Mathematical Statistics

Unit - I

Some Special Distributions.

The Binomial and Related Distributions

Defn:: Bernoulli experiment

A Bernoulli experiment is a random experiment, the outcome of which can be classified in but one of two mutually exclusive and exhaustive ways say, success or failure

Eg:: female or male, life or death; nondefective or defective, Head or Tail.

Remark: A sequence of Bernoulli trials occurs when a Bernoulli experiment is performed several independent times so that the probability to success say p. ~~ex~~ remains the same from trial to trial.

Let p denote the probability of success on each trial.

Defn: Bernoulli Distributions

Note: Random ~~Variable is a~~ Function

Let X be a random variable associated with Bernoulli trial by defining as follows:

$$X(\text{success}) = 1 \quad \text{and} \quad X(\text{failure}) = 0.$$

That is, the two outcomes, success and failure, are denoted by one and Zero respectively.

~~The Pro probability de de~~

The probability density function (p.d.f) of X can be written as $f(x) = p^x (1-p)^{1-x}$, $x = 0, 1$.

This is called Bernoulli Distribution

②

Defn: The expected value of X is

$$\text{Mean } \mu = E(x)$$
$$= \sum_{x=0}^{1} x\, p^x (1-p)^{1-x}$$
$$= 0(1-p) + 1p$$
$$= p$$
$$\therefore \mu = p$$

Defn: The Variance of X is

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$
$$= \sum_{x=0}^{1} (x-p)^2 p^x (1-p)^{1-x}$$
$$= p^2(1-p) + (1-p)^2 p$$
$$= p(1-p)[p + 1 - p]$$
$$= p(1-p)$$

Standard Deviation of X is $\sigma = \sqrt{p(1-p)}$

Remarks: 1, If the sequence of n Bernolli trials, and X_i is the random variable associated with the i'th trial. The ~~Sequen~~ Sequence of n Bernoulli trials with n tuple of zeros and ones.

2, Let the x be the random variable equal the number of observed successive in n Bernoulli trials, the possible values of X are 0, 1, 2, ..., n. If x successes occur, where x = 0, 1, 2, ..., n then n-x failures occur. Then the number of ways of selecting x positions for the x successes in the n trials is

$$^nC_x = \binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

③

Since the trials are independent and since the probabilities of success and and failures on each trial are p and $1-p$ respective, the Probability of each of these ways is $p^x(1-p)^{n-x}$.

~~Ex~~ Defn: Binomial distribution

When n Bernoulli trials are Conducted, each with an identical probability of success p, the experiment is known as a binomial random experiment

Defn: A binomial random experiment satisfies the following Criteria.

i) The random experiment Consists of n identical trials

(ii) The are two possible outcomes for each trial

(iii) The trials are mutually independent.

iv) The probability of success on each trial is identical

$X \sim$ binomial (n, p) (or $X \sim b(n, p)$) models the number of Successes in n independent Bernolli trials, each with probability of Success p, where n is a positive integer

Defn: A discrete random Variable x with p.d.f

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n$$
$$= 0 \quad \text{else where}$$

for some positive integer n and $0 < p < 1$ is $b(n, p)$ random Variable

Note: i) It is clear that $f(x) \geq 0$ and $\sum f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

$$= [(1-p)+p]^n = 1$$

(ii) A random Variable X that has a p.d.f of the form of $f(x)$ is Called binomial distribution, and any such $f(x)$ is Called binomial p.d.f.

④

Note: (iv) A binomial distribution ~~will be~~ is denoted by $b(n, p)$ The constants n and p are called the parameters of the binomial distribution.

Example: If X is $b(5, \frac{1}{3})$ then X is the binomial p.d.f

$$f(x) = \binom{5}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{5-x}, \quad x = 0, 1, 2, \dots, 5$$
$$= 0 \text{ elsewhere}$$

The moment generating function (m.g.f)

The m.g.f of a binomial distribution is

$$M(t) = \sum_{x} e^{tx} f(x)$$
$$= \sum_{x=0}^{n} e^{tx}\binom{n}{x} p^{x}(1-p)^{n-x}$$
$$= \sum_{x=0}^{n}\binom{n}{x}(pe^{t})^{x}(1-p)^{n-x}$$
$$= [(1-p) + pe^{t}]^{n} \text{ for all values of } t.$$

Find mean and Variance of binomial distribution by using m.g.f

Sol Let X be $b(n, p)$

Then m.g.f $M(t) = [(1-p) + pe^{t}]^{n}$ for all real values of t

To find mean μ

$$M'(t) = n[(1-p) + pe^{t}]^{n-1}(pe^{t}), \quad \text{where } M'(t) = \frac{d}{dt}[M(t)]$$
$\hookrightarrow$ ①

and $$M''(t) = n[(1-p) + pe^{t}]^{n-1}(pe^{t}) + (pe^{t})n(n-1)[(1-p) + pe^{t}]^{n-2}(pe^{t})$$

5

Now

$$M''(t) = n[(1-p)+pe^t]^{n-1}(pe^t) + n(n-1)[(1-p)+pe^t]^{n-2}(pe^t)^2 \quad \to (2)$$

Put $t=0$ in (1)

$$\text{Mean } \mu = M'(0) = n[(1-p)+pe^0]^{n-1}(pe^0)$$

$$= n[1-\cancel{p}+\cancel{p}]^{n-1} p$$

$$\boxed{\mu = np}$$

Put $t=0$ in (2)

$$M''(t) = n[(1-p)+pe^0]^{n-1} pe^0 + n(n-1)[(1-p)+pe^0]^{n-2}(pe^0)^2$$

$$= n[(1-p)+p]^{n-1} p + n(n-1)[(1-p)+p]^{n-2} p^2$$

$$= np + n(n-1)p^2$$

$$\text{Variance }(X) = \sigma^2 = M''(0) - \mu^2$$

$$= np + n(n-1)p^2 - (np)^2$$

$$= np[1 + (n-1)p - np]$$

$$= np[1 + \cancel{np} - p + \cancel{np}]$$

$$= np(1-p)$$

Problem ① Let X be the number of heads (successes) in $n=7$ independent tosses of an unbiased Coin. Find the binomial p.d.f of x, $P(0 \leq X \leq 1)$ and $P(X=5)$

Solution: Given $n = 7$

probality of getting head (success) $p = \frac{1}{2}$

The p.d.f of binomial distribution is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2 \ldots n$$

$$= 0 \text{ elsewhere}$$

Now, $$f(x) = \binom{7}{x} \left(\frac{1}{2}\right)^x \left(1-\frac{1}{2}\right)^{7-x}, \quad x = 0,1,2,\ldots 7$$

$$= \binom{7}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}, \quad x = 0,1,2,\ldots 7 \quad \to ①$$

$$= 0 \text{ elsewhere}$$

$\cancel{\text{∴}}$. Then x has the m.g.f

$$M(t) = [(1-p) + pe^t]^n$$

$$= \left[\frac{1}{2} + \frac{1}{2}e^t\right]^7$$

Mean $\mu = np$ $= 7/2$

Variance $\sigma^2 = np(1-p)$ $= 7 \cdot \frac{1}{2} \left(\frac{1}{2}\right)$ $= \frac{7}{4}$

Now $$P(0 \leq X \leq 1) = \sum_{x=0}^{1} f(x)$$

$$= \sum_{x=0}^{1} \binom{7}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}$$

$$= \binom{7}{0}\left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^7 + \binom{7}{1}\left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{7-1}$$

$$= \left(\frac{1}{2}\right)^7 + 7\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^6$$

$$= \frac{1}{128} + \frac{7}{128} = \frac{8}{128} = \frac{1}{16}$$

$$P(X=5) = f(5)$$

$$= \binom{7}{5}\left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{7-5}$$

$$= 21 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^2$$

$$= 21 \left(\frac{1}{2}\right)^7 = \frac{21}{128}$$

$$\binom{7}{5} = \binom{7}{2} = \frac{7 \times \cancel{6}^3}{1 \times \cancel{2}}$$

(2) If the m.g.f of a random Variable X is $M(t) = (2/3 + \frac{1}{3}e^t)^5$ then find p.d.f and mean and Variance of the distribution?

Sol Let $M(t) = (\frac{2}{3} + \frac{1}{3}e^t)^5$

This is of the form $[(1-p) + pe^t]^5$

Here $p = \frac{1}{3}, n = 5$

$\therefore$ The p.d.f of binomial distribution

$$f(x) = \binom{5}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{5-x}, \quad x = 0, 1, 2, \cdots 5$$

$= 0$ elsewhere

Mean $\mu = np$
$= 5/3$

Variance $\sigma^2 = np(1-p)$
$= 5\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$
$= \frac{10}{9}$

(3) If the m.g.f of a random Variable X is $\left(\frac{1}{3}\right)$ $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$, find $P(X = 2 \text{ or } 3)$

Sol Let m.g.f of a random Variable X is
$$\left(\frac{1}{3} + \frac{2}{3}e^t\right)^5 \rightarrow (1)$$

Equation (1) is of the form $[(1-p) + pe^t]^n$

Here $p = \frac{2}{3}$ $n = 5$

$\therefore$ The p.d.f of binomial distribution is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots n$$

Then $f(x) = \binom{5}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{5-x} \quad x = 0, 1, 2 \ldots 5$

$P(X = 2 \text{ or } 3) = f(2) + f(3) \rightarrow (2)$

Now $f(2) = \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{5-2}$ $\qquad \frac{5\times\cancel{4}^2}{1\times\cancel{2}}$

$= (10)\left(\frac{4}{9}\right)\left(\frac{1}{3}\right)^3$ $\qquad \begin{array}{r}6\\27\\9\\ \hline 243\end{array}$

$= 10\left(\frac{4}{9}\right)\left(\frac{1}{27}\right)$

$= \frac{40}{243}$ $\qquad \frac{5\times\cancel{4}\times\cancel{3}}{1\times\cancel{2}\times\cancel{3}}$

$f(3) = \binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^{5-3}$ $\qquad \begin{array}{r}6\\27\\9\\ \hline 243\end{array}$

$= 10\left(\frac{8}{27}\right)\left(\frac{1}{9}\right)$

$= \frac{80}{243}$

using in ①

$P(x=2 \text{ or } 3) = \frac{40}{243} + \frac{80}{243} = \frac{120}{243}$

④ The m.g.f of a random Variable X is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$. Show that $P(\mu-2\sigma < X < \mu+2\sigma) = \sum_{x=1}^{5}\binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}$

Sol

The m.g.f of random Variable x is

$\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9 \rightarrow ①$

Equation ① is of the form $[(1-p)+pe^t]^9$

Here, $p = \frac{1}{3}$ $\quad n = 9$

Now, mean $\mu = np$

$= 9\times\frac{1}{3} = 3$

Variance $\sigma^2 = np(1-p)$

$= \cancel{9}\left(\frac{1}{\cancel{3}}\right)\left(\frac{2}{\cancel{3}}\right)$

$= 2$

$\sigma = \sqrt{2}$

Now $\mu - 2\sigma = 3-2\sqrt{2}$ and $\mu+2\sigma = 3+2\sqrt{2}$

$P(\mu-2\sigma < x < \mu+2\sigma) = P(3-2\sqrt{2} < x < 3+2\sqrt{2})$

$= P(x = 1, 2, 3 \ldots 5)$

$= \sum_{x=1}^{5}\binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}$

(5) If x is $b(n, p)$ show that

$$E\left(\frac{x}{n}\right) p \text{ and } E\left[\left(\frac{x}{n} - p\right)^2\right] = \frac{p(1-p)}{n}.$$

Sol

$$E\left(\frac{x}{n}\right) = \frac{1}{n} E(x)$$
$$= \frac{1}{n} \mu$$
$$= \frac{1}{n} np$$
$$= p$$

$$E\left[\left(\frac{x}{n} - p\right)^2\right] = \frac{1}{n^2} E\left[(x - np)^2\right] \qquad \because E\left[(x-np)^2\right] = Var(x)$$
$$= \frac{1}{n^2} np(1-p)$$
$$= \frac{p(1-p)}{n}$$

Negative binomial Distribution :.

Consider a sequence of independent repetitions of a random experiment with constant probability p of success. Let the random variable Y denote the total number of failures in this sequence the rth success; that is $Y + r$ is equal to the number of trials necessary to produce exactly r successes. Here r is a fixed positive integer. Let y be an element of $\{y : y = 0, 1, 2 \ldots\}$. To find the p.d.f of Y. Then, by the multiplication rule of probabilities $P(Y = y) = g(y)$ is equal to the product of the probability

$$\binom{y+r-1}{r-1} p^{r-1} (1-p)^y$$ of obtaining exactly

$r-1$ successes in the first $y+r-1$ trials and the probability p of a success on the $(y+r)^{th}$ trial. Thus the p.d.f $g(y)$ of Y is given by

$$g(y) = \binom{y+r-1}{r-1} p^r (1-p)^y, \quad y = 0, 1, 2, \ldots$$
$$= 0$$

A distribution with a p.d.f $g(y)$ is called negative binomial distribution.

Note: i) The m.g.f of ~~negatibr~~ negative binomial distribution is $M(t) = p\left[1-(1-p)e^{t}\right]^{-1}$

ii) If $r=1$, then we have Y is a geometric distribution

Note: i) The m.g.f of ~~negative~~ negative binomial distribution is $M(t) = p[1-(1-p)e^t]^{-1}$

(ii) If $r=1$, then we have Y is a geometric distribution

The Poisson Distribution

Defn: Consider the function $f(x)$ defined by

$$f(x) = \frac{m^x e^{-m}}{x!}, \quad x = 0, 1, 2, \ldots$$

$$= 0 \quad \text{elsewhere}, \quad \text{where } m > 0$$

is called Poisson distribution.

Remark: If $m>0$, then $f(x) \geq 0$ and $\sum_x f(x) = \sum_{x=0}^{\infty} \frac{m^x e^{-m}}{x!}$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!}$$

$$= e^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots\right]$$

$$= e^{-m} \cdot e^{m} = e^0 = 1$$

Thus $f(x)$ satisfies the conditions of ~~of~~ a p.d.f of a discrete type of random variable

A random variable that has a p.d.f of the form $f(x)$ is said to be a Poisson ~~distn~~ distribution, and any such $f(x)$ is called a Poisson p.d.f.

The m.g.f of a Poisson distribution Find mean and variance

The m.g.f of a Poisson distribution is given by

$$M(t) = \sum_x e^{tx} f(x)$$

$$= \sum_x e^{tx} \frac{m^x e^{-m}}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!}$$

$$= e^{-m} \cdot \left[1 + \frac{me^t}{1!} + \frac{(me^t)^2}{2!} + \frac{(me^t)^3}{3!} + \cdots\right]$$

$$= e^{-m} e^{me^t}$$

$$\boxed{M(t) = e^{m(e^t - 1)}} \rightarrow (1) \text{ for all real values of } t.$$

$$M(t) = e^{m(e^t - 1)} \rightarrow (1)$$

Diff w.r.t t.

$$M'(t) = e^{m(e^t-1)} (me^t) \rightarrow (2)$$

Again Diff w.r.t t

$$M''(t) = e^{m(e^t-1)} (me^t) + (me^t)\, e^{m(e^t-1)} (me^t)$$

$$= e^{m(e^t-1)} (me^t) + e^{m(e^t-1)} (me^t)^2 \rightarrow (3)$$

Put $t=0$ in (2) and (3)

From (2)

$$M'(0) = \mu = e^{m(e^0-1)} (me^0)$$

$$= e^{m(1-1)} (m\,1)$$

Mean $\boxed{\mu = m}$

From (3)

$$M''(0) = e^{m(e^0-1)} (me^0) + e^{m(e^0-1)} (me^0)^2$$

$$= e^{m(1-1)} (m) + e^{m(1-1)} m^2$$

$$= m + m^2$$

Variance $\sigma^2 = M''(0) - \mu^2$

$$= m + m^2 - m^2$$

$$\sigma^2 = m$$

Variance $\boxed{\sigma^2 = m}$

Note :- Poisson p.d.f written as

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}, \quad x = 0, 1, 2, \dots$$

$$= 0 \quad \text{elsewhere}$$

Problem ① If x ~~ho~~ is a Poisson distribution with $\mu = 2$ find p.d.f of x; mean, Variance and $P(1 \le x)$

Sol Let x be a Poisson distribution with ~~ho~~ mean $\mu = 2$

$$\therefore \text{The p.d.f } f(x) = \frac{\mu^x e^{-\mu}}{x!}, \quad x = 0,1,2,\dots$$
$$= 0 \quad \text{elsewhere}$$

$$\therefore f(x) = \frac{2^x e^{-2}}{x!}, \quad x = 0,1,2,\dots$$
$$= 0 \quad \text{elsewhere}$$

Since x is poisson distribution $\mu = \sigma^2 = 2$

$$P(1 \le x) = 1 - P(x = 0)$$
$$= 1 - f(0)$$
$$= 1 - \frac{2^0 e^{-2}}{0!}$$
$$= 1 - e^{-2}$$
$$= 0.865$$

② If the m.g.f of a random Variable x is $M(t) = e^{4(e^t - 1)}$ find $f(x)$ and $P(x = 3)$

Sol The m.g.f of ~~x~~ random Variable is ~~ho~~ in poisson distribution

Then We have $M(t) = e^{m(e^t - 1)}$

~~Hen~~ Here $m = 4$

$$\text{The p.d.f } f(x) = \frac{m^x e^{-m}}{x!}$$
$$= \frac{4^x e^{-4}}{x!}, \quad x = 0,1,2,\dots$$
$$= 0 \quad \text{elsewhere}$$

$$P(X=3) = \frac{4^3 e^{-4}}{3!}$$

$$= \frac{64\, e^{-4}}{6}$$

$$= \frac{32}{3} e^{-4}$$

$$= 0.195$$

(3) If the random Variable x has a poisson distribution such that $P(x=1) = P(x=2)$, find $P(x=4)$

Sol The p.d.f $f(x) = \frac{\mu^x e^{-\mu}}{x!}$, $x = 0, 1, 2 \ldots$ $\hookrightarrow$ (1)

$= 0$ elsewhere

If $P(x=1) = P(x=2)$ then

$$f(1) = f(2)$$

$$\frac{\cancel{\mu^1} \cancel{e^{-\mu}}}{1!} = \frac{\mu^{\cancel{2}} \cancel{e^{-\mu}}}{2!}$$

$$\mu = 2$$

$P(X=4) = f(4)$ using in (1)

$$= \frac{2^4 e^{-2}}{4!}$$

$$= \frac{16 e^{-2}}{24} = \frac{2}{3} e^{-2}$$

The Gamma and Chi-Square Distributions

The gamma function $\Gamma(\alpha)$

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\,dy \rightarrow (1)$$

$$\left(\frac{e^{-y}}{-1}\right)$$

If $\alpha = 1$

$$\Gamma(1) = \int_0^\infty e^{-y}\,dy$$

$$= \left(-e^{-y}\right)_{y=0}^{\infty} = -e^{-\infty} - (-e^{0}) = 1$$

If $\alpha > 0$,

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\,dy$$

By Integration by parts, we have $\left[\int u\,dv = uv - \int v\,du\right]$

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\,dy$$

$$= \int_0^\infty y^{\alpha-1}\,d(-e^{-y})$$

$$= \left(-y^{\alpha-1} e^{-y}\right)_0^\infty - \int_0^\infty (-e^{-y})(\alpha-1)\,y^{\alpha-2}\,dy$$

$$= 0 + (\alpha-1)\int_0^\infty y^{\alpha-2} e^{-y}\,dy$$

$$= (\alpha-1)\int_0^\infty y^{\alpha-2} e^{-y}\,dy$$

If α is a positive integer greater than 1

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)\cdots(3)(2)(1)\,\Gamma(1)$$

$$= (\alpha-1)(\alpha-2)\cdots(3)(2)(1)(1) \qquad \because \Gamma(1) = 1$$

$$= 1\cdot 2\cdot 3\cdots \alpha-1$$

$$\Gamma(\alpha) = (\alpha-1)!$$

Put $y = \frac{x}{\beta}$, where $\beta > 0$ in ① Then, we have

$$\Gamma(\alpha) = \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta} d(x/\beta)$$

$$= \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta} (1/\beta)\, dx$$

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)} \frac{x^{\alpha-1}}{\beta^{\alpha-1}} e^{-x/\beta} \left(\frac{1}{\beta}\right) dx$$

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad \cancel{0<x<\infty}$$

Since $\alpha > 0$, $\beta > 0$ and $\Gamma(\alpha) > 0$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty$$

$$= 0 \quad \text{elsewhere}$$

$f(x)$ is called p.d.f of gamma distribution with parameters α and β

Note: $f(x)$ is a p.d.f of a random variable of the continuous type.

The m.g.f of gamma distribution. Find mean and Variance

The m.g.f of gamma distribution

$$M(t) = \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(1/\beta - t)} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(1-\beta t)/\beta} dx$$

Put $y = \dfrac{x(1-\beta t)}{\beta}$, $t < \frac{1}{\beta}$ $\quad x = \dfrac{\beta y}{1-\beta t}$

$$dx = \frac{\beta}{1-\beta t} dy$$

Limit $x = 0 \quad y = 0$

$x = \infty \quad y = \infty$

$$\therefore M(t) = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta y}{1-\beta t}\right)^{\alpha-1} e^{-y} \left(\frac{\beta}{1-\beta t}\right) dy$$

$$= \int_0^\infty \frac{\left(\frac{\beta}{1-\beta t}\right)}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta y}{1-\beta t}\right)^{\alpha-1} e^{-y} dy$$

$$= \int_0^\infty \frac{\left(\frac{\beta}{1-\beta t}\right)}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta}{1-\beta t}\right)^{\alpha-1} y^{\alpha-1} e^{-y} dy$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta}{1-\beta t}\right)^{\alpha} y^{\alpha-1} e^{-y} dy$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\cancel{\beta^\alpha}} \frac{\cancel{\beta^\alpha}}{(1-\beta t)^\alpha} y^{\alpha-1} e^{-y} dy$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha} \int_0^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} \int_0^{\infty} y^{\alpha-1} e^{-y} dy$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} \Gamma(\alpha)$$

$$\boxed{M(t) = \frac{1}{(1-\beta t)^{\alpha}}}, \quad t < \frac{1}{\beta} \longrightarrow (1)$$

Mean and Variance

Diff (1) w.r.t ~~x~~ t

$$M(t) = (1-\beta t)^{-\alpha}$$

$$M'(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta) \longrightarrow (2)$$

Diff (2) w.r.t t

$$M''(t) = (-\alpha)(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)^2$$

$$= (\alpha)(\alpha+1)(1-\beta t)^{-\alpha-2}\beta^2 \longrightarrow (3)$$

Put $t=0$ in (2) and (3)

$$M'(0) = \mu = (-\alpha)(1-\beta 0)^{-\alpha-1}(-\beta)$$

$$= \cancel{(-\beta)}(-\alpha)(-\beta)$$

Mean. $\boxed{\mu = \alpha\beta}$

Variance $\sigma^2 = M''(0) - \mu^2$

$$= (\alpha)(\alpha+1)(1-\beta 0)^{-\alpha-2}\beta^2 - (\alpha\beta)^2$$

$$\cancel{= \alpha(\alpha+1}$$

$$= (\alpha)(\alpha+1)\beta^2 - \alpha^2\beta^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

Variance $\boxed{\sigma^2 = \alpha\beta^2}$

Chi-Square Distribution

Consider p.d.f of gamma distribution

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty$$

Put $\alpha = r/2$ where r is a positive integer, and $\beta = 2$

$$f(x) = \frac{1}{\Gamma(r/2)\, 2^{r/2}} x^{r/2 - 1} e^{-x/2}, \quad 0 < x < \infty$$

$$= 0 \qquad \text{elsewhere}$$

is called a Chi-Square distribution, and any $f(x)$ of this form is called a chi-Square p.d.f.

The m.g.f of Chi-Square distribution

The m.g.f of Gamma distribution is

$$M(t) = (1 - \beta t)^{-\alpha}, \quad t < 1/\beta.$$

Put $\alpha = r/2$ and $\beta = 2$, we get

The m.g.f of Chi-square distribution

$$M(t) = (1 - 2t)^{-r/2}, \quad t < 1/2$$

Mean $\mu = \alpha\beta$

$$= \frac{r}{2} \cdot 2 = r.$$

Variance $\sigma^2 = \alpha\beta^2.$

$$= (r/2)\, 2^2$$

$$= 2r.$$

Note: The random variable X is $\chi^2(r)$ mean that the random variable X has a Chi-Square distribution with r degrees of freedom.

Example If x has the p.d.f $f(x) = \frac{1}{4} x e^{-x/2}, \quad 0 < x < \infty$

$= 0$ elsewhere

then x is $\chi^2(4)$

Hence $\mu = 4$, $\sigma^2 = 8$ and $M(t) = (1-2t)^{-2}$, $t < \frac{1}{2}$

Problem Let x have a gamma distribution with $\alpha = \frac{r}{2}$, where r is a positive integer, and $\beta > 0$. Define the random Variable $Y = \frac{2X}{\beta}$. Find the p.d.f of Y.

Sol Let x be a gamma distribution with $\alpha = \frac{r}{2}$ and $\beta > 0$

Let $Y = \frac{2x}{\beta}$ then $X = \frac{\beta Y}{2}$

The distribution function Y is

$$G(y) = P(Y \le y) = \cancel{P(X \le} \frac{\beta y}{2}$$

$$= P\left(X \le \frac{\beta y}{2}\right)$$

If $y \le 0$, then $G(y) = 0$,

If $y > 0$ then

$$G(y) = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx, \quad 0 < x < \infty$$

$$= \int_0^{\beta y/2} \frac{1}{\Gamma(r/2)\beta^{r/2}} x^{r/2-1} e^{-x/\beta} dx \qquad y = \frac{2x}{\beta}, \quad x = \beta/2$$

The p.d.f Y is

$$g(y) = G'(y) = \left[\frac{1}{\Gamma(r/2)\beta^{r/2}} x^{r/2-1} e^{-x/\beta}\right]_0^{\beta y/2}$$

$$= \frac{1}{\Gamma(r/2)\beta^{r/2}} \left(\frac{\beta y}{2}\right)^{r/2-1} e^{-\frac{\beta y/2}{\beta}}$$

$$= \frac{\beta/2}{\Gamma(r/2)\beta^{r/2}} \left(\frac{\beta y}{2}\right)^{r/2-1} e^{-y/2}$$

:20:

$$= \frac{\beta/2}{\Gamma(\frac{r}{2})\beta^{r/2}} \left(\frac{\beta}{2}\right)^{r/2-1} \left(\frac{y}{\cancel{\;}}\right)^{r/2-1} e^{-y/2}$$

$$= \frac{\cancel{\beta/2}}{\Gamma(\frac{r}{2})\beta^{r/2}} \left(\frac{\beta}{2}\right)^{r/2} \left(\cancel{\frac{\beta}{2}}\right)^{-1} \left(\frac{y}{\cancel{\;}}\right)^{r/2-1} e^{-y/2}$$

$$= \frac{1}{\Gamma(\frac{r}{2})\cancel{\beta^{r/2}}} \frac{\cancel{\beta^{r/2}}}{2^{r/2}} \left(\frac{y}{\cancel{\;}}\right)^{r/2-1} e^{-y/2}$$

$$= \frac{1}{\Gamma(\frac{r}{2})2^{r/2}} \cancel{y}\, y^{r/2-1} e^{-y/2}$$

$$= \frac{1}{\Gamma(\frac{r}{2})\, 2^{r/2}} y^{r/2-1} e^{-y/2}$$

if $y>0$ Y is $\chi^2(r)$

The Normal Distribution

Consider the integral $I = \int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{2}\right) dy \rightarrow (1)$

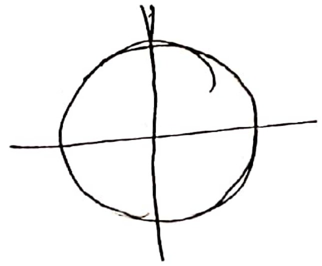
This integral exists because the integrand is positive Continuous function which is bounded by an integrable function; that is $0 < \exp\left(-\frac{y^2}{2}\right) < \exp(-|y|+1)$, $-\infty < y < \infty$

and $\int_{-\infty}^{\infty} \exp(-|y|+1)\, dy = 2e$

Now $$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{y^2+z^2}{2}\right) dy\, dz \rightarrow (2)$$

To changing to polar Co-ordinate

$y = r\cos\theta$ and $z = \sin\theta$



From (2) $$I^2 = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} \exp\left(\frac{-r^2}{2}\right) r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2/2}\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\int_{r=0}^{\infty} e^{-r^2/2}\, r\, dr\right] d\theta$$

Put $v = -r^2/2$

$dv = -2r/2\, dr$

$\therefore r\,dr = -dv$

$r=0, \quad v=0$

$r=\infty, \quad v=\infty$

$$= \int_{\theta=0}^{2\pi}\left[\int_{v=0}^{\infty} e^{-v}(-dv)\right] d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\int_{v=0}^{\infty} -e^{-v}\, dv\right] d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[-\left(\frac{e^{-v}}{-1}\right)_0^{\infty}\right] d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[-(e^{-\infty} - e^{0})\right] d\theta$$

$$I^2 = \int_{\theta=0}^{2\pi} d\theta = 2\pi$$

$$I^2 = 2\pi$$

$$I = \sqrt{2\pi}$$

using in (1)

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{2}\right) dy$$

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \rightarrow (3)$$

Put $y = \frac{x-a}{b}$, $b > 0$ in (3)

$dy = \frac{dx}{b}$ Limit $y = \infty$ $x = \infty$

$y = -\infty$ $x = -\infty$

using in (3)

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x-a)^2}{2b^2}\right] \frac{dx}{b} = 1$$

$$\int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{(x-a)^2}{2b^2}\right] = 1$$

Since $b > 0$,

$$f(x) = \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{(x-a)^2}{2b^2}\right], \quad -\infty < x - \infty$$

Satisfies the condition of p.d.f of a continuous type of random variable. A random variable of the continuous type that has a p.d.f of the form of f(x) is said to be normal distribution, and any f(x) of this form is called a normal p.d.f.

The m.g.f of a normal distribution

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{(x-a)^2}{2b^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[tx - \frac{(x-a)^2}{2b^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[\frac{2b^2tx - (x^2 - 2ax + a^2)}{2b^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{-2b^2tx + x^2 - 2ax + a^2}{2b^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{a^2 - (a+b^2t)^2}{2b^2}\right] \exp\left[-\frac{(x-a-b^2t)^2}{2b^2}\right] dx$$

$$= \exp\left[-\frac{a^2-(a+b^2t)^2}{2b^2}\right] \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{(x-a-b^2t)^2}{2b^2}\right] dx$$

$$= \exp\left[-\frac{a^2-(a+b^2t)^2}{2b^2}\right] \quad (1)$$

$$\because \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{(x-a-b^2t)^2}{2b^2}\right] dx = 1$$

$$= \exp\left[-\frac{a^2-(a^2+2ab^2t+b^4t^2)}{2b^2}\right]$$

$$= \exp\left[-\frac{\cancel{a^2} - \cancel{a^2} - 2ab^2t - b^4t^2}{2b^2}\right]$$

$$= \exp\left[-\frac{-2ab^2t - b^4t^2}{2b^2}\right]$$

$$= \exp\left[\frac{2ab^2t + b^4t^2}{2b^2}\right]$$

$$= \exp\left[\frac{2ab^2t}{2b^2} + \frac{b^4t^2}{2b^2}\right]$$

$$\boxed{M(t) = \exp\left[at + \frac{b^2t^2}{2}\right]}$$

Mean and Variance of Normal Distribution

$$M(t) = e^{(at + b^2t^2/2)} \rightarrow (1).$$

Diff (1) w.r.t t

$$M'(t) = e^{(at + b^2t^2/2)}\left[a + \frac{b^2 2t}{2}\right]$$

$$= M(t)\,(a + b^2t) \rightarrow (2)$$

Diff (2) w.r.t t

$$M''(t) = M(t)(b^2) + M'(t)\,(a + b^2t)$$

$$= M(t)\,b^2 + M(t)\,(a + b^2t)\,(a + b^2t)$$

$$M''(t) = M(t)\,b^2 + M(t)\,[a + b^2t)^2 \rightarrow (3)$$

Put $t = 0$ in (2) and (3)

$$\mu = M'(0) = M(0)\,(a + b^2 0)$$

$$= e^0(a) \qquad \because M(0) = e^{(a0 + \frac{b^2 0}{2})} = e^0 = 1$$

Mean. $\boxed{\mu = a}$

$$\text{Variance } \sigma^2 = M''(0) - \mu^2$$

$$= M(0)\,b^2 + M(0)\,(a + b^2 0)^2 - a^2$$

$$= 1\,b^2 + 1\,(a^2) - a^2$$

$$= b^2 + a^2 - a^2$$

$$\boxed{\text{Variance } \sigma^2 = b^2}$$

Note: i) A normal p.d.f in the form of

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

Here mean is μ and Variance σ^2

Then m.g.f $M(t)$ can be written

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

(2) If a random Variable X is in normal distribution then is it denoted as $X \sim N(\mu, \sigma^2)$

Problem:

(1) If X has the m.g.f $M(t) = e^{2t+32t^2}$ Find the p.d.f of the normal distribution.

Sol Let $M(t) = e^{2t+32t^2} \rightarrow (1)$

The m.g.f normal distribution is

$$M(t) = e^{\mu t + \sigma^2 t^2/2} \rightarrow (2)$$

Comparing (1) and (2), we have

$$\mu = 2, \quad \frac{\sigma^2}{2} = 32$$

$$\sigma^2 = 64, \quad \sigma = 8$$

$\therefore$ The p.d.f $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty$

$$= \frac{1}{8\sqrt{2\pi}} \exp\left[-\frac{(x-2)^2}{2\times 64}\right]$$

$$= \frac{1}{8\sqrt{2\pi}} \exp\left[-\frac{(x-2)^2}{128}\right], \quad -\infty < x < \infty$$

Theorem: If the random variable X is $N(\mu, \sigma^2)$, $\sigma^2 > 0$, then the random variable $W = (X-\mu)/\sigma$ is $N(0,1)$.

Proof: Let $G(w)$ be the distribution function of W

Since $\sigma > 0$

$$G(w) = P\left(\frac{X-\mu}{\sigma} \le w\right)$$
$$= P(X - \mu \le w\sigma)$$
$$= P(X \le w\sigma + \mu)$$

$$G(w) = \int_{-\infty}^{w\sigma+\mu} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \rightarrow ①$$

Put $y = \frac{x-\mu}{\sigma}$

$x = y\sigma + \mu$

$dx = dy\,\sigma$

$\frac{dx}{\sigma} = dy$

Limit

$x = -\infty$; $y = -\infty$

$x = w\sigma + \mu$; $y = \frac{w\sigma + \mu - \mu}{\sigma} = \frac{w\sigma}{\sigma} = w$

$\therefore$ From ①

$$G(w) = \int_{-\infty}^{w} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \cancel{dx}\, dy$$

The p.d.f $g(w) = G'(w)$ of the continuous type random variable W

$$g(w) = G'(w) = \left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)\right]_{y=-\infty}^{w}$$
$$= \left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right)\right], \quad -\infty < w < \infty$$

Thus W is $N(0,1)$

Hence the proof

Note: If X is $N(0,1)$ then $f(x) = \frac{1}{1\sqrt{2\pi}} \exp\left[-\frac{(x-0)^2}{2(1)^2}\right]$
$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

If X is $N(\mu, \sigma^2)$. Then $c_1 < c_2$ we have $P(X = c_1) = 0$

Now $P(c_1 < X < c_2) = P(X < c_2) - P(X < c_1)$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{c_2-\mu}{\sigma}\right) - P\left(\frac{X-\mu}{\sigma} < \frac{c_1-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{(c_2-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw - \int_{-\infty}^{(c_1-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$$

Because $W = (X-\mu)/\sigma$ is $N(0,1)$ [by previous Theorem]

Now $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$ and its derivatives $\Phi'(z) = \phi(z)$

Then, we say that $\Phi(z)$ and $\phi(z)$ are the distribution function and p.d.f of a standard normal distribution $N(0,1)$

Hence $P(c_1 < X < c_2) = \cancel{P\left(\frac{c_2-\mu}{\ }\right)}$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{c_2-\mu}{\sigma}\right) - P\left(\frac{X-\mu}{\sigma} < \frac{c_1-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{c_2-\mu}{\sigma}\right) - \Phi\left(\frac{c_1-\mu}{\sigma}\right)$$

Note: $\Phi(-x) = 1 - \Phi(x)$

Problem ① Let X be $N(2, 25)$. Then Find ~~$P(0 < X < 0$~~ $P(0 < X < 10)$ and $P(-8 < X < 1)$

Sol Given $\mu = 2$, $\sigma^2 = 25$

$\sigma = 5$

$$P(c_1 < X < c_2) = \Phi\left(\frac{c_2-\mu}{\sigma}\right) - \Phi\left(\frac{c_1-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{10-2}{5}\right) - \Phi\left(\frac{0-2}{5}\right)$$

$$= \Phi\left(\frac{8}{5}\right) - \Phi\left(\frac{2}{5}\right)$$

$$= \Phi(1.6) - \Phi(-0.4)$$

$$= 0.945 - (1 - 0.655)$$

$$= 0.600$$

$$P(-\cancel{8} < x < 1) = \phi\left(\frac{1-2}{5}\right) - \phi\left(\frac{-8-2}{5}\right)$$

$$= \phi\left(\frac{-1}{5}\right) - \phi(-2)$$

$$= \phi(-0.2) - \phi(-2)$$

$$= (1-0.579) - (1-0.977)$$

$$= 0.398$$

(2) Let x be $N(\mu, \sigma^2)$

Then $\cancel{P(\mu - 2\sigma)}$

$$P(\mu - 2\sigma < x < \mu + 2\sigma)$$

$$= \phi\left(\frac{\mu + 2\sigma - \mu}{\sigma}\right) - \phi\left(\frac{\mu - 2\sigma - \mu}{\sigma}\right)$$

$$= \phi\left(\frac{2\sigma}{\sigma}\right) - \phi\left(\frac{2\sigma}{\sigma}\right)$$

$$= \phi(2) - \phi(-2)$$

$$= 0.977 - (1 - 0.977)$$

$$= 0.954$$

Note: The mean μ of $N(\mu, \sigma^2)$ is called location parameter and the standard deviation σ is called Scale parameter

Theorem If the random variable x is $N(\mu, \sigma^2)$, $\sigma^2 > 0$, then the random variable $V = (x-\mu)^2/\sigma^2$ is $\chi^2(1)$

Proof: Let $V = W^2$, where $W = (x-\mu)/\sigma$ is $N(0,1)$, the distribution function $G(v)$ of V is, for $v \geq 0$

$$G(v) = P(W^2 \leq v)$$

$$= P(-\sqrt{v} \leq W \leq \sqrt{3})$$

That is $$G(v) = \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw, \quad 0 \leq v$$

$$= 2\int_{0}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw, \quad 0 \leq v \quad \rightarrow (1)$$

If $v < 0$ then $G(v) = 0$

Put $w = \sqrt{y}$ then

$$dw = \frac{1}{2\sqrt{y}}\, dy$$

Limit $w = 0, \quad y = 0$

$w = \sqrt{v} \quad y = v$

∴ Using in ①

$$G(v) = \cancel{2}\int_0^v \frac{1}{\sqrt{2\pi}}\, e^{-y/2}\, \frac{1}{\cancel{2}\sqrt{y}}\, dy$$

$$= \int_0^v \frac{1}{\sqrt{2\pi}\sqrt{y}}\, e^{-y/2}\, dy$$

Thus, the p.d.f $g(v) = G'(v)$ of the continuous type random variable V is

$$g(v) = G'(v) = \left[\frac{1}{\sqrt{2\pi}\sqrt{y}}\, e^{-y/2}\right]_{y=0}^{v}$$

$$= \left[\frac{1}{\sqrt{2\pi}\sqrt{v}}\, e^{-v/2}\right]$$

$$= \left[\frac{1}{\sqrt{2}\sqrt{\pi}}\, v^{1/2-1}\, e^{-v/2}\right]$$

Thus $g(v) = \frac{1}{\sqrt{\pi}\sqrt{2}}\, v^{1/2-1}\, e^{-v/2}, \quad 0 < v < \infty$

$= 0$ elsewhere

Since $g(v)$ is p.d.f $\int_0^\infty g(v)\, dv = 1$

Now, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

and thus V is $\chi^2(1)$.

Bivariate normal distribution

Consider the function

$$f(x,y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} e^{-q/2}, \quad -\infty < x < \infty, \; -\infty < y < \infty$$

where $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 < \rho < 1$

$$q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$$

The constants $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ and ρ are respective parameters of a distribution.

A joint p.d.f of this form is called bivariate normal p.d.f and the random variables X and Y are said to have a bivariate normal distribution.

Unit II

Distributions of Functions of Random Variables

Sampling Theory

Let $X_1, X_2, \cdots X_n$ denote n random variables have the joint p.d.f $f(x_1, x_2, \cdots x_n)$ These variables may or may not be independent.

Let Y be a random variable defined by a function of $X_1, X_2, \cdots X_n$ say $Y = u(X_1, X_2, \cdots, X_n)$ the p.d.f $f(x_1, x_2, \cdots x_n)$ is given, we find the p.d.f of Y.

Defn: A function of one or more random variables that does not depend upon any unknown parameter is called a Statistic

Example (i) The random variable $Y = \sum_1^n X_i$ is a Statistic.

(ii) The random variable $Y = (X_i - \mu)/\sigma$ is not a statistic unless μ and σ are known numbers.

Random sample

Defn: Let $X_1, X_2, \cdots X_n$ denote n independent random variables, each of which has the same but possibly unknown p.d.f $f(x)$. that is, the probability density functions of $X_1, X_2, \cdots X_n$ are respectively, $f_1(x_1) = f(x_1)$, $f_2(x_2) = f(x_2), \cdots, f_n(x_n) = f(x_n)$ so that the joint p.d.f is $f(x_1)f(x_2)\cdots f(x_n)$. The random variables $X_1, X_2, \cdots, X_n$ are said to be random sample from a distribution has p.d.f $f(x)$

Defn: Let $X_1, X_2, \cdots X_n$ denote a random sample of size n from a given distribution. The statistic

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \sum_{i=1}^{n} \frac{X_i}{n}.$$

is called mean of the random sample. and the statistic

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n} = \sum_{i=1}^{n} \frac{X_i^2}{n} - \bar{X}^2$$

is called the variance of the random sample

Transformations of Variables of the Discrete Type

~~He~~ A method of finding the distribution of a function of one ~~or~~ or more random variables is called the change of variable technique.

Problem ① Let X have the poisson p.d.f

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}, \quad x = 0, 1, 2, \cdots \rightarrow ①$$

$$= 0 \text{ elsewhere}$$

find the p.d.f of Y by $Y = 4X$.

Sol Let $y = 4x$

Then a transformation from x ~~to~~ to y.

We say that the transformation maps the space $A = \{x \mid x = 0, 1, 2, \cdots\}$ to the space $B = \{y \mid y = 0, 4, 8, 12, \cdots\}$

Now $y = 4x$, $x = \frac{1}{4} y$

Using in ①

$$g(y) = P(Y = y) = P\left(X = \frac{y}{4}\right)$$

$$= \frac{\mu^{y/4} e^{-\mu}}{(y/4)!}, \quad y = 0, 4, 8, \cdots$$

$$= 0 \text{ elsewhere.}$$

② Let X have the binomial p.d.f

$$f(x) = \frac{3!}{x!(3-x)!} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{3-x}, \quad x = 0, 1, 2, 3, \cdots$$

$$= 0 \text{ elsewhere}$$

Find the p.d.f of $Y = X^2$

Sol Let $y = x^2$

Then a transformation from x to y.

We say that the transformation maps the space $A = \{x \mid x = 0, 1, 2, 3, \cdots\}$ to the space

$$B = \{y \mid y = 0, 1, 4, 9, \cdots\}$$

$y = x^2$

$x = \pm\sqrt{y}$

$x = \sqrt{y}$

Then $g(y) = f(\sqrt{y}) = \dfrac{3!}{(\sqrt{y})!\,(3-\sqrt{y})!}\left(\frac{2}{3}\right)^{\sqrt{y}}\left(\frac{1}{3}\right)^{3-\sqrt{y}}$; $y = 0, 1, 4, 9, \ldots$

$= 0$ elsewhere.

Remark ~~Note~~ : Let $f(x_1, x_2)$ be the joint p.d.f of two discrete type random variables X_1 and X_2 with A (two-dimensional) ~~set~~ set of points at which $f(x_1, x_2) \geq 0$. Let $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a one-to-one transformation that maps A onto B. The joint p.d.f of the two random variables $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$ is given by

$$g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)], \quad (y_1, y_2) \in B$$

$= 0$ elsewhere

Where $x_1 = w_1(y_1, y_2)$ $x_2 = w_2(y_1, y_2)$ is the single valued inverse of $y_1 = u_1(x_1, x_2)$, $y_2 = u_2(x_1, x_2)$. From this joint p.d.f $g(y_1, y_2)$, we may obtain the marginal p.d.f of Y_1 by summing on y_2 or the marginal p.d.f of Y_2 by summing on y_1

Problem ① Let X_1 and X_2 be two independent random variables that have Poisson distribution with means μ_1 and μ_2 respectively. The joint p.d.f of X_1 and X_2 is

$$\frac{\mu_1^{x_1}\mu_2^{x_2}e^{-\mu_1-\mu_2}}{x_1!\,x_2!}, \quad x_1 = 0, 1, 2, 3, \ldots;\ x_2 = 0, 1, 2, 3, \ldots$$

and 0 ~~elsew~~ elsewhere.

Find the p.d.f of $Y_1 = X_1 + X_2$.

Sol We use the change of variable technique

Let $Y_1 = X_1 + X_2$, Consider $Y_2 = X_2$

Then $y_1 = x_1 + x_2$ and $y_2 = x_2$

$A = \{ (x_1, x_2) \mid x_1 = 0, 1, 2, 3, \ldots ; x_2 = 0, 1, 2, 3, \ldots \}$

$B = \{ (y_1, y_2) \mid y_2 = 0, 1, 2, \ldots y_1 \text{ and } y_1 = 0, 1, 2, \ldots \}$

If $(y_1, y_2) \in B$ then $0 \le y_2 \le y_1$.

The inverse functions are given by $x_1 = y_1 - y_2$ and $x_2 = y_2$

Thus the joint p.d.f of Y_1 and Y_2 is

$$g(y_1, y_2) = \frac{\mu_1^{y_1 - y_2} \mu_2^{y_2} e^{-\mu_1 - \mu_2}}{(y_1 - y_2)! \, y_2!} ; \quad (y_1, y_2) \in B$$

$$= 0 \quad \text{elsewhere}$$

The marginal p.d.f of Y_1 is given by

$$g_1(y_1) = \sum_{y_2 = 0}^{y_1} g(y_1, y_2)$$

$$= \frac{e^{-\mu_1 - \mu_2}}{y_1!} \sum_{y_2 = 0}^{y_1} \frac{y_1!}{(y_1 - y_2)! \, y_2!} \mu_1^{y_1 - y_2} \mu_2^{y_2}$$

$$= \frac{(\mu_1 + \mu_2)^{y_1} e^{-\mu_1 - \mu_2}}{y_1!}, \quad y_1 = 0, 1, 2 \ldots$$

and 0 elsewhere

That $Y_1 = X_1 + X_2$ has a Poisson distribution with parameter $\mu_1 + \mu_2$

Problem ① Let X be a random Variable of the Continuous, having p.d.f

$$f(x) = 2x, \quad 0 < x < 1$$
$$= 0 \quad \text{elsewhere}$$

Find the p.d.f of $Y = 8x^3$ by using change the Variable technique

Sol Let $y = 8x^3$

Then a transformation from x to y

The transformation maps the space $A = \{x : 0 < x < 1\}$ to the space $B = \{y / 0 < y < 8\}$

$$y = 8x^3$$
$$x^3 = y/8$$
$$x = \frac{1}{2}\sqrt[3]{y}$$

The event $\frac{1}{2}\sqrt[3]{a} < x < \frac{1}{2}\sqrt[3]{b}$

$$P(a < Y < b) = P\left(\frac{1}{2}\sqrt[3]{a} < x < \frac{1}{2}\sqrt[3]{b}\right)$$

$$= \int_{\frac{\sqrt[3]{a}}{2}}^{\sqrt[3]{b}/2} 2x \, dx \rightarrow ①$$

Limit

$$x = \frac{\sqrt[3]{a}}{2} \qquad \frac{\sqrt[3]{a}}{\cancel{2}} = \frac{1}{\cancel{2}}\sqrt[3]{y}$$

$$y = a$$

$$y = \frac{\sqrt[3]{b}}{2} \qquad y = b$$

Now $x = \frac{1}{2}\sqrt[3]{y}$

$$\frac{dx}{dy} = \frac{1}{2}\,\frac{1}{3}\, y^{1/3 - 1}$$
$$= \frac{1}{6} y^{-2/3}$$
$$= \frac{1}{6 y^{2/3}}$$
$$dx = \frac{1}{6y^{2/3}}\, dy$$

Using in ①

$$P(a<Y<b) = \int_a^b x\left(\frac{\sqrt[3]{y}}{x}\right)\left(\frac{1}{6y^{2/3}}\right)dy$$

$$= \int_a^b y^{1/3}\left(\frac{1}{6y^{2/3}}\right)dy$$

$y^{1/3} \cdot y^{-2/3}$
$= y^{1/3-2/3}$
$= y^{-1/3}$

$$= \int_a^b \frac{1}{6y^{1/3}}\,dy$$

Since this is true for every $0<a<b<\infty$, the p.d.f $g(y)$ of Y is the integrand

$$\therefore\ g(y) = \frac{1}{6y^{1/3}}, \quad 0<y<\infty$$

$$= 0 \quad \text{elsewhere}$$

Problem ② Let x have the p.d.f $f(x) = 1, \quad 0<x<1$
$= 0$ elsewhere

Show that the random variable $Y = -2\log x$ has a Chi-Square distribution with 2 degrees of freedom.

Sol

Let $y = -2\log x = u(x)$

$~~x~~ = \log x = -y/2$

$x = e^{-y/2} = w(y)$

The space A is $A = \{x : 0<x<1\}$ one-to-one transformation y to $B = \{y : 0<y<\infty\}$

The Jacobian of the transformation is

$$J = \frac{dx}{dy} = w'(y) = -\frac{1}{2}e^{-y/2}$$

The p.d.f $g(y)$ of $Y = -2\log x$ is. i

$$g(y) = f[w(y)]\,|J|$$

$$= f[e^{-y/2}]\,|J|$$

$$= (1)\left|-\frac{1}{2}e^{-y/2}\right|$$

$$= \frac{1}{2}e^{-y/2}, \quad 0<y<\infty$$

$= 0$ elsewhere

E3

To Find the p.d.f of the functions of two random Variables of Continuous type

Let $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a one-to-One transformation maps a (two-dimensional) set A in the $x_1 x_2$-plane onto a (two-dimensional) set B in the $y_1 y_2$-plane. If express each of x_1 and x_2 in terms of y_1 and y_2 then $x_1 = w_1(y_1, y_2)$ $x_2 = w_2(y_1, y_2)$

The derminant of order 2 $\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$ is

Called the Jacobian of the transformation denoted by the symbol J.

Problem Let A be the set $A = \{(x_1, x_2) \mid 0 < x_1 < 1,\ 0 < x_2 < 1\}$

Find the set B in $y_1 y_2$-plane that is the mapping of A under the One-to-one transformation $y_1 = u_1(x_1, x_2) = x_1 + x_2$
$y_2 = u_2(x_1, x_2) = x_1 - x_2$

and compute the Jacobian of the transformation.

Sol Let $x_1 = w_1(y_1, y_2) = \frac{1}{2}(y_1 + y_2)$
$x_2 = w_2(y_1, y_2) = \frac{1}{2}(y_1 - y_2)$

$\therefore y_1 + y_2 = 2x_1$
$x_1 = \frac{1}{2}(y_1 + y_2)$
$y_1 - y_2 = 2x_2$
$x_2 = \frac{1}{2}(y_1 - y_2)$

To determine the set B in the plane $y_1 y_2$-plane onto which A is mapped under the transformation. The boundaries of A are transformed as follows into the boundaries of B

$x_1 = 0$ into $0 = \frac{1}{2}(y_1 + y_2)$

$x_1 = 1$ into $1 = \frac{1}{2}(y_1 + y_2)$

$x_2 = 0$ into $0 = \frac{1}{2}(y_1 - y_2)$

$x_2 = 1$ into $1 = \frac{1}{2}(y_1 - y_2)$

$$\text{Jacobian } J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

<u>To find the joint p.d.f of two functions of two continuous type random variables</u>

Let X_1 and X_2 be random variables of continuous type, having joint p.d.f $h(x_1, x_2)$.

Let A be the two dimensional set in the $x_1 x_2$-plane where $h(x_1, x_2) > 0$

Let $Y_1 = u_1(X_1, X_2)$ be a random variable whose p.d.f is to be found.

If $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a one-to-one transformation of A onto a set B in the $y_1 y_2$-plane

$$P[(Y_1, Y_2) \in B] = \iint_B h[w_1(y_1, y_2), w_2(y_1, y_2)]\,|J|\,dy_1 dy_2$$

The joint p.d.f $g(y_1, y_2)$ of Y_1 and Y_2 is

$$g(y_1, y_2) = h[w_1(y_1, y_2), w_2(y_1, y_2)]\,|J|, \quad (y_1, y_2) \in B$$
$$= 0 \quad \text{elsewhere}$$

<u>Problem</u> Let the random variable X have the p.d.f

$$f(x) = 1, \quad 0 < x < 1$$
$$= 0 \quad \text{elsewhere}$$

and let X_1, X_2 denote a random sample of this distribution

The joint p.d.f of X_1 and X_2 is then

$$h(x_1, x_2) = f(x_1) f(x_2) = 1, \quad 0 < x_1 < 1, \; 0 < x_2 < 1$$
$$= 0 \quad \text{elsewhere} \quad \to (1)$$

Consider the two random Variables $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$

To find the joint p.d.f of Y_1 and Y_2

The one-to-one transformation $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$ maps A onto the space B.

$$y_1 + y_2 = 2x_1 \qquad y_1 - y_2 = 2x_2$$

$$x_1 = \frac{1}{2}(y_1 + y_2) \qquad x_2 = \frac{1}{2}(y_1 - y_2)$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -1/2$$

$$g(y_1, y_2) = h\left[\frac{1}{2}(y_1 + y_2) ; \frac{1}{2}(y_1 - y_2)\right] |J|$$

$$= f\left[\frac{1}{2}(y_1 + y_2)\right] f\left[\frac{1}{2}(y_1 - y_2)\right] |J|$$

$$= (1)(1)\frac{1}{2} \; ; \qquad (y_1, y_2) \in B$$

$$= \frac{1}{2}$$

$$= 0 \quad \text{elsewhere}$$

The marginal p.d.f of Y_1 is given by

$$g_1(y_1) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_2$$

$$= \int_{-y_1}^{y_1} \frac{1}{2}\, dy_2$$

$$= \frac{1}{2}\left(y_2\right)_{y_2 = -y_1}^{y_1}$$

$$= \frac{1}{2}(y_1 + y_1)$$

$$= y_1$$

$$g_1(y_1) = \int_{y_1 - 2}^{2 - y_1} 1/2\, dy_2 \quad , \quad 1 < y_1 < 2$$

$$= 2 - y_1$$

$$= 0 \quad \text{elsewhere}$$

111^{ly} The marginal p.d.f $g_2(y_2)$ is given by

$$g_2(y_2) = \int_{-y_2}^{y_2+2} \frac{1}{2}\, dy_1$$

$$= \frac{1}{2}\left(y_1\right)_{y_1=-y_2}^{y_2+2}$$

$$= \frac{1}{2}\left[(y_2+2) + y_2\right]$$

$$= \frac{1}{2}\left[2y_2+2\right]$$

$$= y_2+1 \quad , \quad -1 < y_2 \leq 0$$

$$= \int_{y_2}^{2-y_2} \tfrac{1}{2}\, dy_1 = 1-y_2 \, , \quad 0 < y_2 < 1$$

$$= 0 \text{ elsewhere}$$

The Beta, t and F Distributions:

The Beta distribution:

Let X_1 and X_2 be two independent random variables that have gamma distribution and joint p.d.f

$$h(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}, \quad 0 < x_1 < \infty, \ 0 < x_2 < \infty$$

$$= 0 \quad \text{elsewhere, where } \alpha > 0, \ \beta > 0$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1+X_2)$

Prove that Y_1 and Y_2 are independent and find marginal p.d.f Y_2.

Proof Let $y_1 = u_1(x_1, x_2) = x_1 + x_2 \rightarrow (1)$

$y_2 = u_2(x_1, x_2) = \frac{x_1}{x_1+x_2} \rightarrow (2)$

From (2) $y_2 = \frac{x_1}{x_1+x_2}$

$= \frac{x_1}{y_1}$

$$\boxed{x_1 = y_1 y_2}$$

Now $y_1 = x_1 + x_2$

$= y_1 y_2 + x_2$

$x_2 = y_1 - y_1 y_2$

$$\boxed{x_2 = y_1(1-y_2)}$$

Jacobian $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$

$= \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1 y_2 - y_1(1-y_2)$

$= -y_1 y_2 - y_1 + y_1 y_2$

$= -y_1 \neq 0$

The tromformation is one-to-one, and it maps A onto B
$B = \{ (y_1, y_2) : 0 < y_1 < \infty, 0 < y_2 < 1 \}$ in the $y_1 y_2$-plane.

The joint p.d.f of Y_1 and Y_2 is

$$g(y_1, y_2) = h(x_1, x_2) |J|$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1 - x_2} (y_1)$$

$$= (y_1) \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (y_1 y_2)^{\alpha-1} [y_1(1-y_2)]^{\beta-1} e^{-y_1}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha-1} \cdot y_1 \, y_1^{\beta-1} \, y_2^{\alpha-1} (1-y_2)^{\beta-1} e^{-y_1}$$

$$= \frac{y_2^{\alpha-1}(1-y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha+\beta-1} e^{-y_1}, \quad 0 < y_1 < \infty, \ 0 < y_2 < 1$$

$$= 0 \quad \text{elsewhere}$$

We know that the random ~~to~~ Variables x_1 and x_2 have the joint p.d.f $f(x_1, x_2)$. Then x_1 and x_2 are independent ~~iff~~ if and only if $f(x_1, x_2)$ can be written as a product of a nonnegative function of x_1 alone and nonnegative function x_2 alone That is

$$f(x_1, x_2) = g(x_1) h(x_2) \text{ where } g(x_1) > 0, x_1 \in A_1 \text{ Zero}$$

else where and $h(x_2) > 0$, $x_2 \in A_2$ Zero elsewhere

Thus the random Variables are independent

~~The~~

The marginal p.d.f of Y_2 is

$$g_2(y_2) = \frac{y_2^{\alpha-1}(1-y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_1^{\alpha+\beta-1} e^{-y_1} dy_1$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1}(1-y_2)^{\beta-1}, \quad 0<y_2<1$$

$$\left[\because \int_0^\infty y_1^{\alpha+\beta-1} e^{-y_1} dy_1 = \Gamma(\alpha+\beta)\right]$$

$$= 0 \text{ elsewhere}$$

This p.d.f is the beta distribution with parameter α and β. Since $g(y_1, y_2) = g_1(y_1)\, g_2(y_2)$ it must be that the p.d.f of Y_1 is

$$g_1(y_1) = \frac{1}{\Gamma(\alpha+\beta)} y_1^{\alpha+\beta-1} e^{-y_1}, \quad 0<y_1<\infty$$

$$= 0 \text{ elsewhere}$$

which is that of a gamma distribution with parameter values $\alpha+\beta$ and 1.

Note: Mean and Variance of Y_2, which has a beta ~~distribut~~ distribution α and β are

$$\boxed{\mu = \frac{\alpha}{\alpha+\beta}} \text{ and } \boxed{\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}}$$

The t - distribution

Let W be a random variable that is $N(0,1)$; Let V be a random variable that is $\chi^2(r)$; and let W and V be independent. Then the joint p.d.f of W and V, say $h(w, v)$ is the product of the p.d.f of W and that of V or

$$h(w, v) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma(\frac{r}{2})\, 2^{r/2}} v^{r/2-1} e^{-v/2}, \quad -\infty<w<\infty, \quad 0<v<\infty$$

$$= 0 \text{ elsewhere}$$

Define a new random Variable T by $T = \frac{W}{\sqrt{V/r}}$

Find the p.d.f $g_1(t)$ of T, by using Change of Variable technique

Proof: The equations $t = \frac{w}{\sqrt{v/r}}$ and $u = v$

Define a one-to-one transformation that maps $A = \{(w, v) / -\infty < w < \infty, 0 < v < \infty\}$ onto $B = \{(t, u) / -\infty < t < \infty, 0 < u < \infty\}$

Since $w = \frac{t\sqrt{u}}{\sqrt{r}}$, $v = u$

$$\text{Jacobian } J = \begin{vmatrix} \frac{\partial w}{\partial t} & \frac{\partial w}{\partial u} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial u} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\sqrt{u}}{\sqrt{r}} & \frac{t}{2\sqrt{r}\sqrt{u}} \\ 0 & 1 \end{vmatrix} = \frac{\sqrt{u}}{\sqrt{r}}$$

$$|J| = \frac{\sqrt{u}}{\sqrt{r}}$$

The joint p.d.f of T is $g(t, u) = h\left(\frac{t\sqrt{u}}{\sqrt{r}}, u\right)|J|$

$$= \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma(\frac{r}{2}) 2^{r/2}} v^{r/2 - 1} e^{-v/2} \left(\frac{\sqrt{u}}{\sqrt{r}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(r/2) 2^{r/2}} u^{r/2-1} \exp\left[-\frac{w^2 + v}{2}\right] \frac{\sqrt{u}}{\sqrt{r}}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})\, 2^{r/2}} u^{r/2-1} \exp\left[-\frac{\frac{t^2 u}{r} + u}{2}\right] \frac{\sqrt{u}}{\sqrt{r}}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})\, 2^{r/2}} u^{r/2-1} \exp\left[-\left(\frac{t^2 u}{2r} + \frac{u}{2}\right)\right] \frac{\sqrt{u}}{\sqrt{u}}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma\left(\frac{r}{2}\right) 2^{r/2}} u^{\frac{r}{2}-1} \exp\left[-\frac{u}{2}\left(1+\frac{t^2}{r}\right)\right] \frac{\sqrt{u}}{\sqrt{r}};$$

$$-\infty < t < \infty, \quad 0 < u < \infty$$

$= 0$ elsewhere

The maginal p.d.f of T is

$$g_1(t) = \int_{-\infty}^{\infty} g(t,u)\,du$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi r}\,\Gamma\left(\frac{r}{2}\right) 2^{r/2}} u^{\frac{(r+1)}{2}-1} \exp\left[-\frac{u}{2}\left(1+\frac{t^2}{r}\right)\right] du$$

Put $z = \frac{u\left[1+\left(t^2/r\right)\right]}{2}$

Limit

Put $u = 0$ $\quad z = 0$

$u = \infty$ $\quad z = \infty$

$$u = \frac{2z}{1+t^2/r}$$

$$du = \frac{2\,dz}{1+t^2/r}$$

$$g_1(t) = \int_0^{\infty} \frac{1}{\sqrt{2\pi r}\,\Gamma\left(\frac{r}{2}\right) 2^{r/2}} \left[\frac{2z}{1+t^2/r}\right]^{\frac{r+1}{2}-1} e^{-z} \left(\frac{2}{1+t^2/r}\right) dz$$

$$= \frac{1}{\sqrt{2\pi r}\,\Gamma\left(\frac{r}{2}\right) 2^{r/2}} \int_0^{\infty} 2^{\frac{r+1}{2}-1} \left(\frac{z}{1+t^2/r}\right)^{\frac{r+1}{2}-1} e^{-z} \cdot 2\left(\frac{1}{1+t^2/r}\right) dz$$

$$= \frac{1}{2^{1/2}\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right) 2^{r/2}} 2^{\frac{r+1}{2}} \int_0^{\infty} \frac{z^{\frac{r+1}{2}-1}}{\left(1+t^2/r\right)^{\frac{r+1}{2}-1}} e^{-z} \left(\frac{1}{1+t^2/r}\right) dz$$

$$= \frac{1}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)} \int_0^{\infty} \frac{z^{\frac{r+1}{2}-1}}{\left(1+t^2/r\right)^{\frac{r+1}{2}}} \cdot \cancel{\left(1+\frac{t^2}{r}\right)} e^{-z} \cancel{\left(\frac{1}{1+t^2/r}\right)} dz$$

$$= \frac{1}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)} \frac{1}{\left(1+t^2/r\right)^{r+1/2}} \int_0^{\infty} z^{\frac{r+1}{2}-1} e^{-z}\,dz$$

$$= \frac{1}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)} \frac{1}{\left(1+t^2/r\right)^{\frac{r+1}{2}}} \Gamma\left(\frac{r+1}{2}\right) \qquad \left(\because \Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z}\,dz\right)$$

$$\boxed{g(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}}} \quad -\infty < t < \infty$$

Thus, if W is $N(0,1)$, if V is $\chi^2(r)$ and if W and V are independent then $T = \frac{W}{\sqrt{V/r}}$ has the p.d.f $g_1(t)$

The distribution of the random variable T is called a t-distribution [Student's - distribution] [W.S. Gosset] (Student's t-distribution)

—x—

The F-distribution

Consider two independent Chi-Square random variables U and V having r_1 and r_2 degrees of freedom respectively. The Joint p.d.f $h(u,v)$ of U and V is then

$$h(u,v) = \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{(r_1+r_2)/2}} u^{r_1/2-1} v^{\frac{r_2}{2}-1} e^{-(u+v)/2},$$

$$0 < u < \infty, \quad 0 < v < \infty$$

$$= 0 \quad \text{elsewhere}.$$

We define the new random variable

$$W = \frac{U/r_1}{V/r_2}$$

Find the p.d.f $g_1(w)$

Proof: Let the equations $w = \frac{u/r_1}{v/r_2}$, $z = v$

Define a one-to-one transformation that maps the set $A = \{(u, v) : 0 < u < \infty,\ 0 < v < \infty\}$ onto the set

$B = \{(w, z),\ 0 < w < \infty,\ 0 < z < \infty\}$

Now, $w = \dfrac{u/r_1}{v/r_2}$ and $z = v$

$$\frac{u}{r_1} = \frac{wv}{r_2}$$

$$u = \left(\frac{r_1}{r_2}\right) wv$$

$$\boxed{u = \left(\frac{r_1}{r_2}\right) zw} \quad \because z = v$$

and $\boxed{v = z}$

$$\text{Jacobian } J = \begin{vmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$= \cancel{\begin{vmatrix} \left(\frac{r_1}{r_2}\right) z & 0 \\ 0 & \left(\frac{r_1}{r_2}\right) w \end{vmatrix}}$$

$$= \begin{vmatrix} \left(\frac{r_1}{r_2}\right) z & \left(\frac{r_1}{r_2}\right) w \\ 0 & 1 \end{vmatrix} = \left(\frac{r_1}{r_2}\right) z$$

$$|J| = \left(\frac{r_1}{r_2}\right) z$$

The Joint p.d.f $g(w, z) = h(u, v)\,|J|$

$$= \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{(r_1+r_2)/2}}\, u^{\frac{r_1}{2}-1}\, v^{r_2/2 - 1}\, e^{-(u+v)/2}\, |J|$$

$$= \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{(r_1+r_2)/2}} \left(\frac{r_1}{r_2} zw\right)^{\frac{r_1}{2}-1} z^{\frac{r_2}{2}-1}$$

$$\exp\left[-\frac{\left(\frac{r_1}{r_2} zw + z\right)}{2}\right] \left(\frac{r_1}{r_2}\right) z$$

$$= \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\, 2^{(r_1+r_2)/2}} \left(\frac{r_1 z w}{r_2}\right)^{\frac{r_1}{2}-1} z^{\frac{r_2}{2}-1} \times \exp\left[-\frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)\right] \frac{r_1 z}{r_2}$$

$$= \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\, 2^{(r_1+r_2)/2}} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}-1} z^{\frac{r_1}{2}-1} w^{\frac{r_1}{2}-1} z^{\frac{r_2}{2}-1} \times \exp\left[-\frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)\right] \frac{r_1}{r_2} z$$

$$g(w,z) = \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\, 2^{(r_1+r_2)/2}} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1} z^{(r_1+r_2)/2-1} \times \exp\left[-\frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)\right],$$

$0 < w < \infty$,
$0 < z < \infty$

The marginal $g_1(w)$ of W is

$$g_1(w) = \int_{-\infty}^{\infty} g(w,z)\, dz$$

$$= \int_0^{\infty} \frac{\left(\frac{r_1}{r_2}\right)^{r_1/2} (w)^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\, 2^{(r_1+r_2)/2}} z^{\frac{(r_1+r_2)}{2}-1} \exp\left[-\frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)\right] dz \quad \rightarrow (1)$$

Put $y = \frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)$

Limit

$y = 0 \quad z = 0$
$y = 0 \quad z\ \infty$

$$dy = \frac{1}{2}\left(\frac{r_1 w}{r_2}+1\right) dz$$

$$z = \frac{2y}{\left(\frac{r_1 w}{r_2}+1\right)}$$

Using in (1)

$$g_1(w) = \int_0^\infty \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} (w)^{\frac{r_1}{2}-1}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{(r_1+r_2)/2}} \left(\frac{2y}{\frac{r_1 w}{r_2}+1}\right)^{\frac{r_1+r_2}{2}-1} e^{-y} \left(\frac{2}{\frac{r_1 w}{r_2}+1}\right) dy$$

$$= \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} (w)^{\frac{r_1}{2}-1}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{(r_1+r_2)/2}} \int_0^\infty \frac{2^{\frac{r_1+r_2}{2}-1}\; y^{\frac{r_1+r_2}{2}-1}}{\left(\frac{r_1 w}{r_2}+1\right)^{\frac{r_1+r_2}{2}-1}} \frac{2}{\left(\frac{r_1 w}{r_2}+1\right)} e^{-y}\, dy$$

$$= \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} (w)^{\frac{r_1}{2}-1}\, 2^{\frac{r_1+r_2}{2}}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{\frac{r_1+r_2}{2}}} \frac{1}{\left(\frac{r_1 w}{r_2}+1\right)^{\frac{r_1+r_2}{2}}} \int_0^\infty y^{\frac{r_1+r_2}{2}-1} e^{-y}\, dy$$

$$= \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} (w)^{\frac{r_1}{2}-1}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)} \frac{1}{\left(\frac{r_1 w}{r_2}+1\right)^{(r_1+r_2)/2}} \Gamma\left(\frac{r_1+r_2}{2}\right)$$

$$g_1(w) = \frac{\Gamma\left(\frac{r_1+r_2}{2}\right)\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)} \frac{(w)^{\frac{r_1}{2}-1}}{\left[1+\frac{r_1 w}{r_2}\right]^{\frac{r_1+r_2}{2}}} \;;\quad 0 < w < \infty$$

If U and V are independent Chi-Square variables with r_1 and r_2 degrees of freedom respectively, then $W = \frac{U/r_1}{V/r_2}$ has the p.d.f $g_1(w)$. The distribution of this random variable is called an F-distribution. We ~~calls~~ call the ratio, which is denoted by W, F

That $\boxed{F = \frac{U/r_1}{V/r_2}}$

The moment-generating Function Technique:

Problem ① Let the independent random variables X_1 and X_2 have the same p.d.f

$$f(x) = \frac{x}{6}, \quad x = 1, 2, 3$$
$$= 0 \quad \text{elsewhere}$$

Then the joint p.d.f of X_1 and X_2 is

$$f(x_1) f(x_2) = \frac{x_1 x_2}{36}, \quad x_1 = 1, 2, 3, \; x_2 = 1, 2, 3$$
$$= 0 \quad \text{elsewhere}$$

Now $P(X_1 = 2, X_2 = 3) = \frac{(2)(3)}{36} = 1/6$

$P(X_1 + X_2 = 3)$, here $X_1 + X_2 = 3$

The events are $(X_1 = 1, X_2 = 2)$ and $(X_1 = 2, X_2 = 1)$

Thus $P(X_1 + X_2 = 3) = P(X_1 = 1, X_2 = 2) + P(X_1 = 2, X_2 = 1)$

$$= \frac{(1)(2)}{36} + \frac{(2)(1)}{36} = \frac{4}{36} = \frac{1}{9}$$

Let y represent any of the numbers 2, 3, 4, 5, 6.

The events $X_1 + X_2 = y$,

The Probability of each of the events $X_1 + X_2 = y$, $y = 2, 3, 4, 5, 6$ can be computed as in the case $y = 3$

Let $g(y) = P(X_1 + X_2 = y)$

y	2	3	4	5	6
$g(y)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

$X_1 + X_2 = 4$

$(1, 3), (2, 2), (3, 1)$

$\frac{(1)(3)}{36} + \frac{(2)(2)}{36} + \frac{(3)(1)}{36}$

$\frac{3}{36} + \frac{4}{36} + \frac{3}{36}$

Now the m.g.f of Y is

$$M(t) = E[e^{t(X_1 + X_2)}]$$
$$= E[e^{tX_1} \cdot e^{tX_2}]$$
$$= E[e^{tX_1}] \, E[e^{tX_2}] \quad \rightarrow ①$$

$\because$ X_1 and X_2 are independent

Here x_1 and x_2 have the same distribution, so they have the same m.g.f

$$E(e^{tx_1}) = E(e^{tx_2}) = \frac{1}{6}e^{t} + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$$

Using in ①

$$M(t) = \left(\frac{1}{6}e^{t} + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}\right)^2$$

$$= \frac{1}{36}\left(e^{t} + 2e^{2t} + 3e^{3t}\right)^2$$

$$= \frac{1}{36}\left[e^{2t} + 4e^{4t} + 9e^{6t} + 4e^{t}\cdot e^{2t} + 6e^{t}\cdot e^{3t} + 12e^{2t}\cdot e^{3t}\right]$$

$$= \frac{1}{36}\left[e^{2t} + 4e^{4t} + 9e^{6t} + 4e^{3t} + 6e^{4t} + 12e^{5t}\right]$$

$$= \frac{1}{36}\left[e^{2t} + 4e^{3t} + 10e^{4t} + 12e^{5t} + 9e^{6t}\right]$$

$$= \frac{e^{2t}}{36} + \frac{4e^{3t}}{36} + \frac{10e^{4t}}{36} + \frac{12e^{5t}}{36} + \frac{9e^{6t}}{36}$$

In $M(t)$, the p.d.f $g(y)$ of Y is except 0 ~~and~~ at $y = 2, 3, 4, 5, 6)$ and $g(y)$ assumes the values $\frac{1}{36}, \frac{4}{36}, \frac{10}{36}, \frac{12}{36}, \frac{9}{36}$

② Let x_1 and x_2 be independent with normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively. Define the ~~root~~ random variable Y by $Y = x_1 - x_2$. Find the p.d.f $g(y)$ of Y, using m.g.f technique.

Sol The m.g.f of Y is $M(t) = E(e^{tY})$

$$= E\left[e^{t(x_1 - x_2)}\right]$$

$$= E\left[e^{tx_1 - tx_2}\right]$$

$$= E\left[e^{tx_1}\cdot e^{-tx_2}\right]$$

$$= E[e^{tx_1}]\, E[e^{-tx_2}] \rightarrow ①$$

Since x_1 and x_2 are independent

$$E[e^{tx_1}] = \exp\left[\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right]$$

$$E[e^{-tx_2}] = \exp\left[-\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right]$$

using in ①

$$M(t) = \exp\left[\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right] \exp\left[-\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right]$$

$$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}}\, e^{-\mu_2 t + \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2} - \mu_2 t + \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{(\mu_1 - \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

$$= \exp\left[(\mu_1 - \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}\right]$$

The distribution of y completely determined by its m.g.f $M(t)$, and it is seen that y has the p.d.f $g(y)$ which is $N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

That is, the difference between two independent, normally distributed random variables is itself a random variable which normally distributed with mean equal to the difference of the means and the variance equal to the sum of the variances.

Theorem 1: Let $x_1, x_2, \dots x_n$ be independent random variables having the normal distributions $N(\mu_1, \sigma_1^2)$, $N(\mu_2, \sigma_2^2)$, $\dots$, $N(\mu_n, \sigma_n^2)$ respectively. The random variable $y = k_1 x_1 + k_2 x_2 + \dots + k_n x_n$, where $k_1, k_2, \dots, k_n$ are real constants, is normally distributed with mean $k_1\mu_1 + k_2\mu_2 + \dots + k_n\mu_n$ and variance $k_1^2\sigma_1^2 + k_2^2\sigma_2^2 + \dots + k_n^2\sigma_n^2$. That is, y is $N(\sum_1^n k_i\mu_i, \sum_1^n k_i^2\sigma_i^2)$

Proof: Let $x_1, x_2, \dots x_n$ be the independent random variables having the normal distribution $N(\mu_1, \sigma_1^2)$, $N(\mu_2, \sigma_2^2)$, $\dots$, $N(\mu_n, \sigma_n^2)$ respectively.

Then the m.g.f of x_i is $i = 1, 2, \dots n$

$$M(t) = \cancel{\exp}\, E(e^{tx_i}) = \exp\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right), \quad i = 1, 2, \dots n. \quad \to (1)$$

Consider the random variable

$$y = kx_1 + k_2 x_2 + \dots + k_n x_n.$$

The m.g.f of y is

$$M(t) = E[e^{ty}]$$

$$= E[e^{t(k_1x_1 + k_2x_2 + \dots + k_nx_n)}]$$

$$= E[e^{tk_1x_1 + tk_2x_2 + \dots + tk_nx_n}]$$

$$= E[e^{tk_1x_1} \cdot e^{tk_2x_2} \cdots e^{tk_nx_n}]$$

$$= E[e^{tk_1x_1}]\, E[e^{tk_2x_2}] \dots E[e^{tk_nx_n}] \quad \to (2)$$

$\because x_1, x_2 \dots x_n$ are independent

Now $E[e^{tk_ix_i}] = \exp\left[\mu_i(k_it) + \frac{\sigma_i^2 (k_it)^2}{2}\right]$ from (1), $i = 1, 2, \dots n$ for ~~real~~ k

Using in (2) the m.g.f of y is.

$$M(t) = \exp\left[(k_1\mu_1)t + \frac{(k_1^2\sigma_1^2)t^2}{2}\right] \exp\left[(k_2\mu_2)t + \frac{(k_2^2\sigma_2^2)t^2}{2}\right] \dots \exp\left[(k_n\mu_n)t + \frac{(k_n^2\sigma_n^2)t^2}{2}\right]$$

$$= \prod_{i=1}^{n} \exp\left[(k_i \mu_i) t + \frac{(k_i^2 \sigma_i^2) t^2}{2} \right]$$

$$= \exp\left[\left(\sum_{i=1}^{n} (k_i \mu_i) t + \frac{\left(\sum_{i=1}^{n} k_i^2 \sigma_i^2\right) t^2}{2} \right]$$

Hence y is $N\left(\sum_{i=1}^{n} k_i \mu_i, \sum_{i=1}^{n} k_i^2 \sigma_i^2\right)$

Theorem 2: If $x_1, x_2, \ldots, x_n$ are independent random Variables with respective moment-generating function $M_i(t)$, $i = 1, 2, 3, \ldots n$, then the moment-generation function of $y = \sum_{i=1}^{n} a_i x_i$, where $a_1, a_2 \ldots a_n$ are real Constant is $M_y(t) = \prod_{i=1}^{n} M_i(a_i t)$.

Proof The m.g.f of y is given by

$$M_y(t) = E[e^{ty}]$$

$$= E\left[e^{t(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)}\right]$$

$$= E\left[e^{a_1 t x_1 + a_2 t x_2 + \cdots + a_n t x_n}\right]$$

$$= E\left[e^{a_1 t x_1} \cdot e^{a_2 t x_2} \cdots e^{a_n t x_n}\right]$$

$$= E[e^{a_1 t x_1}] \, E[e^{a_2 t x_2}] \cdots E[e^{a_n t x_n}]$$

$\rightarrow$ ① $\because x_1, x_2, \ldots x_2$ are independent

Since $E(e^{tx_i}) = M_i(t)$,

$$E[e^{a_i t x_i}] = M_i(a_i t)$$

From ①, we have

$$M_y(t) = M_1(a_1 t) M_2(a_2 t) \ldots M_n(a_n t)$$

$$= \prod_{i=1}^{n} M_i(a_i t)$$

Corollary: If $x_1, x_2, \dots x_n$ are observations of a random sample from a distribution with moment-generating function $M(t)$, then

(a) The moment generating function of $y = \sum_{i=1}^{n} x_i$ is

$$M_y(t) = \prod_{i=1}^{n} M(t) = [M(t)]^n$$

(b) The moment generating function of $\bar{x} = \sum_{i=1}^{n} \left(\frac{1}{n}\right) x_i$ is

$$M_{\bar{x}}(t) = \prod_{i=1}^{n} M\left(\frac{t}{n}\right) = [M(t/n)]^n$$

Proof: For ⓐ, Let $a_i = 1$, $i = 1, 2, \dots n$ in Theorem ②

For (b) take $a_i = 1/n$, $i = 1, 2, \dots, n$

Problem:

① Let $x_1, x_2, \dots x_n$ denote the outcomes on n Bernoulli trials. The m.g.f of x_i, $i = 1, 2, 3, \dots, n$ is

$$M(t) = 1 - p + pe^t$$

If $Y = \sum_{i=1}^{n} x_i$ then $M_y(t) = \prod_{i=1}^{n} (1 - p + pe^t)$

$$= (1 - p + pe^t)^n$$

Thus Y is $b(n, p)$

② Let x_1, x_2, x_3 be the observations of a random sample of size $n = 3$ from the exponential distribution having mean β and, the m.g.f $M(t) = \frac{1}{1 - \beta t}$, $t < 1/\beta$

The m.g.f of $Y = x_1 + x_2 + x_3$ is

$$M_y(t) = [(1 - \beta t)^{-1}]^3$$

$$= (1 - \beta t)^{-3}, \quad t < 1/\beta$$

which is a gamma distribution with parameters $\alpha = 3$ and β

Thus y has gamma distribution.

The m.g.f of $\bar{x}$ is

$$M_{\bar{x}}(t) = \left[\left(1 - \frac{\beta t}{3}\right)^{-1}\right]^3$$

$$= \left(1 - \frac{\beta t}{3}\right)^{-3}, \quad t < 3/\beta$$

Hence the distribution of $\bar{x}$ is a gamma distribution with parameters $\alpha = 3$ and $\beta/3$ respectively.

Theorem 3: Let $X_1, X_2, \cdots, X_n$ be independent variables that have the Chi-Square distributions $\chi^2(r_1), \chi^2(r_2), \cdots, \chi^2(r_n)$ respectively. Then the random variable $Y = X_1 + X_2 + \cdots + X_n$ has a Chi-Square distribution with $r_1 + r_2 + r_3 + \cdots + r_n$ degrees of freedom. That is Y is $\chi^2(r_1 + r_2 + \cdots + r_n)$

Proof: Let $X_1, X_2, \cdots X_n$ be independent random variables that have the chi-square distributions $\chi^2(r_1), \chi^2(r_2), \cdots \chi^2(r_n)$

The m.g.f of Chi-Square distribution is.

$$M_i(t) = E(e^{tX_i})$$

$$= (1 - 2t)^{-r_i/2}, \quad t < 1/2, \quad i = 1, 2, \cdots n$$

From the theorem 2 with $a_1 = a_2 = \cdots = a_n = 1$.

$$M(t) = E(e^{tY})$$

$$= E[e^{t(X_1 + X_2 + \cdots + X_n)}]$$

$$= E[e^{tX_1 + tX_2 + \cdots + tX_n}]$$

$$= E[e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}]$$

$$= E[e^{tX_1}]\, E[e^{tX_2}] \cdots E[e^{tX_n}]$$

$$= (1 - 2t)^{-r_1/2} \cdot (1 - 2t)^{-r_2/2} \cdots (1 - 2t)^{-r_n/2}$$

$$= (1 - 2t)^{-(r_1 + r_2 + \cdots + r_n)/2}, \quad t < 1/2$$

This is m.g.f of a distribution $\chi^2(r_1 + r_2 + \cdots + r_n)$

Thus Y has the Chi-Square distribution with $r_1 + r_2 + \cdots + r_n$ degrees of freedom.

Theorem 4:

Let $x_1, x_2, \dots x_n$ denote a random sample of size n from a distribution $N(\mu, \sigma^2)$. The random variable $Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$ has a Chi-Square distribution with n degrees of freedom.

The Distributions of $\bar{x}$ and nS^2/σ^2

Let $x_1, x_2, \ldots, x_n$ be the random sample of size $n \geq 2$ from a distribution $N(\mu, \sigma^2)$. In this section ~~we~~ we investigate the distributions of the mean and the variance of this random sample, that the distributions of the two statistics

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} \text{ and } S^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

From the theorem ① of unit II, we have $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$

$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2 = \sigma^2$ and $k_1 = k_2 = \cdots = k_n = 1/n$, Then $Y = \bar{x}$ has a normal distribution with mean and variance given by $\sum_1^n \left(\frac{1}{n}\mu\right) = \mu$, $\sum_1^n \left[\left(\frac{1}{n}\right)^2 \sigma^2\right] = \frac{\sigma^2}{n}$.

Then $\bar{x}$ is $N\left(\mu, \sigma^2/n\right)$

Properties of $\bar{x}$ and S^2

The sample arises from a distribution $N(\mu, \sigma^2)$:

1) $\bar{x}$ is $N(\mu, \sigma^2/n)$
2) nS^2/σ^2 is $\chi^2(n-1)$
3) $\bar{x}$ and S^2 are independent

Expectations of Functions a Random Variables

Problem ① Let x_i be a random variable with mean μ_i and variance σ_i^2, $i = 1, 2, \ldots n$. Let $x_1, x_2, \ldots x_n$ be independent and let $k_1, k_2, \ldots k_n$ denote real constants. Find the mean and variance of a linear function $Y = k_1x_1 + k_2x_2 + \cdots + k_nx_n$

Sol Since E is a linear operator, the mean of Y is given by

$$\mu_Y = E(Y)$$

$$= E(k_1x_1 + k_2x_2 + \cdots + k_nx_n)$$
$$= E(k_1x_1) + E(k_2x_2) + \cdots + E(k_nx_n)$$
$$= k_1E(x_1) + k_2E(x_2) + \cdots + k_nE(x_n)$$
$$= k_1\mu_1 + k_2\mu_2 + \cdots + k_n\mu_n$$
$$= \sum_{i=1}^{n} k_i\mu_i$$

The Variance of y is given by

$$\sigma_y^2 = E\left\{\left[(k_1x_1 + k_2x_2 + \cdots + k_nx_n) - (k_1\mu_1 + k_2\mu_2 + \cdots + k_n\mu_n)\right]^2\right\}$$
$$= E\left\{\left[k_1(x_1-\mu_1) + k_2(x_2-\mu_2) + \cdots + k_n(x_n-\mu_n)\right]^2\right\}$$
$$= E\left\{\sum_{i=1}^{n} k_i^2(x_i-\mu_i)^2 + 2\sum\sum_{i<j} k_ik_j(x_i-\mu_i)(x_j-\mu_j)\right\}$$
$$= \sum_{i=1}^{n} k_i^2 E[(x_i-\mu_i)^2] + 2\sum\sum_{i<j} k_ik_j E[(x_i-\mu_i)(x_j-\mu_j)] \quad \text{①}$$

Consider $E[(x_i-\mu_i)(x_j-\mu_j)]$, $i<j$. Because x_i and x_j are independent, we have

$$E[(x_i-\mu_i)(x_j-\mu_j)] = E(x_i-\mu_i)\,E(x_j-\mu_j) = 0.$$

using in ①

$$\sigma_y^2 = \sum_{i=1}^{n} k_i^2 E[(x_i-\mu_i)^2]$$
$$= \sum_{i=1}^{n} k_i^2 \sigma_i^2$$

Remark: If $x_1, x_2, \ldots x_n$ are independent, then we have ρ_{ij} is the correlation ~~coeffic~~ coefficient of x_i and x_j.
Thus we have

$$E[(x_i-\mu_i)(x_j-\mu_j)] = \rho_{ij}\sigma_i\sigma_j, \quad i<j$$

Hence $$\mu_y = \sum_{i=1}^{n} k_i\mu_j$$

$$\sigma_y^2 = \sum_{i=1}^{n} k_i^2\sigma_i^2 + 2\sum\sum_{i<j} k_ik_j\rho_{ij}\sigma_i\sigma_j$$

Theorem: Let $X_1, X_2, \dots X_n$ be the random Variables have means $\mu_1, \mu_2, \dots \mu_n$ and Variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$. Let $\rho_{ij}, i \neq j$ denote the Correlation coefficient of X_i and X_j and let $k_1, k_2, \dots k_n$ denote real constants. The mean and the Variance of the linear function $Y = \sum_{i=1}^{n} k_i X_i$ are $\mu_Y = \sum_{i=1}^{n} k_i \mu_i$

and $\sigma_Y^2 = \sum_{i=1}^{n} k_i^2 \sigma_i^2 + 2 \sum_{i<j} \sum k_i k_j \rho_{ij} \sigma_i \sigma_j$

Corollary: Let $X_1, X_2, \dots X_n$ denote the observations of a random Sample of Size n from a distribution that has mean μ and Variance σ^2. The mean and the variance of $Y = \sum_{i=1}^{n} k_i X_i$ are $\mu_Y = \left(\sum_{i=1}^{n} k_i\right)\mu$ and $\sigma_Y^2 = \left(\sum_{i=1}^{n} k_i^2\right)\sigma^2$

Problem:

① Let $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$ denote the mean of a random Sample of Size n from a distribution that has mean μ and variance σ^2. In accordance with the Corollary, we have $\mu_{\bar{X}} = \mu \sum_{i=1}^{n} \left(\frac{1}{n}\right) = \mu$ and $\sigma_{\bar{X}}^2 = \sigma^2 \left[\sum_{1}^{n} \left(1/n\right)^2\right] = \sigma^2/n$.

and then $\bar{X}$ is $N(\mu, \sigma^2/n)$

Limiting Distributions:

Limiting Moment-generating Functions:

To find the limiting distribution function of a random Variable Y_n by use of the definition of limiting distribution function obviously requires that $F_n(y)$ for each positive integer n.

Theorem: ① Let the random Variable Y_n have the distribution function $F_n(Y)$ and the moment-generating function $M(t;n)$ that exists for $-h<t<h$ for all n. If there exists a distribution function $F(Y)$ with Corresponding moment generating function $M(t)$, defined for $|t| \leq h_1 < h$ such that $\lim_{n\to\infty} M(t;n) = M(t)$, then Y_n has a limiting distribution with distribution function $F(Y)$.

Consider the limit $\lim_{n\to\infty}\left[1+\frac{b}{n}+\frac{\psi(n)}{n}\right]^{cn} \rightarrow (1)$

where b and c do not depend upon n and where $\lim_{n\to\infty}\psi(n)=0$

Then, we have $\lim_{n\to\infty}\left[1+\frac{b}{n}+\frac{\psi(n)}{n}\right]^{cn} \rightarrow (2)$

$$= \lim_{n\to\infty}\left(1+\frac{b}{n}\right)^{cn}$$

$$= \lim_{n\to\infty}\left(1+\frac{b}{n}\right)^{nc} \qquad \because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$= e^{bc}$$

For example $\lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3}{n^{3/2}}\right)^{-n/2}$

$$= \lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3/\sqrt{n}}{n}\right)^{-n/2} \rightarrow (3)$$

Comparing (2) and (3) we have

$b=-t^2$, $c=-\frac{1}{2}$ and $\psi(n)=\frac{t^3}{\sqrt{2}}$

Then we get

$$\lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3}{n^{3/2}}\right)^{-n/2}$$

$$= \lim_{n\to\infty}\left(1-\frac{t^2}{n}\right)^{-n/2}$$

$$= e^{\frac{t^2}{2}}$$

<u>Problem</u> Let Y_n have a distribution $b(n,p)$. Suppose that the mean $\mu=np$ is the same for every n; that is $p=\mu/n$ where μ is a constant. Find the limiting distribution of the binomial distribution, when $p=\mu/n$

<u>Sol</u>

$$M(t;n) = E\left(e^{tY_n}\right)$$

$$= [(1-p)+pe^t]^n$$

$$= \left[\left(1-\frac{\mu}{n}\right)+\frac{\mu}{n}e^t\right]^n$$

$$= \left[1-\frac{\mu}{n}+\frac{\mu}{n}e^t\right]^n$$

$$= \left[1 + \frac{\mu(e^t - 1)}{n}\right]^n ; \text{ for all real values of } t$$

Taking limit as $n \to \infty$ on both sides

$$\lim_{n\to\infty} M(t;n) = \lim_{n\to\infty}\left[1 + \frac{\mu(e^t-1)}{n}\right]^n$$

$$= e^{\mu(e^t-1)} \quad \text{for all real values of } t.$$

This is a Poisson distribution with mean μ

Thus Y_n has a limiting Poisson distribution with mean μ.

Problem: ② : Let Z_n be $\chi^2(n)$. Then the m.g.f of Z_n is $(1-2t)^{-n/2}$, $t < 1/2$. The mean and the variance of Z_n are n and $2n$ respectively. Find the limiting distribution of the random variable $Y_n = (Z_n - n)/\sqrt{2n}$

Sol The m.g.f of Y_n is

$$M(t;n) = E[e^{tY_n}]$$

$$= E\left[\exp\left[t\left(\frac{Z_n - n}{\sqrt{2n}}\right)\right]\right\}$$

$$= E\left[e^{-tn/\sqrt{2n}} \cdot e^{tZ_n/\sqrt{2n}}\right]$$

$$= e^{-tn/\sqrt{2n}} E\left[e^{tZ_n/\sqrt{2n}}\right]$$

$$= e^{-\frac{t\sqrt{n}}{\sqrt{2}}\left(\frac{n}{2}\right)}\left[1 - 2\frac{t}{\sqrt{2n}}\right]^{-n/2} ; \quad t < \frac{\sqrt{2n}}{2}$$

$$\therefore M(t;n) = \left[e^{t\sqrt{2/n}} - t\sqrt{\frac{2}{n}}\, e^{t\sqrt{2/n}}\right]^{-n/2}, \quad t < \sqrt{\frac{n}{2}}$$

From Taylor's formula, there exists a number $\xi(n)$ between 0 and $t\sqrt{2/n}$ such that

$$e^{t\sqrt{2/n}} = 1 + t\sqrt{\frac{2}{n}} + \frac{1}{2}\left(t\sqrt{\frac{2}{n}}\right)^2 + \frac{e^{\xi(n)}}{6}\left(t\sqrt{\frac{2}{n}}\right)^3$$

If this sum is substituted for $e^{t\sqrt{2/n}}$ in the last expression for $M(t;n)$

$$M(t;n) = \left(1 - \frac{t^2}{n} + \frac{\psi(n)}{n}\right)^{-n/2} \quad \text{where}$$

$$\psi(n) = \frac{\sqrt{2}\, t^3 e^{\xi(n)}}{3\sqrt{n}} - \frac{\sqrt{2}\cdot t^3}{\sqrt{n}} - \frac{2t^4 e^{\xi(n)}}{3n}$$

As $n \to \infty$, $\xi(n) \to 0$, then $\lim \psi(n) = 0$ for every fixed value of t

$$\therefore \lim_{n\to\infty} M(t;n) = e^{t^2/2} \quad \text{for all real values of } t.$$

Thus, the random variable $Y_n = (Z_n - n)/\sqrt{2n}$ has a limiting standard normal distribution

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Thus, the random variable $Y_n = (Z_n - n)/\sqrt{2n}$ has a limiting standard normal distribution

State and Prove the Central Limit Theorem

Statement:-

Let $X_1, X_2, \dots X_n$ denote the observations of a random sample from a distribution that has mean μ and positive variance σ^2. Then the random variable $Y_n = \left(\sum_{i=1}^{n} X_i - n\mu\right)/\sqrt{n}\,\sigma = \sqrt{n}\,(\bar{X}_n - \mu)/\sigma$ has a limiting distribution that is normal with mean zero and variance 1.

Proof: We assume that m.g.f $M(t) = E(e^{tx})$, $-h < t < h$ → ① of the distribution.

Replace the m.g.f by the characteristic function

$$\phi(t) = E(e^{itx})$$

Consider the function

$$m(t) = E[e^{t(x-\mu)}]$$

$$= E[e^{tx} \cdot e^{-\mu t}]$$

$$= e^{-\mu t} E[e^{tx}]$$

$$= e^{-\mu t} M(t), \quad -h<t<h \quad \text{from ①}$$

Since $m(t)$ is the m.g.f for $x-\mu$,

$$m(t) = e^{-\mu t} M(t) \qquad m'(t) = e^{-\mu t} M'(t) + M(t)\, e^{-\mu t}\cdot(-\mu)$$

$$m(0) = 1 \qquad m'(0) = E(x-\mu) = 0$$

$$m''(0) = E[(x-\mu)^2] = \sigma^2$$

By Taylor's formula there exists a number ξ between 0 and t such that

$$m(t) = m(0) + m'(0)t + \frac{m''(\xi)t^2}{2}$$

$$= 1 + \frac{m''(\xi)t^2}{2}$$

If $\frac{\sigma^2 t^2}{2}$ is added and subtracted, then

$$m(t) = 1 + \frac{\sigma^2 t^2}{2} + \frac{m''(\xi)t^2}{2} - \frac{\sigma^2 t^2}{2}$$

$$m(t) = 1 + \frac{\sigma^2 t^2}{2} + \frac{[m''(\xi) - \sigma^2]t^2}{2} \rightarrow (1)$$

Next, Consider $M(t;n)$,

$$M(t;n) = E\left[\exp\left(t\,\frac{\Sigma x_i - n\mu}{\sigma\sqrt{n}}\right)\right]$$

$$= E\left[\exp\left(t\,\frac{(x_1 + x_2 + \cdots + x_n) - (\mu + \mu + \cdots + \mu)}{\sigma\sqrt{n}}\right)\right]$$

$$= E\left[\exp\left(t\,\frac{(x_1 - \mu) + (x_2 - \mu) + \cdots + (x_n - \mu)}{\sigma\sqrt{n}}\right)\right]$$

$$= E\left[\exp\left(t\,\frac{x_1 - \mu}{\sigma\sqrt{n}}\right) \cdot \exp\left(t\,\frac{x_2 - \mu}{\sigma\sqrt{n}}\right) \cdots \exp\left(t\,\frac{x_n - \mu}{\sigma\sqrt{n}}\right)\right]$$

$$= E\left[\exp\left(t\,\frac{x_1 - \mu}{\sigma\sqrt{n}}\right)\right] E\left[\exp\left(t\,\frac{x_2 - \mu}{\sigma\sqrt{n}}\right)\right] \cdots E\left[\exp\left(t\,\frac{x_n - \mu}{\sigma\sqrt{n}}\right)\right]$$

$$= E\left[\exp\left(t\,\frac{x - \mu}{\sigma\sqrt{n}}\right)\right] E\left[\exp\left(t\,\frac{x - \mu}{\sigma\sqrt{n}}\right)\right] \cdots E\left[\exp\left(t\,\frac{x - \mu}{\sigma\sqrt{n}}\right)\right]$$

$$= \left\{E\left[\exp\left(t\,\frac{x - \mu}{\sigma\sqrt{n}}\right)\right]\right\}^n$$

$$M(t;n) = \left[m\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n ; \quad -h < \frac{t}{\sigma\sqrt{n}} < h \rightarrow (2)$$

In equation (1) replace t by $\frac{t}{\sigma\sqrt{n}}$, we have

$$m\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{\sigma^2 \frac{t^2}{\sigma^2 n}}{2} + \frac{[m''(\xi) - \sigma^2]\frac{t^2}{\sigma^2 n}}{2}$$

$$m\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{t^2}{2n} + \frac{[m''(\xi) - \sigma^2]t^2}{2n\sigma^2} \rightarrow (3)$$

Where now ξ is between 0 and $\frac{t}{\sigma\sqrt{n}}$ with $-h\sigma\sqrt{n} < t < h\sigma\sqrt{n}$

Using (3) in (2)

$$M(t;n) = \left\{ 1 + \frac{t^2}{2n} + \frac{[m''(\xi) - \sigma^2]t^2}{2n\sigma^2} \right\}^n$$

Since $m''(t)$ is continuous at $t = 0$ and since $\xi \to 0$ as $n \to \infty$

we have $\lim_{n \to \infty} [m''(\xi) - \sigma^2] = 0$

$$\lim_{n \to \infty} M(t;n) = \lim_{n \to \infty} \left[1 + \frac{t^2}{2n} + \frac{[m''(\xi) - \sigma^2]t^2}{2n\sigma^2} \right]^n$$

$$= \lim_{n \to \infty} \left[1 + \frac{t^2}{2n} \right]^n$$

$$= \lim_{n \to \infty} \left[1 + \frac{t^2/2}{n} \right]^n$$

$$\lim_{n \to \infty} M(t;n) = e^{t^2/2}$$ for all real values of t.

Thus, the random variable $Y_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting standard normal distribution.

Introduction to Statistical Inference

Defn Parameter Space

Let a random Variable x have a p.d.f, that is of known functional form but in the p.d.f depends upon an unknown parameter θ that may have any value in a set Ω. It is denoted as ~~f(x~~, p.d.f $f(x;\theta)$, $\theta \in \Omega$. The set Ω is called the parameter space

Ex, If $X \sim N(\mu, \sigma^2)$ then the parameter space is

$$\Omega = \{(\mu, \sigma^2) : -\infty < \mu < \infty ; 0 < \sigma < \infty\}$$

In particular, for $\sigma^2 = 1$, the family of probability distributions is given by $\{N(\mu, 1) ; \mu \in \Omega\}$, where $\Omega = \{\mu : -\infty < \mu < \infty\}$.

Note: The estimating functions are then referred to as estimators.

Calculation of Point estimates

Problem: ① Let $x_1, x_2, \dots x_n$ be the random sample from the distribution with p.d.f

$$f(x) = \theta^x (1-\theta)^{1-x}, \quad x = 0, 1$$
$$= 0 \quad \text{elsewhere}, \quad \text{where } 0 \le \theta \le 1.$$

Find the point estimates

Sol The probability ~~X = x~~ that $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$ is the joint p.d.f

$$\theta^{x_1}(1-\theta)^{1-x_1} \theta_2^{x_2}(1-\theta)^{1-x_2} \cdots \theta^{x_n}(1-\theta)^{1-x_n} = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$$

where x_i equals to zero or 1, $i = 1, 2, \dots n$.

$$\text{Let } L(\theta) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}, \quad 0 \le \theta \le 1 \quad \to ①$$

This function is called the likelihood function

The value of θ is to be maximized the probability $L(\theta)$ of obtaining this particular observed sample $x_1, x_2, \ldots, x_n$

Taking logarithm of ① on both sides

$$\log[L(\theta)] = \log\left[\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\right]$$

$$= \log \theta^{\sum x_i} + \log (1-\theta)^{n-\sum x_i}$$

$$\log[L(\theta)] = \sum_1^n x_i \log\theta + (n - \sum_1^n x_i) \log(1-\theta)$$

Diff w.r.t θ on both sides

$$\frac{d[\log(L(\theta))]}{d\theta} = \sum_1^n x_i \frac{1}{\theta} + (n - \sum_1^n x_i)\frac{1}{(1-\theta)}(-1)$$

$$= \frac{\sum x_i}{\theta} - \frac{(n-\sum x_i)}{(1-\theta)}$$

Since θ is maximum or minimum

$$\frac{d[\log(L(\theta))]}{d\theta} = 0$$

$$\therefore \frac{\sum x_i}{\theta} - \frac{(n-\sum x_i)}{1-\theta} = 0;$$ Provided that θ is not equal to zero or 1

$$\frac{(1-\theta)\sum_1^n x_i - \theta(n-\sum x_i)}{\theta(1-\theta)} = 0$$

$$(1-\theta)\sum_1^n x_i - \theta(n-\sum x_i) = 0$$

$$(1-\theta)\sum_1^n x_i = \theta(n - \sum_1^n x_i)$$

$$\sum x_i - \cancel{\theta\sum_1^n x_i} = \theta n - \cancel{\theta\sum_1^n x_i}$$

$$n\theta = \sum_1^n x_i$$

$$\boxed{\theta = \frac{\sum_1^n x_i}{n}}$$

That $\frac{\sum_1^n x_i}{n}$ actually maximizes $L(\theta)$ and $\log L(\theta)$ can be easily verified, ~~in~~ in which all of $x_1, x_2, \ldots x_n$

equal zero together or 1 together.

That is $\frac{\sum_1^n x_i}{n}$ is the value of θ that maximizes θ.

The Corresponding statistics

$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$ is called the maximum likelihood estimator of θ. The observed value of $\hat{\theta} = \frac{\sum_1^n x_i}{n}$ is called the ~~most~~ maximum likelihood estimate of θ.

Likelihood function

Consider a random sample $X_1, X_2, \dots X_n$ from a distribution having p.d.f $f(x;\theta)$, $\theta \in \Omega$. The joint p.d.f of $X_1, X_2, \dots, X_n$ is $f(x_1;\theta) f(x_2;\theta) \dots f(x_n;\theta)$. This joint p.d.f may be regarded as a function of θ. which is denoted by $f(x_1;\theta) f(x_2;\theta) \dots f(x_n;\theta)$, $\theta \in \Omega$

$$L(\theta; x_1, x_2, \dots x_n) = f(x_1;\theta) f(x_2;\theta) \dots f(x_n;\theta)$$

This is called likelihood function.

Problem (2): Let $X_1, X_2, \dots, X_n$ be a random sample from the normal distribution $N(\theta, 1)$, $-\infty < \theta < \infty$. Here

$$L(\theta; x_1, x_2, \dots, x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\sum_1^n \frac{(x_i-\theta)^2}{2}\right] \rightarrow (1)$$

Find the maximum likelihood estimator?

Sol

$$\text{Let } L(\theta; x_1, x_2, \dots, x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\sum_1^n \frac{(x_i-\theta)^2}{2}\right]$$

Taking logarithm on both sides

$$\log L(\theta; x_1, x_2, \dots x_n) = \log\left[\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\sum_1^n \frac{(x_i-\theta)^2}{2}\right]\right]$$

$$= \log\left(\frac{1}{\sqrt{2\pi}}\right)^n + \left[-\frac{\sum_1^n (x_i-\theta)^2}{2}\right]$$

$$= n \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{\sum (x_i-\theta)^2}{2} \rightarrow (2)$$

likelihood function maximize or minimize

$$\frac{d[\log L(\theta; x_1, x_2, \ldots, x_n)]}{d\theta} = 0$$

$$n \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{\sum (x_i - \theta)^2}{2} = 0$$

Diff (2) w.r.t θ both sides

$$\frac{d[\log L(\theta; x_1, x_2, \ldots, x_n)]}{d\theta} = -\frac{\cancel{2}\left[\sum_1 (x_i - \theta)\right]}{\cancel{2}}(-1)$$

$$= \sum_1^n (x_i - \theta)$$

If $$\frac{d[\log L(\theta; x_1, x_2, \ldots, x_n)]}{d\theta} = 0$$

$\therefore$ Then $$\sum_1^n (x_i - \theta) = 0$$

$$(x_1 - \theta) + (x_2 - \theta) + \cdots + (x_n - \theta) = 0$$

$$(x_1 + x_2 + \cdots + x_n) - (n\theta) = 0$$

$$\sum_1^n x_i = n\theta$$

$$\theta = \frac{\sum_1^n x_i}{n}$$

$$\therefore \hat{\theta} = u(x_1, x_2, \ldots x_n) = \frac{1}{n}\sum x_i = \bar{x}$$

is the unique m.l.e of the mean θ.

~~Definiti~~ Definition: Any statistic whose mathematical expectation is equal to a parameter θ is called an unbiased estimator of the parameter θ. Otherwise, the statistic is said to be biased.

i.e, $E(\hat{\theta}) = \theta$

Definition: Any statistic that converges in probability to a parameter θ is called a ~~good~~ consistent estimator of the parameter θ.

Problem ① Let $x_1, x_2, \ldots x_n$ represent a random sample from each ~~de~~ of the distributions having the p.d.f

$$f(x;\theta) = \left(\frac{1}{\theta}\right) e^{-x/\theta}, \quad 0 < x < \infty, \quad 0 < \theta < \infty.$$

$$= \text{zero elsewhere}$$

Find the m.l.e $\hat{\theta}$ of θ?

Sol The likelihood ~~ho~~ function is.

$$L(\theta; x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} \left(\frac{1}{\theta}\right) e^{-x_i/\theta} \qquad \because \frac{1}{\theta} = \theta^{-1}$$

$$= \theta^{-n} e^{-\sum_{i=1}^{n} \frac{x_i}{\theta}}$$

Taking logarithm on both sides

$$\log[L(\theta; x_1, x_2, \ldots x_n)] = \log\left[\theta^{-n} e^{-\sum_{i=1}^{n} \frac{x_i}{\theta}}\right]$$

$$= \log \theta^{-n} + \left(-\sum_{i=1}^{n} \frac{x_i}{\theta}\right)$$

$$\log[L(\theta; x_1, \ldots x_n)] = -n \log \theta - \frac{1}{\theta} \sum_{i=1}^{n} x_i \rightarrow ①$$

Diff ① w.r.t θ

$$\frac{d[\log L(\theta; x_1, x_2, \ldots x_n)]}{d\theta} = \frac{-n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i$$

If $\dfrac{d[\log L(\theta; x_1, x_2, \ldots x_n)]}{d\theta} = 0$

Then $\dfrac{-n}{\theta} + \dfrac{1}{\theta^2} \sum x_i = 0$

$$\frac{1}{\theta^2} \sum x_i = \frac{n}{\theta}$$

$$\frac{1}{\theta} \sum x_i = n.$$

$$\theta = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\theta = \frac{\sum x_i}{n} = \bar{x}$$

$\therefore$ The m.l.e $\hat{\theta}$ of θ ~~x~~ is $\bar{x}$

Problem Let $f(x;\theta) = \frac{1}{\theta}$, $0 < x \leq \theta$, $0 < \theta < \infty$

$= 0$, elsewhere

Obtain the maximum likelihood estimator $\hat{\theta}$ for θ.

Sol The likelihood function is

$$L(\theta; x_1, x_2, \ldots x_n) = \frac{1}{\theta}\frac{1}{\theta}\cdots\frac{1}{\theta} = \frac{1}{\theta^n}, \quad 0 < x_i < \theta$$

Taking Logarithm on both Sides

$$\log[L(\theta; x_1, x_2, \ldots x_n)] = \log\left(\frac{1}{\theta^n}\right)$$

$$= \log(\theta^{-n})$$

$$\log[L(\theta; x_1, x_2, \ldots x_n)] = -n\log\theta \rightarrow (1)$$

Diff w.r.t θ on both sides

$$\frac{d[\log(L(\theta; x_1, x_2, \ldots x_n)]}{d\theta} = \frac{-n}{\theta}$$

Now $\frac{d[\log(L(\theta; x_1, x_2, \ldots x_n))]}{d\theta} = 0$

$$\frac{-n}{\theta} = 0$$

$$\Rightarrow \hat{\theta} = \infty$$

In this Case $\theta \geq$ each x_i in particular

$\theta \geq \max(x_i)$

Thus L can be made no ~~lar~~ larger than $\frac{1}{[\max(x_i)]^n}$

and the unique m.l.e $\hat{\theta}$ of θ is the n^{th} order ~~st~~ statistic $\max(x_i)$. That is $E[\max(x_i)] = \frac{n\theta}{n+1}$

Thus the m.l.e of θ is biased.

Note: The property unbiasedness is not ~~a~~ in general property of a m.l.e.

Problem Let $X_1, X_2, \dots X_n$ be a random sample from a distribution $N(\theta_1, \theta_2)$, $-\infty < \theta_1 < \infty$, $0 < \theta_2 < \infty$. Find the $\hat{\theta}_1$ and $\hat{\theta}_2$, the maximum likelihood estimators of θ_1 and θ_2.

Sol The Logarithm of the likelihood function may be written in the form

$$\log L(\theta_1, \theta_2; x_1, x_2, \dots x_n) = -\frac{\sum_1^n (x_i - \theta_1)^2}{2\theta_2} - \frac{n \log(2\pi\theta_2)}{2} \to (1)$$

We maximize by differentiation of (1)

$$\frac{\partial}{\partial\theta_1}[\log L] = -\frac{2\sum_1^n (x_i - \theta_1)^2}{2\theta_2}(-1) - 0$$

$$= \frac{\sum (x_i - \theta_1)^2}{\theta_2} \to (2)$$

$$\frac{\partial}{\partial\theta_2}[\log L] = -\frac{\sum (x_i - \theta_1)^2}{2}\left(\frac{-1}{\theta_2^2}\right) - \frac{n}{2}\frac{\cancel{2\pi}}{\cancel{2\pi}\theta_2}$$

$$= \frac{\sum_1^n (x_i - \theta_1)^2}{2\theta_2^2} - \frac{n}{\cancel{4}\,2\,\theta_2}$$

$$\frac{\partial}{\partial\theta_2}[\log L] = \frac{\sum_1^n (x_i - \theta_1)^2}{2\theta_2^2} - \frac{n}{2\theta_2} \to (3)$$

Now $\frac{\partial}{\partial\theta_1} = 0$, and $\frac{\partial}{\partial\theta_2} = 0$

From (2) and (3), we have

$$\frac{\sum (x_i - \theta_1)^2}{\theta_2} = 0$$

$$\frac{\sum (x_i - \theta_1)^2}{2\theta_2^2} - \frac{n}{2\theta_2} = 0$$

$$\frac{\sum (x_i - \theta_1)^2}{\cancel{2}\theta_2^{\cancel{2}}} = \frac{n}{\cancel{2}\cancel{\theta_2}}$$

$$\frac{\sum (x_i - \theta_1)^2}{\theta_2} = n$$

$$\theta_2 = \frac{\sum (x_i - \theta_1)^2}{n} \rightarrow (4)$$

Now $\sum (x_i - \theta_1)^2 = 0$

$$\sum (x_i - \theta_1) = 0$$

$$(x_1 + x_2 + \cdots + x_n) - n\theta_1 = 0$$

$$n\theta = \sum x_i$$

$$\theta_1 = \frac{\sum x_i}{n} = \bar{x}$$

using in (4)

$$\theta_2 = \frac{\sum (x_i - \bar{x})^2}{n} = s^2$$

Hence $\theta_1 = \frac{\sum x_i}{n} = \bar{x}$

$$\theta_2 = \frac{\sum (x_i - \bar{x})^2}{n} = s^2$$

Thus the maximum likelihood estimators of $\theta_1 = \mu$ and $\theta_2 = \sigma^2$ are the mean and variance of sample $\hat{\theta}_1 = \bar{x}$ and $\hat{\theta}_2 = s^2$ respectively.

Results: Here $\hat{\theta}_1$ is an unbiased estimator of θ_1, the estimator $\hat{\theta}_2 = s^2$ is biased

$$E(\hat{\theta}_2) = \frac{\sigma^2}{n} E\left(\frac{n\hat{\theta}_2}{\sigma^2}\right)$$

$$= \frac{\sigma^2}{n} E\left[\frac{ns^2}{\sigma^2}\right]$$

$$= \frac{(n-1)\sigma^2}{n} = \frac{(n-1)\theta_2}{n}$$

Confidence Intervals for Means

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, σ^2 is known. Consider the maximum likelihood estimator of μ is $\hat{\mu} = \bar{X}$. Then $\bar{X}$ is $N(\mu, \sigma^2/n)$ and $\dfrac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}$ is

$N(0, 1)$. Thus

$$P\left[-2 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 2\right] = 0.954$$

Now $\quad -2 < \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 2$

$$-\frac{2\sigma}{\sqrt{n}} < \bar{X}-\mu < \frac{2\sigma}{\sqrt{n}}$$

$\bar{X} - \dfrac{2\sigma}{\sqrt{n}} < \mu < \bar{X} + \dfrac{2\sigma}{\sqrt{n}}$ are equivalent.

$$\therefore P\left(\bar{X} - \frac{2\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{2\sigma}{\sqrt{n}}\right) = 0.954$$

Since σ is known number, each of the random variables $\bar{X} - \dfrac{2\sigma}{\sqrt{n}}$ and $\bar{X} + \dfrac{2\sigma}{\sqrt{n}}$ is a statistic. The interval $\left(\bar{X} - \dfrac{2\sigma}{\sqrt{n}}, \bar{X} + \dfrac{2\sigma}{\sqrt{n}}\right)$ is a random interval

Note: The number 0.954 is called the confidence ~~of~~ coefficient

Result: The confidence coefficient is equal to the probability that the random interval includes the parameter.

We obtain an 80, a 90, or a 99 percent confidence interval for μ ~~byu~~ by using 1.282, 1.645 or 2.576, respectively, instead of the constant 2.

Problem If $n = 40$, $\sigma^2 = 10$ and $\bar{x} = 7.164$ then

$$\left(7.164 - 1.282\sqrt{\frac{10}{40}},\ 7.164 + 1.282\sqrt{\frac{10}{40}}\right)$$

$(6.523, 7.805)$ is 80 percent confidence interval for μ

Thus, we have an interval estimate of μ

Problem Let $\bar{X}$ denote the mean of a random sample of size 25 from a distribution having variance $\sigma^2 = 100$ and μ. Since $\frac{\sigma}{\sqrt{n}} = 2$ then approximately

$$P\left(-1.96 < \frac{\bar{X}-\mu}{2} < 1.96\right) = 0.95$$

$$P(-3.92 < \bar{x} - \mu < 3.92) = 0.95$$

$$P(\bar{X} - 3.92 < \mu < \bar{X} + 3.92) = 0.95$$

Let the observed mean of the sample be $\bar{x} = 67.53$

Now $\bar{x} - 3.92 = 67.53 - 3.92$

$= 63.61$

$\bar{x} + 3.92 = 67.53 + 3.92$

$= 71.45$

Thus $(63.61, 71.45)$ is approximate 95 percent Confidence interval for the mean μ.

Problem Let the observed value of the mean of random sample of size 20 from a distribution that $N(\mu, 80)$ be 81.2. Find a 95 percent confidence interval for μ

Ans $(77.28, 85.12)$

Sol Given $n = 20$ $\bar{x} = 81.2$ $\sigma^2 = 80$

So $\frac{\sigma}{\sqrt{n}} =$ ~~$\frac{\sigma^2}{n} = \frac{80}{20} = 4$~~ $\frac{\sigma}{\sqrt{n}} = \frac{8.94}{4.47} = 2.01$

$= 2$

$$P\left(-1.96 < \frac{\bar{X}-\mu}{2} < 1.96\right) = 0.95$$

$$P(-3.92 < \bar{X} - \mu < 3.92) = 0.95$$

$$P(\bar{x} - 3.92 < \mu < \bar{x} + 3.92) = 0.95$$

Let the observed mean of sample be $\bar{x} = 81.2$

Now $\bar{x} - 3.92 = 81.2 - 3.92$
$= 77.28$

$\bar{x} + 3.92 = 81.2 + 3.92$
$= 85.12$

Thus $(77.28, 85.12)$ is approximate 95 percent confindence interval for mean μ

Finding a confidence interval for the mean μ of a normal distribution (t-distribution)

$$T = \frac{\sqrt{n}(\bar{x} - \mu)\sigma}{\sqrt{ns^2/[\sigma^2(n-1)]}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$$

has a t-distribution with $n-1$ degrees of freedom, whatever the value of $\sigma^2 > 0$

Given positive integer n and a ~~propabo~~ probability of 0.95, we can find a number b ~~from~~ such that

$$P\left(-b < \frac{\bar{x} - \mu}{s/\sqrt{n-1}} < b\right) = 0.95$$

$$P\left(-\frac{bs}{\sqrt{n-1}} < \bar{x} - \mu < \frac{bs}{\sqrt{n-1}}\right) = 0.95$$

$$P\left(\bar{x} - \frac{bs}{\sqrt{n-1}} < \mu < \bar{x} + \frac{bs}{\sqrt{n-1}}\right) = 0.95$$

Then the interval $[\bar{x} - (bs/\sqrt{n-1}), \bar{x} + (bs + \sqrt{n-1})]$ is a random interval having probability 0.95 of including the unknown fixed point μ

Result: If the ~~ex~~ experimental values of $X_1, X_2, \dots, X_n$ are $x_1, x_2, \dots, x_n$ with $s^2 = \sum_1^n (x_i - \bar{x})^2 / n$ where $\bar{x} = \frac{\sum_1^n x_i}{n}$

then the interval $[\bar{x} - (bs/\sqrt{n-1}), \bar{x} + (bs + \sqrt{n-1})]$ is 95 percent ~~Confind~~ Confidence interval for μ for every $\sigma^2 > 0$

Problem If $n = 10$ $\bar{x} = 3.22$ and $S = 1.17$ then the interval $[3.22 - (2.262)(1.17)/\sqrt{9},\ 3.22 + (2.262)(1.17)/\sqrt{9}]$ $(2.34, 4.10)$ is a 95 percent confidence interval for μ.

$$P(\bar{x} - 3.92 < \mu < \bar{x} + 3.92) = 0.95$$

Let the observed mean of sample be $\bar{x} = 81.2$

Now $\bar{x} - 3.92 = 81.2 - 3.92$

$= 77.28$

$\bar{x} + 3.92 = 81.2 + 3.92$

$= 85.12$

Thus $(77.28, 85.12)$ is approximate 95 percent Confindence interval for mean μ

<u>Finding a Confidence interval for the mean μ of a normal distribution (t-distribution)</u>

$$T = \frac{\sqrt{n}(\bar{x} - \mu)\sigma}{\sqrt{ns^2/[\sigma^2(n-1)]}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$$

has a t-distribution with $n-1$ degrees of freedom, whatever the value of $\sigma^2 > 0$

Given positive integer n and a ~~propaba~~ probability of 0.95, we can find a number b ~~from~~ such that

$$P\left(-b < \frac{\bar{x} - \mu}{s/\sqrt{n-1}} < b\right) = 0.95$$

$$P\left(-\frac{bs}{\sqrt{n-1}} < \bar{x} - \mu < \frac{bs}{\sqrt{n-1}}\right) = 0.95$$

$$P\left(\bar{x} - \frac{bs}{\sqrt{n-1}} < \mu < \bar{x} + \frac{bs}{\sqrt{n-1}}\right) = 0.95$$

Then the interval $[\bar{x} - (bs/\sqrt{n-1}), \bar{x} + (bs + \sqrt{n-1})]$ is a random interval having probability 0.95 of including the unknown fixed point μ

Result: If the ~~ex~~ experimental values of $X_1, X_2, \ldots, X_n$ are $x_1, x_2, \ldots, x_n$ with $s^2 = \sum_1^n (x_i - \bar{x})^2/n$ where $\bar{x} = \frac{\sum_1^n x_i}{n}$

then the interval $[\bar{x} - (bs/\sqrt{n-1}), \bar{x} + (bs + \sqrt{n-1})]$ is 95 percent ~~Confind~~ Confidence interval for μ for every $\sigma^2 > 0$

Problem If $n = 10$ $\bar{x} = 3.22$ and $s = 1.17$ then the interval $[3.22 - (2.262)(1.17)/\sqrt{9}, 3.22 + (2.262)(1.17)/\sqrt{9}]$ $(2.34, 4.10)$ is a 95 percent confidence interval for μ.

Confindence Intervals for Differences of Means

The random variable T is also used to obtain a confidence interval for the difference $\mu_1 - \mu_2$ between the means of two normal ~~distris~~ distributions, say $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ when the have the same, but unknown, variance σ^2

To Find the Confidence interval for $\mu_1 - \mu_2$ as follows

Let $X_1, X_2, \ldots X_n$ and $Y_1, Y_2, \ldots, Y_m$ be the independent random samples from the two distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ respectively.

Let the means of the samples be $\bar{X}$ and $\bar{Y}$ and the variances of the samples be S_1^2 and S_2^2

Then, we know that $\bar{X}$, S_1^2 and $\bar{Y}$ and S_2^2 are ~~instance~~ independence

Thus $\bar{X}$ and $\bar{Y}$ are normally and independently distributed with mean μ_1 and μ_2 and variances $\frac{\sigma^2}{n}$ and $\frac{\sigma^2}{m}$ respectively

Therefore, their difference $\bar{X} - \bar{Y}$ is normally distributed with mean $\mu_1 - \mu_2$ and variances $\frac{\sigma^2}{n} + \frac{\sigma^2}{m}$

Then the random Variable $\dfrac{(\bar{x}-\bar{y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma^2}{n}+\dfrac{\sigma^2}{m}}}$

is normally distributed with zero mean and unit Variance. Further, $\dfrac{nS_1^2}{\sigma^2}$ and $\dfrac{mS_2^2}{\sigma^2}$ have independent Chi-Square distribution with $n-1$ and $m-1$ degrees of freedom respectively, So that their Sum $(nS_1^2+mS_2^2)/\sigma^2$ has a Chi-Square distribution with $n+m-2$ degrees of freedom, provided $m+n-2>0$

∴ The random Variable

$$T = \frac{(\bar{x}-\bar{y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{nS_1^2+mS_2^2}{n+m-2}\left(\dfrac{1}{n}+\dfrac{1}{m}\right)}}$$ has a ~~t-d~~

t - distribution with $n+m-2$ degrees of freedom.

Result: Find a positive number b [from Table IV of Appendix B such that

$$P(-b<T<b)=0.95$$

If $R = \sqrt{\dfrac{nS_1^2+mS_2^2}{n+m-2}\left(\dfrac{1}{n}+\dfrac{1}{m}\right)}$

Then the probability is written in form

$$P\left[(\bar{x}-\bar{y})-bR<\mu_1-\mu_2<(\bar{x}-\bar{y})+bR\right]=0.95$$

$$\left[(\bar{x}-\bar{y})-b\sqrt{\frac{nS_1^2+mS_2^2}{n+m-2}\left(\frac{1}{n}+\frac{1}{m}\right)},\ (\bar{x}-\bar{y})+b\sqrt{\frac{nS_1^2+mS_2^2}{n+m-2}\left(\frac{1}{n}+\frac{1}{m}\right)}\right]$$

has probability 0.95 of including the unknown fixed point $(\mu_1-\mu_2)$

Verification of Confidence interval for differences of probabilities of means of binomial distribution

Statement: Let Y_1 and Y_2 be two independent random variables with binomial distributions $b(n_1, p_1)$ and $b(n_2, p_2)$ respectively. Find the confidence intervals for the difference $p_1 - p_2$ of the means of $\frac{Y_1}{n_1}$ and $\frac{Y_2}{n_2}$, when n_1 and n_2 are known.

Proof: Since the means and variance of $\frac{Y_1}{n} - \frac{Y_2}{n_2}$ are $p_1 - p_2$ and $\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$ respectively, then the random variable given by the ratio

$$\frac{\left(\frac{Y_1}{n_1} - \frac{Y_2}{n_2}\right) - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}}$$

has mean 0 and Variance 1 for all positive integers n_1 and n_2

Since both Y_1 and Y_2 have approximate normal distributions for the large n_1 and n_2

If $\frac{n_1}{n_2} = c$, where c is a fixed positive constant

we have

$$\frac{\left(\frac{Y_1}{n_1}\right)\left(1 - \frac{Y_1}{n_1}\right)/n_1 + \left(\frac{Y_2}{n_2}\right)\left(1 - \frac{Y_2}{n_2}\right)/n_2}{p_1(1-p_1)/n_1 + p_2(1-p_2)/n_2}$$

Converges in probability to 1 as $n_2 \to \infty$

Suppose $n_1 \to \infty$, from $\frac{n_1}{n_2} = c$ $\because c > 0$

$\therefore$ The random Variable

$$W = \frac{\left(\frac{Y_1}{n_1} - \frac{Y_2}{n_2}\right) - (p_1 - p_2)}{U}$$

where $U = \sqrt{\left(\frac{Y_1}{n_1}\right)\left(1 - \frac{Y_1}{n_1}\right)/n_1 + \left(\frac{Y_2}{n_2}\right)\left(1 - \frac{Y_2}{n_2}\right)/n_2}$ has a limiting distribution, that is $N(0, 1)$

Result: The event $-2 < W < 2$, the probability of which is approximately equal to 0.954, is equivalent the event

$$\frac{Y_1}{n_1} - \frac{Y_2}{n_2} - 2U < p_1 - p_2 < \frac{Y_1}{n_1} - \frac{Y_2}{n_2} + 2U$$

Then the experimental values y_1 and y_2 of Y_1 and Y_2 respectively, will provide an approximate 95.4 percent confidence interval for $p_1 - p_2$

Problem

(1) If $n_1 = 100$, $n_2 = 400$, $y_1 = 30$, $y_2 = 80$. then the experimental values of $\frac{Y_1}{n_1} - \frac{Y_2}{n_2}$ and U are 0.1

$$U = \sqrt{\left(\frac{y_1}{n_1}\right)\left(1 - \frac{y_1}{n_1}\right)\Big/ n_1 + \left(\frac{y_2}{n_2}\right)\left(1 - \frac{y_2}{n_2}\right)\Big/ n_2}$$

$$= \sqrt{\frac{(0.3)(0.7)}{100} + \frac{(0.2)(0.8)}{400}}$$

$$= \sqrt{\frac{0.21}{100} + \frac{0.16}{400}}$$

$$U = 0.05$$

Thus the interval $(0, 0.2)$ is an approximate 95.4 percent confidence interval for $p_1 - p_2$

(2) Two independent random samples each of size 10 from two normal distribution $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ yield $\bar{x} = 4.8$, $S_1^2 = 8.64$, $\bar{y} = 5.6$, $S_2^2 = 7.88$. Find a 95% confidence interval for $\mu_1 - \mu_2$

Sol Given: $\bar{x} = 4.8$ $\quad s_1^2 = 8.64$

$\bar{y} = 5.6$ $\quad s_2^2 = 7.88$

$$T = \frac{(\bar{x} - \bar{y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{n s_1^2 + m s_2^2}{n+m-2}\left(\frac{1}{n} + \frac{1}{m}\right)}}$$

$$= \frac{(5.6 - 4.8)}{\sqrt{\frac{10(8.64)+10(7.88)}{10+10-2}\left(\frac{1}{10} + \frac{1}{10}\right)}}$$

$$= \frac{0.8}{\sqrt{\frac{165.2}{18}(0.2)}} = 0.5905$$

Problem 3 : Let two independent random variables Y_1 and Y_2 with binomial distributions that have parameters $n_1 = n_2 = 100$ p_1 and p_2 respectively be observed to be equal to $Y_1 = 50$ and $Y_2 = 40$. Determine and approximate 90% Confidence interval for $p_1 - p_2$

Sol
$$U = \frac{Y_1}{n_1} - \frac{Y_2}{n_2}$$

$$= \frac{50}{100} - \frac{40}{100}$$

$$= 0.5 - 0.4 = 0.1$$

$$U = \sqrt{\frac{(0.5)(0.5)}{100} - \frac{(0.4)(0.6)}{100}}$$

$$= \sqrt{\frac{0.25}{100} - \frac{0.24}{100}}$$

$$= 0.07$$

$$U = \frac{Y_1}{n_1} - \frac{Y_2}{n_2}$$

$$= \frac{50}{100} - \frac{40}{100}$$

$$= 0.5 - 0.4$$

$$U = 0.1$$

$$U = \sqrt{\frac{(0.5)(0.5)}{100} - \frac{(0.4)(0.6)}{100}}$$

$$= \sqrt{\frac{0.25}{100} - \frac{0.24}{100}}$$

$$U = 0.07.$$

23/9/17.

Chi - square test

In this section is discuss test of Statistical hypothesis called Chi - square test.

Let the random variable x_i be $N(\mu, \sigma^2)$, $i = 1, 2, \ldots n$ and let $x_1, x_2, \ldots x_n$ we mutually independent thus the joint pdf of these variable is

$$\frac{1}{\sigma_1 \sigma_2 \cdots \sigma_n (2\pi)^{n/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2\right]$$

The random variable is defined by the exponent (apart from the co-efficient $-\frac{1}{2}$) is $\sum_{i=1}^{n}(x_i - \mu_i)^2/\sigma_i^2$ and this random variable is $\chi^2(n)$.

Note :

we generalized this joint normal distribution of probability from n random variable that are dependent and we called the distribution a multi variate normal distribution.

1. Let us discuss some random variables that have approximate chi-square distribution :

Let x_1 be $b(n, P_1)$

Since the random variable $Y = (x_1 - nP_1) / \sqrt{nP_1(1-P_1)}$ has $n \to \infty$ a limiting distribution is $N(0,1)$

$\therefore$ we strongly suspect the limiting distribution of $Z = Y^2$ is $\chi^2(1)$.

If $G_n(y)$ is the distribution function of Y

$$\therefore \lim_{n \to \infty} G_n(y) = \phi(y) \quad , -\infty < y < \infty$$

where $\phi(y)$ is the distribution function of a distribution $N(0,1)$

Let $H_n(z)$ be the distribution function of $Z = Y^2$, for each +ve integer n. If $z \geq 0$ we have

$$H_n(z) = P(Z \leq z) = P(-\sqrt{z} \leq Y \leq \sqrt{z})$$
$$= G_n(\sqrt{z}) - G_n((-\sqrt{z}))$$

Since $\phi(y)$ is continuous every where

$$\lim H_n(z) = \lim_{n \to \infty} G_n(\sqrt{z}) - \lim_{n \to \infty} G_n[-(\sqrt{z})]$$

If we change variable of integration put $w^2 = v$

$2w\,dw = dv$

$dw = \frac{dv}{2\sqrt{v}}$

then $\lim_{n\to\infty} H_n(z) = \sqrt{2}\int_0^z \frac{1}{\sqrt{\pi}} e^{-v/2} \frac{dv}{2\sqrt{v}}$

$w = 0 \Rightarrow v = 0$

$w = \sqrt{z} \Rightarrow v = z$

$$= \int_0^z \frac{1}{\sqrt{2\pi}\, v^{1/2}} e^{-v/2}\, dv$$

$$= \int_0^z \frac{1}{\sqrt{2\pi}} v^{-1/2} e^{-v/2}\, dv$$

$[\because \sqrt{\pi} = \Gamma(1/2)]$

$$\lim_{n\to\infty} H_n(z) = \int_0^z \frac{1}{\Gamma(1/2)\, 2^{1/2}} v^{1/2 - 1} e^{-v/2}\, dv.$$

In provided that $z \geq 0$

Suppose if $z < 0$ then $\lim_{n\to\infty} H_n(z) = 0$.

$\therefore \lim_{n\to\infty} H_n(z)$ is equal to the distribution function of a random variable $\chi^2(1)$.

Result:

Let us consider X_1 is in $b(n, p_1)$, Let $X_2 = n - X_1$ and let $p_2 = 1 - p_1$

If we denote y^2 by Q_1 instead of Z

$$\therefore Q_1 = \frac{(X_1 - np_1)^2}{np_1(1-p_1)} = \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_1 - np_1)^2}{n(1-p_1)}$$

$$= \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_1 - np_1)^2}{np_2}$$

Since $(X_1 - np_1)^2 = (n - X_2 - n + np_2)^2$

$$= (X_2 - np_2)^2$$

Since Q_1 has a limiting χ^2 distribution with one degree of freedom.

We say that when n is a positive integer that Q_1 has an approximate chi-square distribution with one degree of freedom.

The above result can be generalized as follows.

26/9/17. Let $X_1, X_2, \dots X_{K-1}$ have multinomial distribution with parameters $n, P_1, P_2, \dots P_{K-1}$

Let $X_K = n - (X_1 + X_2 + \dots + X_{K-1})$ and

let $P_K = 1 - (P_1 + P_2 + \dots + P_{K-1})$

Defined Q_{K-1} By $Q_{K-1} = \sum_{i=1}^{K} \frac{(X_i - nP_i)^2}{nP_i}$

as $n \to \infty$ Q_{K-1} has limiting distribution of $\chi^2 (K-1)$ we can say that when $n \to \infty$ Q_{K-1} has approximate χ^2 - distribution with $K-1$ degrees of freedom. when n is a +ve integer

Problem:

1. One of the 1st six +ve integers is to be chosen by a random experiment.

Let $A_i = \{x : x = i\}$ $i = 1, 2, 3, \dots, 6$ the hypothesis H_0 (null hypothesis)

$H_0 : P(A_i) = P_{i0} = 1/6$, $i = 1, 2, \dots 6$ will be tested at the approximate 5% significant left against all alternatives.

Soln:

To make the test the random experiment will be repeated under the same condition 60 independent times

Soln:

$k=6$, $n=60$, $np_{i0} = 60 \times (1/6)$

$np_{i0} = 10$

Let x_i denote the frequency with which the random experiment terminates (stop) with the out come in A_i , $i = 1, 2, \ldots 6$

Let $Q_{k-1} = \sum_{i=1}^{k} \frac{(x_i - np_i)^2}{np_i}$

$Q_5 = \sum_{i=1}^{6} \frac{(x_i - 10)^2}{10}$

If H_0 is true, from the table with $k-1 = 6-1 = 5$ degrees of freedom we have probability of $(Q_5 \geq 11.1) = 0.05$

Now, the experimental frequency of $A_1, A_2, \ldots A_6$ are respectively 13, 19, 11, 8, 5 and 4

The observed value of Q_5

$$Q_5 = \sum_{i=1} \frac{(13-10)^2}{10} + \frac{(19-10)^2}{10} + \frac{(11-10)^2}{10} + \frac{(8-10)^2}{10} + \frac{(5-10)^2}{10} + \frac{(4-10)^2}{10}$$

$$= \frac{9}{10} + \frac{81}{10} + \frac{1}{10} + \frac{4}{10} + \frac{25}{10} + \frac{36}{10}$$

$$= 0.9 + 8.1 + 0.1 + 0.4 + 2.5 + 3.6$$

$$Q_5 = 15.6$$

$\therefore$ $15.6 \geq 11.1$

Thus our null hypothesis is rejected.

ie) $P(A_i) = 1/6$, $i = 1, 2, \ldots 6$ is rejected at 5%.

level of Significants.

2. A point is to be selected from the unit interval $\{x : 0 < x < 1\}$ by a random process. Let $A_1 = \{x : 0 < x \le 1/4\}$, $A_2 = \{x : 1/4 < x \le 1/2\}$, $A_3 = \{x : 1/2 < x \le 3/4\}$, $A_4 = \{x : 3/4 < x < 1\}$. Let the probability is P_i, $i = 1, 2, 3, 4$ assigned to these sets under the hypothesis be determined by the pdf $2x$, $0 < x < 1$, 0 elsewhere. Then these probability are respectively

$P_{10} = \int_0^{1/4} 2x \, dx$, $P_{20} = \frac{3}{16}$, $P_{30} = 5/16$, $P_{40} = 7/16$

Thus the hypothesis is said to be tested is that P_1, P_2, P_3 and P_4 have the preceeding values in a multinumial distribution with $k = 4$.

29/4/17

Soln:

The Hypothesis is said to be tested at an approximate 0.025 significants level by repeating the random experiment $n = 80$ independent times under the same conditions.

Here the young P_{i0}, $i = 1, 2, 3, 4$ are respectively 5, 15, 25 and 35

The observed frequecies of A_1, A_2, A_3 & A_4 are 6, 8, 20 & 36 respectively,

Then the observed value of $Q_3 = \sum_1^4 \frac{(x_i - nP_{i0})^2}{nP_{i0}}$

$$= \frac{(6-5)^2}{5} + \frac{(18-15)^2}{15} + \frac{(20-25)^2}{25} + \frac{(36-35)^2}{35}$$

$$= \frac{64}{35}$$

$= 1.83$ approximately

from the table with $4-1 = 3$ degrees of freedom; the values corresponding to a 0.025 significants level is 9.35

Since the observed value Q_3 is less than 9.35 the hypothesis is accepted at 0.025 level of significants

Bayesian Estimation:

We shall now describe the Bayesian approach to the problem of estimation

2m (This approach takes into account any prior knowledge of experiment that the statistician has and it is one application of a principle of statistical inference that may be called Bayesian statistics)

1. Explain Bayesian statistics estimation

Consider a random variable x has the distribution of the probability are depends on the θ.

where θ is an element of well defined set Ω

Let us introduce a random variable Θ has a distribution of the probability over the set Ω

We consider x as a possible value of the random variable X but θ as a possible value of the random variable Θ

$\therefore$ The distribution of x depends upon θ

an experimental value of the random variable Θ.

we shall denote the pdf of Θ by $h(\theta)$ and we take $h(\theta) = 0$ when θ is not an element of Ω.

Next, we denote the pdf of X by $f(x/\theta)$, since it is a conditional pdf of X given $\Theta = \theta$.

Let $X_1, X_2, \ldots X_n$ is a random sample from this conditional distribution of X

Thus we can write the joint conditional pdf of $X_1, X_2, \ldots X_n$ given $\Theta = \theta$

$$f(x_1/\theta) \cdot f(x_2/\theta) \ldots f(x_n/\theta)$$

Thus the joint pdf of $X_1, X_2, \ldots X_n$ and Θ is

$$g(x_1, x_2 \ldots x_n, \theta) = f(x_1/\theta) \cdot f(x_2/\theta) \ldots f(x_n/\theta) \cdot h(\theta).$$

If Θ is a random variable continuous random variable the joint marginal pdf of $X_1, X_2, \ldots X_n$ is given by

$$g_1(x_1, x_2, \ldots x_n) = \int_{-\infty}^{\infty} g(x_1, x_2, \ldots x_n, \theta)\, d\theta.$$

If Θ is a random variable of the discrete type, integration would be replaced by Σ. $\left(\sum_{-\infty}^{\infty} g(x_1, x_2, \ldots x_n, \theta)\right)$

In each case the conditional pdf of Θ given $X_1 = x_1, X_2 = x_2, \ldots X_n = x_n$ is

$$\boxed{k(\theta/x_1, x_2, \ldots x_n) = \frac{g(x_1, x_2, \ldots x_n, \theta)}{g_1(x_1, x_2, \ldots x_n)} = \frac{f(x_1/\theta) \cdot f(x_2/\theta) \ldots f(x_n/\theta)\, h(\theta)}{g_1(x_1, x_2, \ldots x_n)}}$$

This relationship is another form of Bayes's formula

03/10/17.

Problem:

1. Let $X_1, X_2, \ldots X_n$ be a random sample from a poisson distribution with mean θ where θ is the observed value of random variable Θ having a gamma distribution with known parameters α and β. Thus $g(x_1, x_2, \ldots x_n, \theta) =$

$$\left[\frac{\theta^{x_1} e^{-\theta}}{x_1!} \cdots \frac{\theta^{x_n} e^{-\theta}}{x_n!}\right]\left[\frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma\alpha\, \beta^{\alpha}}\right]$$

provided $x_i = 0, 1, 2, \ldots$ $i = 1, 2, \ldots n$ $0 < \theta < \infty$ and is equal to zero elsewhere.

Now,

$$g_1(x_1, x_2, \ldots x_n) = \int_0^\infty \frac{\theta^{\sum x_i + \alpha - 1} e^{-\left(\frac{n\beta+1}{\beta}\right)\theta}}{x_1! \ldots x_n! \, \Gamma(\alpha)\, \beta^{\alpha}}$$

$$= \frac{\Gamma\left(\sum_i^n x_i + \alpha\right)}{x_1!\, x_2! \ldots x_n!\, \Gamma(\alpha)\, (n\beta+1/\beta)^{\sum x_i + \alpha}}$$ the

Conditional pdf of Θ

Given $X_1 = x_1 \ldots x_n = x_n$

$$K(\theta / x_1, x_2, \ldots x_n) = \frac{g(x_1, x_2 \ldots x_n, \theta)}{g_1(x_1, x_2, \ldots x_n)}$$

$$= \frac{\theta^{\sum x_i + \alpha - 1} e^{-\theta / \left[\frac{\beta}{n\beta+1}\right]}}{\Gamma(\sum x_i + \alpha) \left[\beta / (n\beta+1)\right]^{\sum x_i + \alpha}} \quad 0 < \theta < \infty$$

$= 0$ elsewhere

2. Let $x_1, x_2, \ldots x_n$ denote a random sample from a distribution $b(1, \theta)$. $0 < \theta < 1$ we find a decision function δ ie) a Bayes solution

Soln:

Let $Y = \Sigma x_i$ and y is $B(1, \theta)$

Now, The Conditional pdf of Y is given $\Theta = \theta$

$g(y/\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$ $y = 0, 1, 2, \ldots n$

$= 0$ elsewhere

we consider the prior pdf of the random variable Θ to be $h(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha \, \Gamma\beta} \theta^{\alpha-1} (1-\theta)^{\beta-1}$ $0 < \theta < 1$

$= 0$ elsewhere

Where α and β are +ve constants

Thus the conditional pdf of Θ given $Y = y$ is at points of positive probability density w.r.t $P(y) \alpha$

$k(\theta/y) \propto \theta^y (1-\theta)^{n-y} \theta^{\alpha-1} (1-\theta)^{\beta-1}$, $0 < \theta < 1$

$k(\theta/y) = \frac{\Gamma(n+\alpha+\beta)}{\Gamma(\alpha+y) \, \Gamma(n+\beta-y)} \theta^{\alpha+y-1} (1-\theta)^{\beta+n-y-1}$, $0 < \theta < 1$

and $y = 0, 1, 2, \ldots n$

Unit - V

Theory of Statistical Tests :

In this unit we consider some methods of constructing good statistical test the begin with testing a simple hypothesis H_0 (null hypothesis) against a simple alternative hypothesis H_1.

(X) The following are three important conditions

2m for the tests

(i) Define a best test for H_0 against H_1?

(ii) Prove a theorem that provides the method of determine a best test?

(iii) Give 2 examples?

06/10/17.

Results :

Let $f(x, \theta)$ be the pdf of a random variable x

Let $x_1, x_2, \ldots x_n$ be a random sample from this distribution consider the two simple hypothesis

$H_0 : \theta = \theta'$ $\quad H_1 : \theta = \theta''$.

Thus $\Omega = \{\theta : \theta = \theta', \theta''\}$

Now, we define a best critical region for testing simple hypothesis H_0. against the alternative simple hypothesis H_1.

Note :

The symbols probability of $[(x_1, x_2, \ldots x_n) \in C ; H_0]$

$P[(x_1, x_2, \dots x_n) \in C ; H_1]$. It means that $P[(x_1 x_2 \dots x_n) \in C]$ when H_0 & H_1 are true respectively,

Definition:

Let c be a subset of sample space then c is called best critical region of size α for testing simple hypothesis $H_0 : \theta = \theta'$ against the alternative simple hypothesis $H_1 : \theta = \theta'$. If for every subset A of the sample space for which $P[(x_1, x_2, \dots, x_n) \in A ; H_0] = \alpha$

$(\because H_0 : P(x_1, x_2 \dots x_n)) = \alpha$

(i) $P[(x_1, x_2, \dots x_n \in C ; H_0] = \alpha$

(ii) $P[(x_1, x_2 \dots x_n \in C : H_1] \geq P[(x_1, x_2, \dots, x_n) \in A : H_1]$.

First we assume H_0 to be true in general there will be a multiplicity of subsets A of the sample space such that $P[(x_1, x_2, \dots, x_n) \in A)] = \alpha$.

Suppose that there is one of these subsets say C such that when H_1 is true.

The power of the test associated with C is atleast as great as the power of the test associated with each other A. Then C is defined as a best critical region of size α for testing H_0 against H_1

(7)

Problem:

Consider the one of random variable x has a binomial distribution with $n=5$ and $p=\theta$ then $f(x,\theta)$ be the pdf of x and let $H_0: \theta = \frac{1}{2}$ and $H_1: \theta = \frac{3}{4}$

The following table gives at points of probability density the values of $f(x, \frac{1}{2})$ $f(x, \frac{3}{4})$ the ratio $f(x; \frac{1}{2}) / f(x; \frac{3}{4})$

x	0	1	2	3	4	5
$f(x; \frac{1}{2})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$f(x; \frac{3}{4})$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{1024}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$
$f(x; \frac{1}{2}) / f(x; \frac{3}{4})$	32	$\frac{32}{3}$	$\frac{32}{9}$	$\frac{32}{27}$	$\frac{32}{81}$	$\frac{32}{243}$

Soln:

Let x be a random variable

$H_0: \theta = \frac{1}{2}$ against the alternative simple hypothesis

$H_1: \theta = \frac{3}{4}$.

First we assign a significant level of the test to be $\alpha = \frac{1}{32}$.

We find a best critical region of size $\alpha = \frac{1}{32}$.

If $A_1 = \{x : x = 0\}$

$A_2 = \{x : x = 5\}$

$P(x \in A_1 : H_0) = \frac{1}{32}$

$P(x \in A_2 : H_0) = \frac{1}{32}$ and

there is no other subset A_3 of the space $\{x : x = 0,1,2,3,4,5\}$ such that probability of $(x \in A_3 : H_0) = 1/32$

Then either A_1 or A_2 is the best critical region C of size $\alpha = 1/32$ for testing H_0 against H_1

Now,

$P(X \in A_1 : H_0) = 1/32$

$P(X \in A_1 : H_1) = 1/1024$

$\therefore$ If the set A_1 is used as a critical region of size $\alpha = 1/32$.

The probability of rejecting H_0 when H_1 is true is much less than the probability of rejecting H_0 when H_0 is true.

(ii) Suppose we use the A_2 If the set A_2 is used as critical region, Then $P[x \in A_2 : H_0] = 1/32$ and

$P[x \in A_2 : H_1] = \frac{243}{1024}$

ie) The probability of rejecting H_0 when H_1 is true is much greater than the probability of rejecting H_0 when H_1 is true.

(7) 10m

Neyman - Pearson theorem

Let $X_1, X_2, \ldots X_n$ where n is a fixed +ve integers, denote a random sample from a distribution that has pdf $f(x, \theta)$ then the joint pdf of $X_1, X_2, \ldots, X_n$ is $L(\theta; x_1, x_2, \ldots x_n) = f(x_1; \theta)\, f(x_2; \theta) \ldots f(x_n; \theta)$

Let θ' and θ'' be distinct fixed values of θ.

So that $\Omega = \{\theta : \theta = \theta', \theta''\}$, and let k be a +ve number. Let C be a subset of sample space such that

(i) $\dfrac{L(\theta'; x_1, x_2, \ldots, x_n)}{L(\theta''; x_1, x_2, \ldots x_n)} \leq k$ for each point $(x_1, x_2, \ldots, x_n) \in C$

(ii) $\dfrac{L(\theta'; x_1, x_2, \ldots, x_n)}{L(\theta''; x_1, x_2, \ldots x_n)} \geq k$ for each point $(x_1, x_2, \ldots x_n) \in C^*$

(iii) $\alpha = P[(X_1, X_2, \ldots X_n) \in C;\ H_0]$.

Then C is a best critical region of size α for testing the simple hypothesis $H_0 : \theta = \theta'$ against the alternative simple hypothesis $H_1 : \theta = \theta''$.

Proof:

We prove the theorem when the random variables are continuous type

If C is only critical region of size α, the theorem is proved.

If there is another critical region of size α denoting by A. we denote $\int \cdots \int_R L(\theta; x_1, x_2, \ldots x_n) dx_1 \ldots dx_n$ by $\int_R L(\theta)$.

In this notation we prove that $\int_C L(\theta'') - \int_A L(\theta'') \geq 0$

Since C is the union of disjoint sets $C \cap A$ and $C \cap A^*$ and A is the union of disjoint sets $A \cap C$ and $A \cap C^*$

$$\int_C L(\theta'') - \int_A L(\theta'') = \int_{C\cap A} L(\theta'') + \int_{C\cap A^*} L(\theta'') - \int_{A\cap C} L(\theta'') - \int_{A\cap C^*} L(\theta'').$$

$$= \int_{C\cap A^*} L(\theta'') - \int_{A\cap C^*} L(\theta''). \quad \text{from (i)}$$

However, by the hypothesis of the theorem

$L(\theta'') \geq \frac{1}{K} L(\theta')$ at each point of C. and

hence at each point of $C \cap A^*$.

$$\therefore \int_{C\cap A^*} L(\theta'') \geq \frac{1}{K} \int_{C\cap A^*} L(\theta')$$

But. $L(\theta'') \leq \frac{1}{K} L(\theta')$ at each point of C^* from (ii)

and hence at each point of $A \cap C^*$

$\therefore$ we have $\int_{A\cap C^*} L(\theta'') \leq \frac{1}{K} \int_{A\cap C^*} L(\theta')$

$\left[-\int_{A\cap C^*} L(\theta'') \geq -\frac{1}{K} \int_{A\cap C^*} L(\theta') \right.$

$\therefore$ we have $\int_{C\cap A^*} L(\theta'') - \int_{A\cap C^*} L(\theta'') \geq \frac{1}{K} \int_{C\cap A^*} L(\theta') - \frac{1}{K} \int_{A\cap C^*} L(\theta')$

$$= \frac{1}{K}\int_{C\cap A^*} L(\theta') - \int_{A\cap C^*} L(\theta') \longrightarrow (1)$$

$$\int_C L(\theta'') - \int_A L(\theta'') \geq \frac{1}{K}\left[\int_{C\cap A^*} L(\theta') - \int_{A\cap C^*} L(\theta')\right] \longrightarrow (2)$$

But

$$\int_{C\cap A^*} L(\theta') - \int_{A\cap C^*} L(\theta')$$

$$= \int_{C\cap A^*} L(\theta') + \int_{C\cap A} L(\theta') - \int_{A\cap C} L(\theta') - \int_{A\cap C^*} L(\theta')$$

$$= \int_{C\cap A^*} L(\theta') - \int_{A\cap C^*} L(\theta').$$

$$= \int_C L(\theta') - \int_A L(\theta')$$

$$= \alpha - \alpha$$

$$= 0$$

using in (2) get,

$$\int_C L(\theta'') - \int_A L(\theta'') \geq 0$$

<u>Note</u>:

If the random variables are of the discrete type the proof is the same with integration replaced by Σ.

Result:

As stated in the statement of this theorem conditions (i), (ii) & (iii) are sufficient once for region C to be a best critical region of size α, however they are all so necessary.

Problem: (X) 5m

Let $x_1, x_2 \dots x_n$ denote a random sample from the distribution that has the pdf $f(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x-\theta)^2}{2}\right]$ $-\infty < x < \infty$.

using the N.P theorem test pdf,

Soln:

Simple hypothesis $H_0 = \theta = \theta' = 0$ and alternative simple hypothesis $H_1 = \theta = \theta'' = 1$.

Now, $\dfrac{L(\theta', x_1, x_2, \dots x_n)}{L(\theta'', x_1, x_2, \dots x_n)}$

$$= \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\sum_{i=1}^{n} x_i^2/2\right]}{(1/\sqrt{2\pi})^n \exp\left[-\sum_{i=1}^{n} (x_i-1)^2/2\right]}$$

$$= \exp\left(-\sum x_i^2 + n/2\right)$$

If $K > 0$ the set of all points $(x_1, x_2, \dots, x_n)$ such that $\exp\left(-\sum_{i=1}^{n} x_i + n/2\right) \le K$ is the best critical region, This inequality holds iff $-\sum_{i=1}^{n} x_i + n/2 \leq \log k$

$$\sum x_i \geq n/2 - \log k = c$$

In this case a best critical region is the set C

$= \{(x_1, x_2, \ldots, x_n) ; \sum x_i \geq c\}$

where c is a constant that can be determined so that, the size of critical region is a desired number α.

11/10/17.

1. Let $X_1, X_2, \ldots X_n$ denote a random sample which has a pdf $f(x)$ is +ve on and only on the non-negative integers it is desired to test a simple hypothesis

(*) 5m

$$H_0 : f(x) = \frac{e^{-1}}{x!}, \quad x = 0, 1, 2, \ldots$$

$$= 0 \quad \text{elsewhere}$$

Against the alternative simple hypothesis H_1

$$H_1 : f(x) = (1/2)^{x+1} \quad x = 0, 1, 2, \ldots$$

$$= 0 \quad \text{elsewhere}$$

Soln:

By using Neymann pearson's theorem

$$\frac{g(x_1, x_2, \ldots, x_n)}{h(x_1, x_2, \ldots x_n)} = \frac{e^{-n}/(x_1!\, x_2! \ldots x_n!)}{(1/2)^n (1/2)^{x_1 + x_2 + \cdots + x_n}}$$

$$= \frac{e^{-n} / x_1!\, x_2! \cdots x_n!}{(1/2^n)\,(1/2)^{x_1+x_2+\cdots+x_n}}$$

$$= \frac{e^{-n}\, 2^n \cdot 2^{x_1+x_2+\cdots+x_n}}{\prod_{i=1}^{n} x_i!}$$

$$= \frac{(e^{-1}\cdot 2)^n\, 2^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

If $K>0$ the set of points $x_1, x_2, \ldots, x_n$ such that taking logarithm we have,

$$\frac{(e^{-1}2)^n\, 2^{\sum x_i}}{\prod_{i=1}^{n} x_i!} \le K$$

$$\log\left[\frac{(e^{-1}2)^n\, 2^{\sum x_i}}{\prod_{i=1}^{n} x_i!}\right] \le \log K$$

$$\log\left[(e^{-1}2)^n\, 2^{\sum x_i}\right] - \log \prod_{i=1}^{n} x_i! \le \log K$$

$$\log (e^{-1}2)^n + \log 2^{\sum x_i} - \log\left[\prod_{i=1}^{n} x_i!\right] \le \log K$$

$$n.\log(e^{-1}2) + \sum x_i \log 2 - \log\left[\prod_{i=1}^{n} x_i!\right] \le \log K.$$

$$(\sum x_i)\log 2 - \log\left[\prod_{i=1}^{n} x_i!\right] \le \log K - n\log(e^{-1}.2) = c \text{ (say)}$$

is a best critical region C

Consider $K=1$ and $n=1$ we have,

$$\frac{2^{x_1}}{x_1!} \le e/2$$

This inequality satisfied by all the points in the

Set C $\quad C = \{ x_1 ; x_1 = 0, 3, 4, 5, \dots \}$

Thus the power of the test when H_0 is true

ie) $P(X_1 \in C : H_0] = 1 - P[X_1 = 1, 2 ; H_0]$

$= 0.448$ (approximately)

The power of the test when H_1 is true

$P(X_1 \in C : H_1) = 1 - P[X_1 = 1, 2 ; H_1]$

$= 1 - (1/4 - 1/8)$

$= 0.625.$

Uniformly most powerful test :

Defn:

2m The Critical region C is a uniformly most powerful critical region of size α for testing the simple hypothesis H_0 against the an alternative composite hypothesis H_1 if the set C is critical region of size α for testing H_0 against each simple hypothesis in H_1.

2m Uniformly most powerful test

A test is defined by this critical region c is called a uniformly most powerful test with significant level α for testing the simple hypothesis H_0 against the alternative composite hypothesis H_1.

Likelihood Ratio test

(✱) 2m

The notion of using the magnitude of the ratio of two probability density functions as the basis of a best test (or) of a uniformly most powerful test can be modified and to provide a method of constructing a test of a composite hypothesis against an alternative composite hypothesis. When a uniformly most powerful test does not exist. This method leads to tests called likelihood ratio test.

A likelihood ratio test as just remarked is not necessarily a uniformly most powerful test.