of f with respect to t and u and so on. The values $u''(t_j), u'''(t_j), \cdots$ can be computed by substituting

$t = t_j$. Therefore, if t_j and $u(t_j)$ are known exactly then (2) can be used to compute u_{j+1} with an error

$$\frac{h^{p+1}}{(p+1)!} u^{(p+1)}(t_j + \theta h)$$

The number of terms to be included in (2) is fixed by the permissible error. If this error is ϵ and the series is truncated at the term $u^{(p)}(t_j)$ then

$$h^{p+1} |u^{(p+1)}(t_j + \theta h)| < (p+1)! \, \epsilon$$

(or)

$$h^{p+1} |f^{(p)}(t_j + \theta h)| < (p+1)! \, \epsilon. \quad \text{——— (3)}$$

For a given h, (3) will determine p, and if p is specified then it will give an upper bound on h.

Since $t_j + \theta h$ is not known, $|f^{(p)}(t_j + \theta h)|$ in (3) is replaced by on maximum value in $[t_0, b]$. The maximum values of this quantity in $[t_0, b]$ gives a rough required bound.

Example - 6.5·

Find the three Term Taylor series solution for the third order initial value problem.

$$w''' + ww'' = 0 \qquad w(0) = 1$$

$$w'(0) = 0 \qquad w''(0) = 1.$$

Find the bound on the error for $t \in [0, 0.2)$

we find,

$$w''' = -ww''$$

$$\boxed{w'''(0) = 0}$$

$$w^{(4)} = -(ww''' + w'w'')$$

$$w^{(4)}(0) = -(w(0)w'''(0) + w'(0)w''(0))$$

$$= -(1(0) + 0(1))$$

$$\boxed{w^{(4)} = 0}$$

$$w^{(5)} = -[ww^{(4)} + w'''w' + w'w''' + w''w'']$$

$$= -[ww^{(4)} + 2w'w''' + (w'')^2]$$

$$w^{(5)}(0) = -[w(0)w^{(4)}(0) + 2w'(0)w'''(0) + (w''(0))^2]$$

$$= -[1(0) + 2(0)(0) + (1)^2]$$

$$= -[1]$$

$$\boxed{w^{(5)}(0) = -1}$$

$$w^{(6)} = -\left[ww^{(5)} + w^{(4)}w' + 2\left[w'w^{(4)} + w'''w''\right] + 2w''.2\right]$$

$$w^{(6)}(0) = -\left[w(0)w^{5}(0) + w^{(4)}(0)w'(0) + 2\left[w'(0)w^{(4)}(0) + w'''(0)w''(0)\right] + 2w''(0)2\right]$$

$$= -\left[0(-1) + (0)(0) + 2\left[0(0) + (0)(0)\right] + 2(0).2\right]$$

$$\boxed{w^{(6)}(0) = 0}$$

$$w^{(7)} = -\left[ww^{(6)} + w^{(5)}w' + w^{(4)}w'' + w'w^{(5)} + 2\left(w'w^{(5)} + w^{(4)}w'' + w'''w''' + w''w^{(4)}\right) + 4w'''\right]$$

$$w^{(7)}(0) = -\left[w(0)w^{(6)}(0) + w^{(5)}(0)w'(0) + w^{(4)}(0)w''(0) + w'(0)w^{(5)}(0) + 2\left(w'(0)\cdot w^{(5)}(0) + w^{(4)}(0)w''(0) + (w'''(0))^2 + w''(0)w^{(4)}(0)\right) + 4w'''(0)\right]$$

$$\boxed{w^{7}(0) = 0}$$

$$w^{(8)} = -\left[ww^{(7)} + w^{6}w' + 2\left(w^{5}w'' + w'w^{(6)}\right) + w^{(4)}w''' + w''w^{(5)} + 2\left(w'w^{(6)} + w^{(5)}w'' + 2\left(w''w^{(5)} + w^{(4)}w'''\right) + 2w'''.2\right) + 4w^{(4)}\right]$$

$$w^{(8)}(0) = -\left[0 + 0 + 2\left((-1) + 0\right) + 0 + (-1) + 2\left(0 + (-1) + 2\left((-1) + 0\right) + 2(0).2\right) + 4(0)\right]$$

$$= -(-11)$$

$$\boxed{w^{(8)}(0) = 11}$$

III^{ly} $w^{(9)}(0) = 0$, $w^{(10)}(0) = 0$ $w^{(11)}(0) = -375$

The Taylor series solution is

$$w(t) = \frac{1t^2}{2!} - \frac{1t^5}{5!} + \frac{11t^8}{8!} + E$$

where
$$|E| \leq \max |w^{(9)}(t)| \frac{t^9}{9!}$$

writting the next term we have,

$$w(t) = \frac{1t^2}{2!} - \frac{1t^5}{5!} + \frac{11}{8!}t^8 - \frac{375}{11!}t^{11}$$

we find $\quad w^{(9)}(t) = -\frac{375}{2}t^2$

and
$$\max_{0 \leq t \leq 0.2} |w^{(9)}(t)| = 7.5$$

Hence
$$|E| \leq \frac{7.5\,(0.2)^9}{9!} \leq (1.06)\,10^{-11}.$$