Sets :-

Set is a collection of well-defined objects.

A = {1, a, 2, 3}

B : {collection of mathematics book in our library}.

collection of brilliant students in our class room is not a set.

Subsets :-

A ⊆ B if for every element in A is also in B.

A = {1, 2} B = {1, 2, 3, 4}

A ⊆ B = 4

Union :-

$A \cup B \{x / x \in A, \text{ or } x \in B\}$

A = {1, 2, 3} B = {1, 2, 5, 7}

$A \cup B = \{1, 2, 3, 5, 7\}$

Intersection :-

$A \cap B = \{x / x \in A \text{ and } x \in B\}$

A = {1, 2, 3} B = {1, 2, 5, 7}

$A \cap B = \{1, 2\}$

DIFFERENCE :-

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$

$A = \{1, 2, 3\}$ $B = \{1, 2, 5, 7\}$

$A - B = \{3\}$.

CARTESIAN PRODUCT :-

$A \times B = \{(a, b) \mid a \in A, b \in B\}$.

$A = \{1, 2\}$ $B = \{5, 7\}$

$A \times B = \{(1,5) (1,7) (2,5) (2,7)\}$.

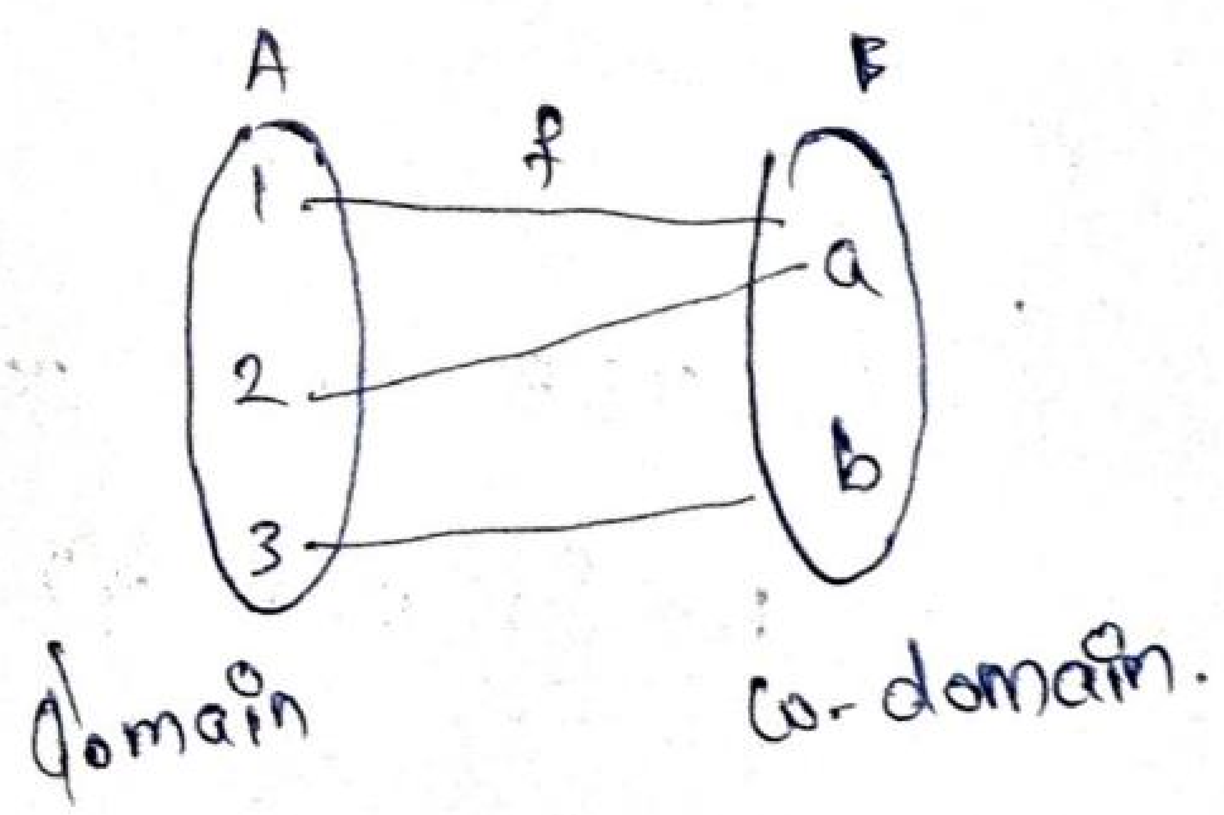
FUNCTION :-

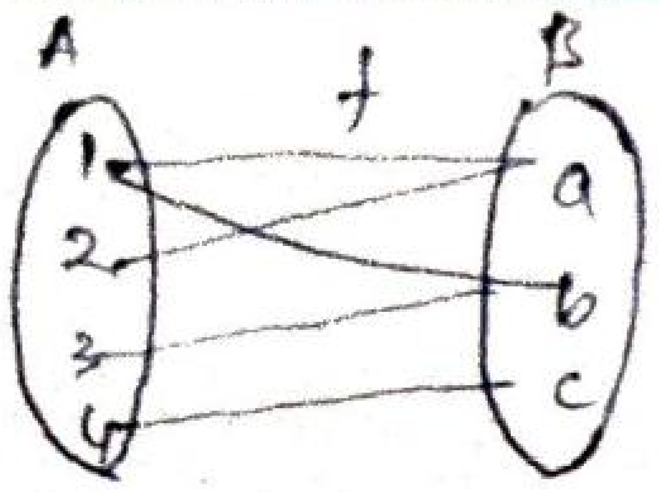
A and B are two non-empty sets if we define a function $f : A \to B$ such that for every element in A assign to unique element in B.

$A = \{1, 2, 3\}$ $B = \{a, b\}$

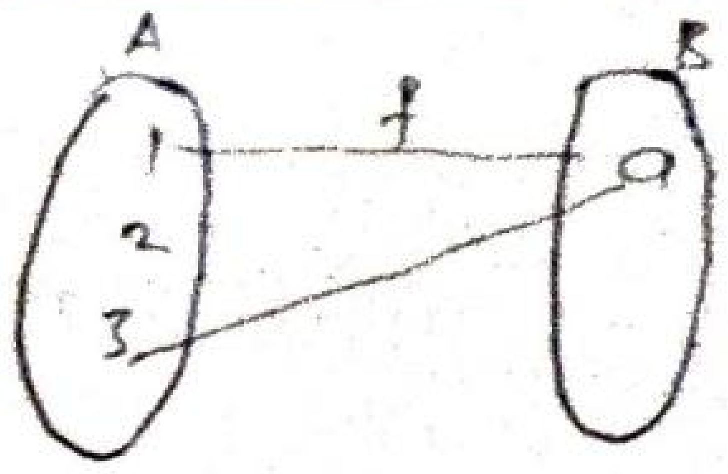
$f : A \to B$ defined as,

$f(1) = a;$ $f(2) = a;$ $f(3) = b$.





∴ one element has 2 images.



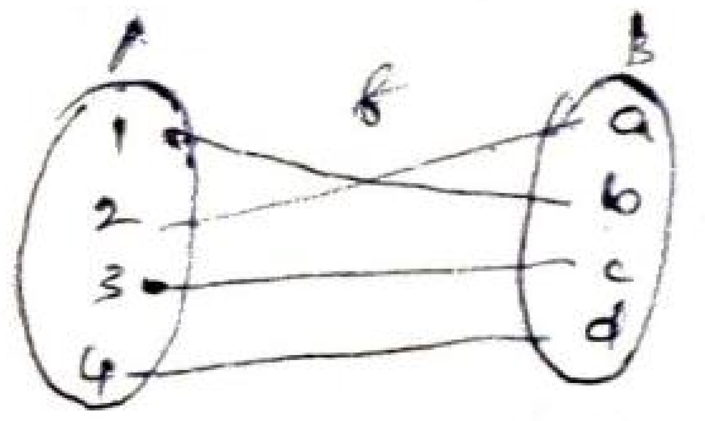
∴ The element 2 has no image.

It is not a function.

ONE - ONE FUNCTION :-

f : A → B is said to be one to one if for every element in A assign to different element in B i.e.,

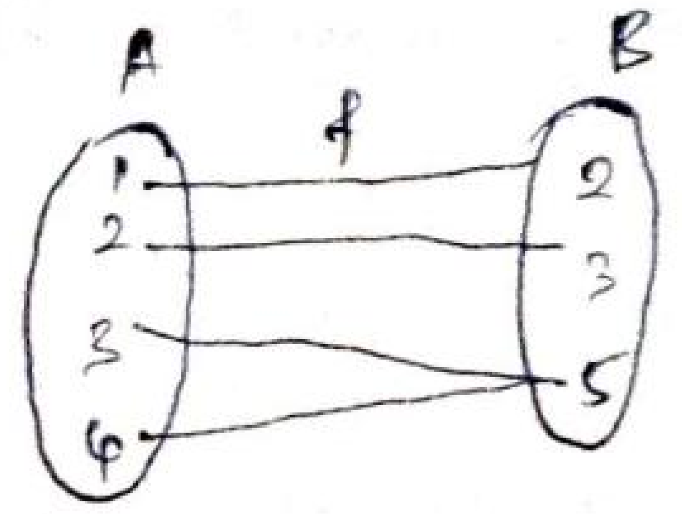
for every a, b ∈ A ∋ : f(a) ≠ f(b) ⇒ a ≠ b.



ONTO FUNCTION :-

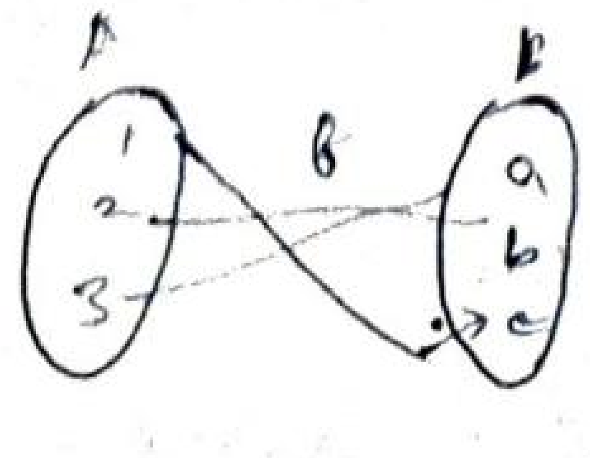
f : A → B is said to be onto function if any element in B is a image of A.

i.e., for any element y ∈ B ∋ f(x) = y ⇒ x ∈ A.



BIJECTION :-

f: A → B is said to be a bijection if f has one - one and onto functions.



INDEX SETS :-

Let I be a non - empty set we defined $\{A_i \mid i \in I\}$ is called a family a index set

$A_i = \{i, i+1, i+2 \ldots\}$

$A_1 = \{1, 2, 3..\}$

$A_2 = \{2, 3, 4 \ldots\}$

COUNTABLE SETS :-

Finite set :-

A set A is said to be finite if f : A → N where N is some natural numbers, and is a bijection.

Equivalent sets :-

Two sets A and B are said to be ~~said~~ equivalent, if there exists a bijection $f : A \to B$.

$A = \{1, 2, 3, \ldots\} = N$

$B = \{-1, -2, -3 \ldots\}$

$f : A \to B$ defined as $f(n) = -n$, is a bijection.

Hence A and B are equivalent sets.

Countably infinite :-

A set A is said to be countably infinite if the set A equivalent to the set of all natural numbers.

Ex :- Let W be the whole numbers. To prove that W is countably infinite.

Sol:-

$f : N \to W$ defined by $f(n) = n - 1$. is a bijection.

$\therefore$ W is equivalent to N.

$\therefore$ W is countably infinite.

(6) To prove that Z is countably infinite.

Sol :-

Let $Z = \{0, \pm 1, \pm 2, \ldots\}$

$f : N \to Z$ defined as $f(n) = \begin{cases} n/2, & n \text{ is e[illegible]} \\ \frac{1-n}{2}, & n \text{ is od[illegible]} \end{cases}$

is a bijection.

$\therefore$ Z is equivalent to N.

$\therefore$ Z is countably infinite.

COUNTABLE SETS :-

A set A is said to be countable if it is finite or countably infinite.

(i) $A = \{1, 2, 3, \ldots 50\}$ is finite is obviously.

(ii) N, W, Z, Q are all countable sets as well as countable infinite.

NOTE :-

(i) A set A is said to be countably infinite if it can be labled by using the set of all natural numbers.

$A = \{a_1, a_2, a_3 \ldots\}$ can be labled.

THEOREM:- 1.1 Every subset of a countable set is countable.

PROOF:- Let A be a countable and B is a subset of A.

Case-(i):-

If A is finite, then B is also finite.

Hence B is countable.

Case-(ii):-

Let A is countably infinite and B is infinite.

Hence $A = \{a_1, a_2, a_3 \ldots\}$.

If we take least element $a_{n_1} \in A$ such that $a_{n_1} \in B$.

Next we take next least number $a_{n_2} \in A$ such that $a_{n_2} \in B$.

Proceeding like this, we get

$B = \{a_{n_1}, a_{n_2}, a_{n_3} \ldots\}$

Hence all the elements of B are labled

$\therefore$ B is countably infinite.

$\therefore$ Every subset of a countable set is countable.

THEOREM :- 1.2

To prove that Q^+ is countable

PROOF :-

$Q^+ = \{ p/q \mid p, q \in Z^+ \text{ and } q \neq 0 \}$.

First we take positive rational numbers whose numerator and denominator add upto 2.

we get only one number namely $\{1/1\}$

Next we take positive rational numbers whose numerator and denominator add upto 3.

we get only ~~one~~ two number namely $\{1/2, 2/1\}$

Next we take positive rational numbers whose numerator and denominator add upto 4.

we have $\{1/3, 3/1, 2/2\}$

Proceeding like this we can list out all the positive rational numbers from the beginning to omitting whose those are repeated one.

Hence we get $= \{1/1, 1/2, 2/1, 1/3, 3/1, \ldots\}$

(9) Therefore all the +ve rational numbers is labeled.

$\therefore$ Q^+ is countable.

Theorem: 1.3 To prove that Q is countable.

Proof:-

Let us take $Q = \{0, \pm q_1, \pm q_2, \dots\}$

Now we take defined $f: N \to Q$ by

$f(1) = 0$; $f(2n) = q_n$; $f(n+1) = -q_n$.

Hence f is bijection.

$\therefore$ Hence Q is equivalent to N.

$\therefore$ Q is countably infinite.

Theorem:- 1.4 To prove that $N \times N$ is countable.

Proof :-

$N \times N = \{(i, j) / i, j \in N\}$

First we take $(i, j) \in N \times N$,

such that $i + j = 2$, we get $(1, 1)$.

Next we take $(i, j) \in N \times N$,

such that $i + j = 3$

we have $\{(1, 2), (2, 1)\}$

Now we take $(i,j) \in N \times N$ such that

$i+j = 4$.

we have $\{(1,3), (2,2), (3,1)\}$

proceeding like this we can list out all the elements of $N \times N$.

$\therefore$ $N \times N$ is countably infinite.

$\therefore$ $N \times N$ is countable.

THEOREM 1.5 Let A, B be countable sets then prove $A \times B$ is countable.

proof :-

We assume that A and B are countably infinite.

We define $f : N \times N \to A \times B$ by

$f(i,j) = (a_i, b_j)$

we claim that f is a bijection.

Suppose $x = (p,q) \in N \times N$ and

$y = (u,v) \in N \times N$.

Now,

$f(x) = f(y) \Rightarrow (a_p, b_q) = (a_u, b_v)$

$\Rightarrow a_p = a_u, \quad b_q = b_v$

$\Rightarrow p = u$ and $q = v$

$\Rightarrow (p,q) = (u,v)$

$\Rightarrow x = y$.

Inj.

$\therefore f$ is 1-1.

Now, suppose $(a_m, a_n) \in A \times B$.

Then $(m, n) \in N \times N$ and $f(m,n) = (a_m, a_n)$

$\therefore f$ is onto.

$\therefore f$ is bijection.

Hence $A \times B$ is equivalent to $N \times N$ which is countable.

Hence $A \times B$ is countable.

THEOREM:- 1.6

Let A be a countably infinite set and f be a mapping of A onto B then B is countable.

Proof :-

Let A be a countably infinite set and $f : A \to B$ is onto.

Let $b \in B$ such that $a \in A$,

there exists $f(a) = b$ ($\because f$ is onto)

Now we define $g : B \to A$ such that $g(b) = a$

clearly g is 1-1.

$\therefore$ B is equivalent to a subset of the countable set A.

$\therefore$ B is countable

THEOREM:- 1.7 countable union of countable set is countable.

Proof :-

Let $D = \{A_1, A_2, A_3 \ldots\}$ be a countable family of countable sets

Case (i) :-

Let assume A_i are all countably infinite sets.

Suppose,

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$$

$$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$$

$$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

Now we define that $f : N \times N \to \cup A_n$ defined as $f(i, j) = a_{ij}$.

Clearly f is onto.

Also by theorem 1.4, $N \times N$ is countably infinite.

Hence by theorem 1.6, $\cup A_n$ is countably infinite.

(18)

Case (ii) :-

Let each A_i be countable.

For each A_i choose a set B_i such that B_i is a countably infinite set and $A_i \subseteq B_i$

Then $\cup A_i \subseteq \cup B_i$

By case (i), $\cup B_i$ is countably infinite

By known theorem, Every subset of a countable set is countable.

$\therefore \cup A_i$ is countable.

UNCOUNTABLE SETS

A set which is not countable is called uncountable.

Ex :- R, S is uncountable.

THEOREM 1.8

To prove that $(0,1]$ is uncountable.

Proof :-

Every real number in $(0,1]$ can be written uniquely as a non-terminating decimal $0.a_1a_2a_3\dots$ where $0 \le a_i \le 9$.

For each i subject to the following restrictions that any terminating decimal $0.a_1a_2\dots a_n0000\dots$ is written as

$0.\ a_1 a_2 \cdots (a_n - 1)\, 9999 \ldots$

(14) Suppose $(0,1]$ is countable.

Now we can labelled all the elements of $(0,1]$

i.e., $x_1, x_2, x_3 \ldots\ldots x_n$

where $x_1 = 0.\ a_{11}\ a_{12}\ a_{13} \ldots\ldots$

$x_2 = 0.\ a_{21}\ a_{22}\ a_{23} \ldots\ldots$

$\vdots \qquad \vdots \qquad \vdots$

$x_n = 0.\ a_{n1}\ a_{n2}\ a_{n3} \ldots\ldots$

Now, for each positive integers n choose an integers be ~~an~~ b_n such that

$$b_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases}$$

let $y = 0 \cdot b_1\ b_2\ b_3 \ldots\ldots$

clearly $y \in (0,1]$.

Also y is different from x_i atleast in the i^{th} place.

Hence $y_i \neq x_i$ for each i, which is contradiction. ($\Rightarrow\Leftarrow$)

$\therefore$ our assumption is wrong.

$\therefore$ $(0,1]$ is uncountable.

(15)

Corollary :-

To prove R is uncountable.

proof :-

let us take $(0,1] \subseteq R$.

$\therefore$ We know that $[0,1]$ is uncountable.

$\therefore$ A superset of uncountable subset is uncountable.

$\therefore$ R is uncountable.

To prove S is uncountable.

proof :-

let $R = Q \cup S$.

we know that Q is countable.

suppose Q is countable,

$\Rightarrow$ R is countable.

But by (theorem 1.7), R is uncountable.

which is contradiction.

$\therefore$ S is uncountable.

INEQUALITIES OF HOLDER AND MINKOWSKI

THEOREM 1.9 HOLDER'S INEQUALITY :-

If $p>1$ and q is such that $\frac{1}{p}+\frac{1}{q}=1$ then,

$$\sum_{i=1}^{n} |a_i b_i| \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$$

where $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ are real numbers.

Proof :-

First we shall prove the inequality.

$$x^{1/p} y^{1/q} \leq \frac{x}{p}+\frac{y}{q} \quad \text{where } x \geq 0 \text{ and } y \geq 0.$$

This inequality is trivial if $x=0$ or $y=0$.

Now, let $x, y > 0$

consider the function,

$f(t) = t^{\lambda} - \lambda t + \lambda - 1$ where $\lambda = \frac{1}{p}$ and $t \geq 0$.

Then $f'(t) = \lambda t^{\lambda-1} - \lambda$

$= \lambda (t^{\lambda-1} - 1)$

$\therefore f(t) = f'(t) = 0.$

Also $f'(t) > 0$ for $0 < t < 1$ and

$f'(t) < 0$ for $0 > 1$.

$\therefore$ $f(t) \leq 0$ for all $t \geq 0$ and in

particular $f\left(\frac{x}{y}\right) \leq 0$.

$$\therefore \left(\frac{x}{y}\right)^{\lambda} - \lambda\left(\frac{x}{y}\right) + \lambda - 1 \leq 0.$$

$$\therefore \left(\frac{x}{y}\right)^{\frac{1}{p}} - \frac{1}{p}\left(\frac{x}{y}\right) + \frac{1}{p} - 1 \leq 0.$$

multiplying by y we get,

$$x^{1/p}\, y^{1/q} \leq \frac{x}{p} + \frac{y}{q} \quad \left(\because 1 - \frac{1}{p} = \frac{1}{q}\right)$$

Now to prove Holder's inequality, we apply the above inequality to the numbers:

$$x_i = \frac{|a_j|^p}{\sum_{i=1}^{n} |a_i|^p} \qquad y_i = \frac{|b_j|^q}{\sum_{i=1}^{n} |b_i|^q} \quad \text{for each } j = 1, 2 \ldots n$$

we get,

$$\frac{|a_j|\ |b_j|}{\left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} |b_i|^q\right]^{\frac{1}{q}}} \leq \frac{x_j}{p} + \frac{y_j}{q}$$

for all $j = 1, 2, \ldots n$.

Adding these inequalities we get,

$$\frac{\sum_{i=1}^{n} |a_i| \, |b_i|}{\left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} |b_i|^q\right]^{\frac{1}{q}}} \leq \sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) \rightarrow (1)$$

Now,

$$\sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) = \frac{1}{p}\left(\sum_{j=1}^{n} x_j\right) + \frac{1}{q}\left(\sum_{j=1}^{n} y_j\right)$$

$$= \frac{1}{p}\left(\frac{\sum_{j=1}^{n} |a_j|^p}{\sum_{j=1}^{n} |a_j|^p}\right) + \frac{1}{q}\left(\frac{\sum_{j=1}^{n} |b_j|^p}{\sum_{j=1}^{n} |b_j|^p}\right)$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

Using this (1) we get,

$$\sum_{i=1}^{n} |a_i b_i| \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} |b_i|^q\right]^{\frac{1}{q}}$$

Corollary : (Cauchy - Schwarz inequality)

$$\sum_{i=1}^{n} |a_i b_i| < \left[\sum_{i=1}^{n} |a_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |b_i|^2\right]^{\frac{1}{2}}$$

Proof :-

Take $p = q = 2$ in Holder's inequality.

THEOREM 1.10

(19)

MINKOWSKI'S INEQUALITY :-

If $p \geq 1$,

$$\left[\sum_{i=1}^{n} |a_i b_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p}$$

where $a_1, a_2, \dots a_n$ and $b_1, b_2, \dots b_n$ are real numbers.

Proof :-

This inequality is trivial when $p=1$.

let $p>1$ clearly

$$\left[\sum_{i=1}^{n} |a_i+b_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{\frac{1}{p}} \rightarrow (1)$$

Now,

$$\sum_{i=1}^{n} (|a_i| + |b_i|)^p = \sum_{i=1}^{n} (|a_i| + |b_i|)^{p-1} (|a_i| + |b_i|)$$

$$= \sum_{i=1}^{n} |a_i| (|a_i| + |b_i|)^{p-1} + \sum_{i=1}^{n} |b_i| (|a_i| + |b_i|)^{p-1}$$

$$\leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^{(p-1)q}\right]^{\frac{1}{q}}$$

$$+ \left[\sum_{i=1}^{n} |b_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^{(p-1)q}\right]^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ (using Holder's inequality)

(20)

Now since $\frac{1}{p} + \frac{1}{q} = 1$ we have

$p+q = pq$.

Hence $(p-1)q = p$

$\therefore$ Dividing by $\left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{\frac{1}{q}}$ we get

$$\left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1-\frac{1}{q}} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p}$$

$$\therefore \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p} \rightarrow (2)$$

From (1) and (2) we get the required inequality.

CHAPTER 2

METRIC SPACES

2.0 INTRODUCTION

The concept of convergence of sequences of real numbers depends on the absolute value of the difference between any two real numbers. We observe that this obsolute value is nothing but the distance between the two numbers when they are considered as points on the real line. For the study of the concepts like continuity and convergence the algebraic properities of **R** are irrelevant. This situation necessitates the study of sets in which a reasonable notion of distance is defined. A set equipped with a reasonable concept of distance is called a *metric space*. In this chapter we develop in a systematic manner the main facts about metric spaces.

2.1 DEFINITION AND EXAMPLES

Definition : A **metric space** is a non-empty set M together with a function $d : M \times M \to \mathbf{R}$ satisfying the following conditions.

(i) $d(x, y) \geq 0$ for all $x, y \in M$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$

(triangle inequality)

d is called a **metric or distance function** and $d(x, y)$ is called the **distance** between x and y.

Note. The metric space M with the metric d is denoted by (M, d) or simply by M when the underlying metric d is clear from the context.

Example 1. In **R** we define $d(x, y) = |x - y|$. Then d is a metric on **R**. This is called the **usual metric** on **R**.

Proof. Clearly $d(x, y) = |x - y| \geq 0$.

Also $d(x, y) = 0 \Leftrightarrow |x - y| = 0$.

$\Leftrightarrow x = y$.

$$d(x, y) = |x - y|$$
$$= |y - x|$$
$$= d(y, x).$$

Now, let $x, y, z \in \mathbf{R}$.

Then $d(x, z) = |x - z| = |x - y + y - z|$

$$\leq |x - y| + |y - z|$$
$$= d(x, y) + d(y, z).$$

$$\therefore \quad d(x, z) \leq d(x, y) + d(y, z).$$

Hence d is a metric on **R**.

Note. In what follows whenever we consider **R** as a metric space the underlying metric is taken to be the usual metric unless otherwise stated.

Example 2. In **C** we define $d(z, w) = |z - w|$. Then d is a metric on **C**. This is called the **usual metric** on **C**.

Note. If the complex number $z = x + iy$ is identified with the point (x, y) of the two dimensional Euclidean plane then the above distance formula takes the form

$d(z, w) = \sqrt{(x - u)^2 + (y - v)^2}$ where $z = x + iy$ and $w = u + iv$.

This is nothing but the usual distance between the points (x, y) and (u, v) in the plane.

Example 3. On any non-empty set M we define d as follows

$$d(x, y) = \begin{cases} 0 \text{ if } & x = y \\ 1 \text{ if } & x \neq y . \end{cases}$$

Then d is a metric on M. This is called the **discrete metric** on M.

Proof. Clearly $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.

Also $d(x, y) = d(y, x) = \begin{cases} 0 \text{ if } & x = y \\ 1 \text{ if } & x \neq y . \end{cases}$

$\therefore$ $d(x, y) = d(y, x)$ for all $x, y \in M$.

Now let $x, y, z \in M$.

Case (i) $x = z$

Then $d(x, z) = 0$.

Also, $d(x, y) + d(y, z) \geq 0$.

$\therefore$ $d(x, z) \leq d(x, y) + d(y, z)$

Case (ii) $x \neq z$

Then $d(x, z) = 1$

Also, since x, z are distinct, y can not be equal to both x and z.

Hence either $y \neq x$ or $y \neq z$.

$\therefore$ $d(x, y) + d(y, z) \geq 1$.

$\therefore$ $d(x, z) \leq d(x, y) + d(y, z)$.

Thus $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

Hence d is a metric on M.

Example 4. In $\mathbf{R}^n$ we define $d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}$

where $x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots\ldots y_n)$.

Then d is a metric on $\mathbf{R}^n$. This is called the **usual metric on** $\mathbf{R}^n$.

Proof. $d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} \geq 0$.

$$d(x, y) = 0 \Leftrightarrow \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} = 0.$$

$$\Leftrightarrow (x_i, - y_i)^2 = 0 \text{ for all } i = 1, 2, \ldots\ldots, n.$$

$$\Leftrightarrow x_i = y_i \text{ for all } i = 1, 2, \ldots, n.$$

$$\Leftrightarrow (x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots\ldots, y_n)$$

$$\Leftrightarrow x = y.$$

Also, $d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$

$$= \left[\sum_{i=1}^{n} (y_i - x_i)^2\right]^{1/2}$$

$$= d(y, x).$$

To prove the triangle inequality, take

$a_i = x_i - y_i$ $b_i = y_i - z_i$ and $p = 2$ in Minkowski's inequality.

We get,

$$\left[\sum_{i=1}^{n} (x_i - z_i)^2\right]^{1/2} \leq \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} + \left[\sum_{i=1}^{n} (y_i - z_i)^2\right]^{1/2}$$

i.e., $d(x, z) \leq d(x, y) + d(y, z)$.

$\therefore$ d is a metric on $\mathbf{R}^n$.

Example 5. Consider $\mathbf{R}^n$. Let $p > 1$.

We define $d(x, y) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$ $x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots\ldots, y_n)$. Then d is a metric on $\mathbf{R}^n$.

The proof is similar to that of example 4.

Example 6. Let $x, y \in \mathbf{R}^2$. Then $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in \mathbf{R}$. We define $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Then d is a metric on $\mathbf{R}^2$.

Proof. $d(x, y) = |x_1 - y_1| + |x_2 - y_2| \geq 0$.

$$\begin{aligned} d(x, y) = 0 &\Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0 \\ &\Leftrightarrow |x_1 - y_1| = 0 \text{ and } |x_2 - y_2| = 0 \\ &\Leftrightarrow x_1 = y_1 \text{ and } x_2 = y_2 \\ &\Leftrightarrow (x_1, x_2) = (y_1, y_2) \\ &\Leftrightarrow x = y. \end{aligned}$$

$$\begin{aligned} d(x, y) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |y_1 - x_1| + |y_2 - x_2| \\ &= d(y, x). \end{aligned}$$

Let $x, y, z \in \mathbf{R}^2$.

$$\begin{aligned} d(x, z) &= |x_1 - z_1| + |x_2 - z_2| \\ &= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2| \\ &\leq \{|x_1 - y_1| + |y_1 - z_1|\} + \{|x_2 - y_2| + |y_2 - z_2|\} \\ &= \{|x_1 - y_1| + |x_2 - y_2|\} + \{|y_1 - z_1| + |y_2 - z_2|\} \end{aligned}$$

$$= d(x, y) + d(y, z)$$

Thus, $d(x, z) \le d(x, y) + d(y, z)$.

Hence, d is a metric on $\mathbf{R}^2$.

Note. More generally in $\mathbf{R}^n$ we define $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ where $x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots\ldots, y_n)$.

Then d is a metric on $\mathbf{R}^n$.

The proof is straight forward and is left to the reader.

Example 7. In $\mathbf{R}^n$ we define

$d(x, y) = \max\{|x_i - y_i|, i = 1, 2, \ldots\ldots, n\}$ where

$x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then d is a metric on $\mathbf{R}^n$.

Proof. $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots\ldots, |x_n - y_n|\} \ge 0$.

$$d(x, y) = 0 \Leftrightarrow \max\{|x_1 - y_1|, \ldots\ldots, |x_n - y_n|\} = 0$$

$$\Leftrightarrow x_i - y_i = 0 \text{ for all } i = 1, 2, \ldots, n$$

$$\Leftrightarrow x_i = y_i \quad \text{for all } i = 1, 2, \ldots, n$$

$$\Leftrightarrow (x_1, x_2, \ldots\ldots, x_n) = (y_1, y_2, \ldots\ldots, y_n)$$

$$\Leftrightarrow x = y$$

Also, $d(x, y) = \max\{|x_i - y_i|\}$

$$= \max\{|y_i - x_i|\}$$

$$= d(y, x).$$

Now, let $x, y, z \in \mathbf{R}^n$. Since each $x_i, y_i, z_i \in \mathbf{R}$

we have, $|x_i - z_i| \le |x_i - y_i| + |y_i - z_i|$ for all $i = 1, 2, \ldots, n$.

$\therefore \max |x_i - z_i| \le \max |x_i - y_i| + \max |y_i - z_i|$

$\therefore\ d(x, z) \leq d(x, y) + d(y, z)$.

Hence d is a metric on $\mathbf{R^n}$.

Example 8. Let $c_1, c_2, \ldots\ldots, c_n$ be given fixed positive real numbers.

Let $x, y \in \mathbf{R^n}$ where $x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

We define $d(x, y) = \sum_{i=1}^{n} c_i \,|x_i - y_i|$

Then d is a metric on $\mathbf{R^n}$. (Prove)

Note. A non-empty set M can be provided with different metrics.

For example, $\mathbf{R^n}$ has been provided with five different metrics as seen from the examples 4 to 8.

Example 9. Let $p \geq 1$. Let l_p denote the set of all sequences (x_n) such that $\sum_{1}^{\infty} |x_n|^p$ is convergent. Define $d(x, y) = \left[\sum_{n=1}^{\infty} |x_n - y_n|^p\right]^{1/p}$ where $x = (x_n)$ and $y = (y_n)$.

Then d is a metric on l_p.

Proof. Let $a, b \in l_p$.

First we prove $d(a, b)$ is a real number.

By Minkowski's inequality we have

$$\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p} \quad \ldots\ldots\ldots(1)$$

Since $a, b \in l_p$ the right hand side of (1) has a finite limit as $n \to \infty$.

$$\therefore \left(\sum_{i=1}^{\infty} |a_i + b_i|^p\right)^{1/p} \text{ is a convergent series.}$$

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Similarly we can prove that $\left(\sum_{i=1}^{\infty} |a_i - b_i|^p\right)^{1/p}$ is also a convergent series and hence $d(a, b)$ is a real number.

Now, taking limit as $n \to \infty$ in (1) we get

$$\left(\sum_{i=1}^{\infty} |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |b_i|^p\right)^{1/p} \quad \text{.......(2)}$$

Obviously $d(x, y) \geq 0$,

$d(x, y) = 0$ iff $x = y$

and $d(x, y) = d(y, x)$.

Now, let $x, y, z \in l_p$.

Taking, $a_i = x_i - y_i$ and $b_i = y_i - z_i$. in (2) we get

$$\left(\sum_{i=1}^{\infty} |x_i - z_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i - z_i|^p\right)^{1/p}$$

$\therefore\ d(x, z) \leq d(x, y) + d(y, z)$.

Hence d is a metric on l_p.

Note. In particular, l_2 is a metric space with the metric defined by

$$d(x, y) = \sum_{n=1}^{\infty} \{|x_n - y_n|^2\}^{1/2}$$

Example 10. Let M be the set of all bounded real valued functions defined on a non-empty set E. Define $d(f, g) = \sup\{|f(x) - g(x)| / x \in E\}$.

d is a metric on M.

Proof. $d(f, g) = \sup\{|f(x) - g(x)|\} \geq 0$.

Also, $d(f, g) = 0 \Leftrightarrow \sup\{|f(x) - g(x)|\} \geq 0$

$$\Leftrightarrow |f(x) - g(x)| = 0 \text{ for all } x \in E$$
$$\Leftrightarrow f(x) = g(x) \text{ for all } x \in E$$
$$\Leftrightarrow f = g.$$

Also, $d(f, g) = \sup \{ |f(x) - g(x)| \}$

$= \sup \{ |g(x) - f(x)| \}$

$= d(g, f)$.

Now, let $f, g, h \in M$.

We have $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$

$\therefore \sup \{|f(x) - h(x)|\} \leq \sup \{|f(x) - g(x)|\} + \sup \{|g(x) - h(x)|\}$

$\therefore d(f, h) \leq d(f, g) + d(g, h)$.

Hence d is a metric on M.

Example 11. Let M be the set of all sequences in $\mathbf{R}$.

Let $x, y \in M$ and let $x = (x_n)$ and $y = (y_n)$.

Define $$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)}$$

Then d is a metric on M.

Proof. Let $x, y \in M$. First we prove that $d(x, y)$ is a real number ≥ 0.

We have $$\frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)} \leq \frac{1}{2^n} \text{ for all } n.$$

Also, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series.

$\therefore \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)}$ is a convergent series.

(by comparison test)

$\therefore d(x, y)$ is a real number and $d(x, y) \geq 0$.

Now, $d(x, y) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)} = 0$

$$\Leftrightarrow |x_n - y_n| = 0 \text{ for all } n$$

$$\Leftrightarrow x_n = y_n \text{ for all } n$$

$$\Leftrightarrow x = y.$$

Also, $d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)}$

$$= \sum_{n=1}^{\infty} \frac{|y_n - x_n|}{2^n(1 + |y_n - x_n|)}$$

$$= d(y, x).$$

Now, let $x, y, z \in M$. Then

$$\frac{|x_n - z_n|}{1 + |x_n - z_n|} = 1 - \frac{1}{1 + |x_n - z_n|} \leq 1 - \frac{1}{(1 + |x_n - y_n| + |y_n - z_n|)}$$

$$= \frac{|x_n - y_n| + |y_n - z_n|}{1 + |x_n - y_n| + |y_n - z_n|}$$

$$= \frac{|x_n - y_n|}{1 + |x_n - y_n| + |y_n - z_n|} + \frac{|y_n - z_n|}{1 + |x_n - y_n| + |y_n - z_n|}$$

$$\leq \frac{|x_n - y_n|}{1 + |x_n - y_n|} + \frac{|y_n - z_n|}{1 + |y_n - z_n|}.$$

Multiplying both sides of this inequality by $\frac{1}{2^n}$ and taking the sum from $n = 1$ to ∞ we get $d(x, z) \leq d(x, y) + d(y, z)$.

$\therefore$ d is a metric on M.

Example 12. Let l^{∞} denote the set of all bounded sequences of real numbers. Let $x = (x_n)$ and $y = (y_n) \in l^{\infty}$ define d on l^{∞} as

$$d'(x\,,\ y) = lub\,|x_n - y_n|$$

Then d is a metric on l^{∞}.

Solution. $d(x\,,\ y) = lub\,|x_n - y_n| \geq 0$

$$\begin{aligned} d(x\,,\ y) = 0 &\Leftrightarrow lub\,|x_n - y_n| = 0 \\ &\Leftrightarrow |x_n - y_n| = 0 \text{ for } 1 \leq n < \infty \\ &\Leftrightarrow x_n = y_n \text{ for } 1 \leq n < \infty \\ &\Leftrightarrow (x_{n)} = (y_n) \\ &\Leftrightarrow x = y\,. \end{aligned}$$

$$\begin{aligned} \text{Now, } d(x\,,\ y) &= lub\,|x_n - y_n| \\ &= lub\,|y_n - x_n| \\ &= d(y\,,\ x). \end{aligned}$$

Let $z = (z_n)$.

$$\begin{aligned} \text{Now, } \quad |x_n - z_n| &\leq |x_n - y_n| + |y_n - z_n| \\ &\leq lub\,|x_n - y_n| + lub\,|y_n - z_n| \\ &= d(x\,,\ y) + d(y\,,\ z)\,. \end{aligned}$$

$\therefore\ lub\,|x_n - z_n| \leq d(x\,,\ y) + d(y\,,\ z)$

$\therefore\ d(x\,,\ z) \leq d(x\,,\ y) + d(y\,,\ z)$.

$\therefore\ d$ is a metric on l^{∞}.

Solution. $d(x, y) = d_1(x, y) + d_2(x, y) \geq 0$.

$$d(x, y) = 0 \Leftrightarrow d_1(x, y) + d_2(x, y) = 0.$$

$$\Leftrightarrow d_1(x, y) = 0 \text{ and } d_2(x, y) = 0$$

$$\Leftrightarrow x = y$$

Now, $d(x, y) = d_1(x, y) + d_2(x, y)$

$= d_1(y, x) + d_2(y, x)$

$= d(y, x)$.

Let $x, y, z \in M$. Then we have

$d_1(x, z) \leq d_1(x, y) + d_1(y, z)$ and

$d_2(x, z) \leq d_2(x, y) + d_2(y, z)$

Adding, we get $d(x, z) \leq d(x, y) + d(y, z)$

$\therefore$ d is a metric on M.

Problem 2. Determine whether $d(x, y)$ defined on **R** by $d(x, y) = (x - y)^2$ is a metric or not.

Solution. Let $x, y \in \mathbf{R}$.

$d(x, y) = (x - y)^2 \geq 0$.

$d(x, y) = (x - y)^2 = (y - x)^2$

$= d(y, x)$.

But triangle inequality does not hold.

Take $x = -5$, $y = -4$ and $z = 4$

Then $d(x, y) = (-5 + 4)^2 = 1$

$d(y, z) = (-4 - 4)^2 = 64$

$d(x, z) = (4 + 5)^2 = 81$.

Here $d(x, z) > d(x, y) + d(y, z)$

Hence triangle inequality does not hold.

$\therefore$ d is not a metric on $\mathbf{R}$.

Problem 3. If d is a metric on M, is d^2 a metric on M?

Solution. Consider $d(x, y)$ defined on $\mathbf{R}$ by $d(x, y) = |x - y|$.

We know that d is a metric on $\mathbf{R}$ (refer example 1).

$d^2(x, y) = |x - y|^2 = (x - y)^2$.

But d^2 is not a metric (refer solved problem 2).

Problem 4. If d is a metric on M, prove that $\sqrt{d}$ is a metric on M.

Solution. Let $x, y, z \in M$.

$\therefore$ We have $\sqrt{d(x, y)} \geq 0$ (since $d(x, y) \geq 0$)

Also, $\sqrt{d(x, y)} = \sqrt{d(y, x)}$

Now, $d(x, z) \leq d(x, y) + d(y, z)$

$\therefore$ $\sqrt{d(x, z)} \leq \sqrt{d(x, y) + d(y, z)}$

$\leq \sqrt{d(x, y)} + \sqrt{d(y, z)}$ (since $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$)

Hence $\sqrt{d}$ is a metric on M.

Problem 5. Let (M, d) be a metric space. Define

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$ Prove that d_1 is a metric on M.

Solution $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)} \geq 0$. (since $d(x, y) \geq 0$)

$$d_1(x, y) = 0 \Leftrightarrow \frac{d(x, y)}{1 + d(x, y)} = 0$$

$$\Leftrightarrow d(x, y) = 0$$

$$\Leftrightarrow x = y. \text{ (since } d \text{ is a metric)}$$

Also, $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$

$$= \frac{d(y, x)}{1 + d(y, x)}$$

$$= d_1(y, x).$$

Now, let $x, y, z \in M$.

$$\text{Then } d_1(x, z) = \frac{d(x, z)}{1 + d(x, z)}$$

$$= 1 - \frac{1}{1 + d(x, z)}$$

$$\leq 1 - \left[\frac{1}{1 + d(x, y) + d(y, z)}\right]$$

$$= \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)}$$

$$= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)}$$

$$\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)}$$

$$= d_1(x, y) + d_1(y, z).$$

Thus $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$.

$\therefore$ d_1 is a metric on M.

Problem 6. Let (M, d) be a metric space. Define $d_1(x, y) = \min\{1, d(x, y)\}$. Prove that d_1 is a metric on M.

Solution. $d_1(x, y) = \min\{1, d(x, y)\} \geq 0$.

$\therefore$ $d_1(x, y) \geq 0$.

$$d_1(x, y) = 0 \Leftrightarrow \min\{1, d(x, y)\} = 0$$

$$\Leftrightarrow d(x, y) = 0.$$

$$\Leftrightarrow x = y.$$

Also $d_1(x, y) = \min\{1, d(x, y)\}$

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$= \min \{1, d(y, x)\}$

$= d_1(y, x).$

Now, let $x, y, z \in M$.

Then $d_1(x, z) = \min \{1, d(x, z)\} \leq 1$.

To prove $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$.

If $d_1(x, y) = 1$ or $d_1(y, z) = 1$ the inequality is obvious.

Let $d_1(x, y) < 1$ and $d_1(y, z) < 1$. Then

$$
\begin{aligned}
d_1(x, y) + d_1(y, z) &= \min \{1, d(x, y)\} + \min \{1, d(y, z)\} \\
&= d(x, y) + d(y, z) \\
&\geq d(x, z) \\
&\geq \min \{1, d(x, z)\}. \\
&= d_1(x, z).
\end{aligned}
$$

Thus $d_1(x, y) + d_1(y, z) \geq d_1(x, z)$

$\therefore$ d_1 is a metric on M.

Problem 7. Let M be a non empty set., Let $d : M \times M \to \mathbf{R}$ be a function such that (i) $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in M$.

Prove that d is a metric on M.

Solution. Put $y = x$ in (ii).

We have $d(x, x) \leq d(x, z) + d(x, z)$.

$\therefore$ $0 \leq 2\, d(x, z)$ (by (i))

$\therefore$ $d(x, z) \geq 0$.

Now, to prove $d(x, y) = d(y, x)$.

Putting $z = x$ in (ii) we get $d(x, y) \leq d(x, x) + d(y, x)$

(i.e.) $d(x, y) \leq d(y, x)$ [using (i)]

Since this is true for all $x, y \in M$ we have $d(y, x) \leq d(x, y)$.

Hence $d(x, y) = d(y, x)$.

Now (ii) can be written as $d(x, y) \leq d(x, z) + d(z, y)$ which is the triangle inequality.

$\therefore$ d is a metric on M.

Problem 8. If (M_1, d_1), (M_2, d_2), ..., (M_n, d_n) are metric spaces then $M_1 \times M_2 \times \ldots\ldots\ldots \times M_n$ is a metric space with metric d defined by

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \text{ where } x = (x_1, x_2, \ldots, x_n)\,;\ y = (y_1, y_2, \ldots, y_n).$$

Solution. $d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \geq 0.$

$$\begin{aligned}
\text{Also } d(x, y) = 0 &\Leftrightarrow \sum_{i=1}^{n} d_i(x_i, y_i) = 0 \\
&\Leftrightarrow d_i(x_i, y_i) = 0 \text{ for all } i = 1, 2, \ldots, n \\
&\Leftrightarrow x_i = y_i \text{ for all } i = 1, 2, \ldots, n \\
&\Leftrightarrow (x_1, \ldots, x_n) = (y_1, \ldots, y_n) \\
&\Leftrightarrow x = y.
\end{aligned}$$

$$\begin{aligned}
\text{Also } d(x, y) &= \sum_{i=1}^{n} d_i(x_i, y_i) \\
&= \sum_{i=1}^{n} d_i(y_i, x_i) \\
&= d(y, x).
\end{aligned}$$

Now, let $x, y, z \in M$.

Then $d(x, z) = \sum_{i=1}^{n} d_i(x_i, z_i)$

$$\leq \sum_{i=1}^{n} \left[d_i(x_i, y_i) + d_i(y_i, z_i) \right]$$

$$= \sum_{i=1}^{n} d_i(x_i, y_i) + \sum_{i=1}^{n} d_i(y_i, z_i)$$

$$= d(x, y) + d(y, z).$$

$$\therefore\ d(x, z) \leq d(x, y) + d(y, z).$$

Hence d is a metric on M.

Problem 9. In a metric space (M, d) prove that

$$|d(x, z) - d(y, z)| \leq d(x, y) \text{ for all } x, y, z \in M.$$

Solution. Let $x, y, z \in M$.

We have $d(x, z) \leq d(x, y) + d(y, z)$

$\therefore\ d(x, z) - d(y, z) \leq d(x, y)$. .... (1)

Interchanging x and y in (1) we get

$$d(y, z) - d(x, z) \leq d(y, x)$$
$$= d(x, y)$$

$\therefore\ d(y, z) - d(x, z) \leq d(x, y)$. ....(2)

From (1) and (2) we get $|d(x, z) - d(y, z)| \leq d(x, y)$.

Exercises

1. Let $M = \{a, b, c\}$ We define d on M as follows.

$d(a, b) = d(b, a) = 3 : d(b, c) = d(c, b) = 4$

$d(c, a) = d(a, c) = 5$ and $d(a, a) = d(b, b) = d(c, c) = 0$.

Prove that d is a metric on M.

2. If d is a metric on M prove that

(i) $2d$ is a metric on M.

(ii) nd is a metric on M where $n \in \mathbf{N}$.

2.2 BOUNDED SETS IN A METRIC SPACE

Definition. Let (M, d) be a metric space. We say that a subset A of M is **bounded** if there exists a positive real number k such that

$$d(x, y) \leq k \text{ for all } x, y \in A.$$

Example 1. Any finite subset A of a metric space (M, d) is bounded.

Proof. Let A be any finite subset of M.

If $A = \Phi$ then A is obviously bounded.

Let $A \neq \Phi$. Then $\{ d(x, y) / x, y \in A \}$ is a finite set of real numbers.

Let $k = \max \{ d(x, y) / x, y \in A \}$.

Clearly $d(x, y) \leq k$ for all $x, y \in A$.

$\therefore$ A is bounded.

Example 2. $[0,1]$ is a bounded subset of **R** with usual metric since $d(x, y) \leq 1$ for all $x, y \in [0, 1]$.

More generally any finite interval and any subset of **R** which is contained in a finite interval are bounded subsets of **R**.

Example 3. $(0, \infty)$ is an unbounded subset of **R**.

Example 4. If we consider **R** with discrete metric, then $(0, \infty)$ is a bounded subset of **R**, since $d(x, y) \leq 1$ for all $x, y \in (0, \infty)$.

More generally any subset of a discrete metric space M is a bounded subset of M.

Example 5. In l_2 let $e_1 = (1, 0, \ldots\ldots, 0, \ldots\ldots), e_2 = (0, 1, 0, \ldots., 0 \ldots..)$,

$$e_3 = (0, 0, 1, \ldots.. 0, \ldots\ldots), \ldots\ldots\ldots\ldots\ldots$$

Let $A = \{e_1, e_2, \ldots., e_n, \ldots..\}$.

Then A is a bounded subset of l_2.

Proof. $d(e_n, e_m) = \begin{cases} \sqrt{2} & \text{if } n \neq m \\ 0 & \text{if } n = m. \end{cases}$

$\therefore\ d(e_n, e_m) \leq \sqrt{2}$ for all $e_n, e_m \in A$.

$\therefore\ A$ is abounded set in l_2.

Example 6. Let (M, d) be a metric space. Define $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$.

We know that (M, d_1) is also a metric space.

Also $d_1(x, y) < 1$ for all $x, y \in M$.

Hence (M, d_1) is a bounded metric space.

Definition. Let (M, d) be a metric space. Let $A \subseteq M$ Then the diameter of A, denoted by $d(A)$, is defined by $d(A) =$ l.u.b $\{d(x, y) / x, y \in A\}$.

Note 1. A non-empty set A is a bounded set iff $d(A)$ is finite.

Note 2. Let $A, B \subseteq M$. Then $A \subseteq B \Rightarrow d(A) \leq d(B)$.

Example 1. The diameter of any non-empty subset in a discrete metric space is 1.

Example 2. In **R** the diameter of any interval is equal to the length of the interval. For example the diameter of $[0, 1]$ is 1.

Example 3. In any metric space, $d(\Phi) = -\infty$.

Exercises.

1. Let (M, d) be a metric space. Define $d_1(x, y) = \min \{d(x, y), 1\}$

Prove that (M, d_1) is a bounded metric space.

2. Let (M, d) be a bounded metric space. Define $d_1(x, y) = 2\, d(x, y)$.

Prove that (M, d_1) is a bounded metric space.

3. Prove that in a metric space any subset of a bounded set is bounded.

4. Find the diameter of the following subset of **R** with usual metric.

(i) $\{1, 3, 5, 7, 9\}$ (ii) $\{0, 1, 2, 3, \ldots, 100\}$

(iii) $[-3, 5]$ (iv) $[-\frac{1}{2}, \frac{1}{2}]$

(v) **N**. (vi) **Q**.

(vii) $[1, 2] \cup [5, 6]$ (viii) $[3, 6] \cap [4, 8]$

2.3 OPEN BALL (OPEN SPHERE) IN A METRIC SPACE

Definition. Let (M, d) be a metric space. Let $a \in M$ and r be a positive real number. Then the **open ball** or the **open sphere** with centre a and radius r denoted by $B_d(a, r)$ is the subset of M given by

$$B_d(a, r) = \{x \in M / d(a, x) < r\}$$

When the metric d under consideration is clear we write $B(a, r)$ instead of $B_d(a, r)$.

Note 1. $B(a, r)$ is always non-empty since it contains at least its centre a.

Note 2. $B(a, r)$ is a bounded set.

For, let $x, y \in B(a, r)$.

$\therefore\ d(a, x) < r$ and $d(a, y) < r$

$\therefore\ d(x, y) \leq d(x, a) + d(a, y) < r + r = 2r$

Thus $d(x, y) < 2r$. Hence $B(a, r)$ is bounded.

Example 1. Consider **R** with usual metric. Let $a \in$ **R**.

$$\begin{aligned} \text{Then } B(a, r) &= \{x \in \mathbf{R} / d(a, x) < r\} \\ &= \{x \in \mathbf{R} / |a - x| < r\} \\ &= \{x \in \mathbf{R} / a - r < x < a + r\} \\ &= (a - r, a + r). \end{aligned}$$

Example 2. Consider **C** with usual metric. Let $a \in \mathbf{C}$.

Then $B(a, r) = \{z \in \mathbf{C} / d(a, z) < r\}$

$= \{z \in \mathbf{C} / |z - a| < r\}$.

This is the interior of the circle with centre a and radius r.

Example 3. In $\mathbf{R}^2$ with usual metric $B(a, r)$ is the interior of the circle with centre a and radius r.

Example 4. Let d be the discrete metric on M.

$$\text{Then } B(a, r) = \begin{cases} M & \text{if } r > 1 \\ \{a\} & \text{if } r \leq 1 \end{cases}$$

Proof. We have $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Let $a \in M$. Let r be any positive real number.

Case (i) Let $r > 1$. Then $B(a, r) = \{x \in M / d(a, x) < r\}$.

Clearly every point $x \in M$ is such that $d(a, x) < r$.

Hence $B(a, r) = M$.

Case (ii) Let $r \leq 1$. In this case for any point $x \neq a$, $d(a, x) = 1 \geq r$.

Hence $x \notin B(a, r)$ so that $B(a, r) = \{a\}$

$$\therefore\ B(a, r) = \begin{cases} M & \text{if } r > 1 \\ \{a\} & \text{if } r \leq 1 \end{cases}$$

Example 5. Consider $M = [0, 1]$ with usual metric $d(x, y) = |x - y|$.

Here $B(0, 1/2) = \{x \in [0, 1] / d(0, x) < 1/2\}$

$= \{x \in [0, 1] / |x| < 1/2\}$

$= [0, 1/2)$.

Example 6. Consider $\mathbf{R}^2$ with the metric d given by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

Then $B((0, 0), 1) = \{(x, y) \in \mathbf{R}^2 / |x - 0| + |y - 0| < 1\}$

2.4 OPEN SETS

Definition. Let (M, d) be a metric space. Let A be a subset of M. Then A i said to be open in M if for every $x \in A$ there exists a positive real numbe r such that $B(x, r) \subseteq A$.

Example 1. In **R** with usual metric $(0, 1)$ is an open set.

Proof. Let $x \in (0, 1)$.

Choose $r = \min\{x - 0, 1 - x\} = \min\{x, 1 - x\}$.

Clearly $r > 0$ and $B(x, r) = (x - r, x + r) \subseteq (0, 1)$.

$\therefore$ $(0, 1)$ is open.

Example 2. In **R** with usual metric $[0, 1)$ is not open since no open ball wi centre 0 is contained $[0, 1)$.

Example 3. Consider $M = [0,2)$ with usual metric.

Let $A = [0, 1) \subseteq M$. Then A is open in M.

Proof. Let $x \in [0,1)$

If $x = 0$ then $B(0,\frac{1}{2}) = [0,\frac{1}{2}) \subseteq A$.

If $x \neq 0$ choose $r = \min\{x, 1 - x\}$.

Clearly $r > 0$ and $B(x, r) = (x - r, x + r) \subseteq [0,1)$.

$\therefore$ A is open in M.

Example 4. Any open interval (a, b) is an open set in **R** with usual metric

Proof. Let $x \in (a, b)$.

Let $r = \min\{x - a, b - x\}$

Then $B(x, r) \subseteq (a, b)$. Hence (a, b) is an open set.

Note. Similarly we can prove that $(-\infty, a)$ and (a, ∞) are open sets.

Example 5. In **R** with usual metric the set $\{0\}$ is not an open set since, a open ball with centre 0 is not contained in $\{0\}$.

Theorem 2.2. In any metric space (M, d) each open ball is an open set.

Proof. Let $B(a, r)$ be an open ball in M.

Let $x \in B(a, r)$.

Then $d(a, x) < r$.

$\therefore\ r - d(a, x) > 0$.

Let $r_1 = r - d(a, x)$.

We claim that $B(x, r_1) \subseteq B(a, r)$.

Let $y \in B(x, r_1)$

$\therefore\ d(x, y) < r_1 = r - d(a, x)$.

$\therefore\ d(x, y) + d(a, x) < r$ .... (1)

Now, $d(a, y) \leq d(a, x) + d(x, y) < r$ (by (1)).

$\therefore\ d(a, y) < r$.

$\therefore\ y \in B(a, r)$

Hence $B(x, r_1) \subseteq B(a, r)$

$\therefore\ B(a, r)$ is an open set.

Theorem 2.3. In any metric space the union of *any family* of open sets
open.

Proof. Let (M, d) be a metric space.

Let $\{A_i / i \in I\}$ be a family of open sets in M.

Let $A = \bigcup_{i \in I} A_i$

If $A = \Phi$ then A is open.

$\therefore$ Let $A \neq \Phi$. Let $x \in A$.

Then $x \in A_i$ for some $i \in I$.

Since A_i is open there exists an open ball $B(x, r)$ such that $B(x, r) \subseteq A$

$\therefore\ B(x, r) \subseteq A$.

Hence A is open.

Theorem 2.4. In any metric space the intersection of a *finite* number of open sets is open.

Proof. Let (M, d) be a metric space.

Let $A_1, A_2, \ldots, A_n$ be open sets in M.

Let $A = A_1 \cap A_2 \cap \ldots \cap A_n$.

If $A = \Phi$ then A is open.

$\therefore$ Let $A \neq \Phi$. Let $x \in A$.

$\therefore\ x \in A_i$ for each $i = 1, 2, \ldots, n$.

Since each A_i is an open set there is a positive real number r_i such that $B(x, r_i) \subseteq A_i$ .... (1)

Let $r = \min\{r_1, r_2, \ldots, r_n\}$

Obviously r is a positive real number and $B(x, r) \subseteq B(x, r_i)$ for all $i = 1, 2, \ldots, n$

Hence $B(x, r) \subseteq A_i$ for all $i = 1, 2, \ldots, n$. (by 1)

$$\therefore\ B(x, r) \subseteq \bigcap_{i=1}^{n} A_i .$$

$\therefore\ B(x, r) \subseteq A$.

$\therefore\ A$ is open.

Note. The intersection of an *infinite* number of open sets in a metric space need not be open.

For example, consider **R** with usual metric.

Let $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$.

Let $z \in B(x, \frac{1}{4}r) \cap B(y, \frac{1}{4}r)$.

$\therefore\ z \in B(x, \frac{1}{4}r)$ and $z \in B(y, \frac{1}{4}r)$.

$\therefore\ d(x, z) < \frac{1}{4}r$ and $d(y, z) < \frac{1}{4}r$.

Now, $d(x, y) \leq d(x, z) + d(z, y)$.

$\therefore\ r \leq \frac{1}{4}r + \frac{1}{4}r = \frac{1}{2}r$ which is a contradiction.

Hence $B(x, \frac{1}{4}r) \cap B(y, \frac{1}{4}r) = \Phi$.

Problem 2. Let (M, d) be a metric space. Let $x \in M$. Show that $\{x\}^c$ is open.

Solution. Let $y \in \{x\}^c$. Then $y \neq x$.

$\therefore\ d(x, y) = r > 0$.

Clearly $B(y, \frac{1}{2}r) \subseteq \{x\}^c$.

$\therefore\ \{x\}^c$ is open.

Problem 3. Let (M, d) be a metric space. Show that every subset of M is open iff $\{x\}$ is open for all $x \in M$.

Solution. Suppose every subset of M is open.

Then obviously $\{x\}$ is open for all $x \in M$.

Conversely let $\{x\}$ be open for all $x \in M$.

Let A be any subset of M.

If $A = \Phi$ then A is open.

Let $A \neq \Phi$. Then $A = \bigcup_{x \in A} \{x\}$.

By hypothesis $\{x\}$ is open.

Hence A is open.

Problem 4. Let $A = \left\{(a_n) \,/\, (a_n) \in l_2 \text{ and } \left[\sum_{n=1}^{\infty} a_n{}^2\right]^{1/2} < 1\right\}$

Prove that A is an open subset of l_2.

Solution. We first prove that $A = B(\mathbf{0}, 1)$ where $\mathbf{0} = (0, 0, 0, \ldots.)$

Let $x \in A$. Hence $\sum_{n=1}^{\infty} x_n^2 < 1$.

$$\therefore \; d(x, \mathbf{0}) = \left[\sum_{n=1}^{\infty} (x_n - 0)^2\right]^{1/2} = \left[\sum_{n=1}^{\infty} x_n^2\right]^{1/2} < 1$$

Thus $d(x, \mathbf{0}) < 1$.

$\therefore \; x \in B(\mathbf{0}, 1)$

$\therefore \; A \subseteq B(\mathbf{0}, 1)$ .... (1)

Now, let $y \in B(0, 1)$

$\therefore \; d(\mathbf{0}, y) < 1$.

$$\therefore \; \left[\sum_{n=1}^{\infty} (y_n - 0)^2\right]^{1/2} < 1$$

$$\therefore \; \left[\sum_{n=1}^{\infty} y_n{}^2\right]^{1/2} < 1.$$

$\therefore \; y \in A$.

$\therefore \; B(\mathbf{0}, 1) \subseteq A$. .... (2)

By (1) and (2) we get $A = B(\mathbf{0}, 1)$ .... (3)

Now, the open ball $B(\mathbf{0}, 1)$ is an open set. (By theorem 2.2)

$\therefore \; A$ is an open set.

Problem 5. Prove that any open subset of **R** can be expressed as the union of a countable number of mutually disjoint open intervals.

Solution. Let A be an open subset of **R**. Let $x \in A$. Then there exists a positive real number r such that $B(x, r) = (x - r, x + r) \subseteq A$.

Thus there exists an open interval I such that $x \in I$ and $I \subseteq A$

Let I_x denote the largest open interval such that $x \in I$ and $I_x \subseteq A$

Clearly $\bigcup_{x \in A} I_x = A$.

Now let $x, y \in A$.

We claim that $I_x = I_y$ or $I_x \cap I_y = \Phi$

Supose $I_x \cap I_y \neq \Phi$

Then $I_x \cup I_y$ is an open interval contained in A.

But I_x is the largest open interval such that $x \in I_x$ and $I_x \subseteq A$.

$\therefore$ $I_x \cup I_y = I_x$ so that $I_y \subseteq I_x$

Similarly $I_x \subseteq I_y$.

$\therefore$ $I_x = I_y$. Thus the intervals I_x are mutually disjoint.

We claim that the set $F = \{I_x / x \in A\}$ is countable.

Now for each $I_x \in F$ choose a rational number $r_x \in I_x$.

Since the intervals I_x are mutually disjoint $I_x \neq I_y \Rightarrow r_x \neq r_y$.

$\therefore$ $f : F \to Q$ defined by $f(I_x) = r_x$ is 1-1.

$\therefore$ F is equivalent to a sub set of **Q** which is countable.

$\therefore$ F is countable.

Equivalent metrics.

Definition. Let d and ρ be the two metrics on M. Then the metrics d and ρ are said to be **equivalent** if the open sets of (M, ρ) are the open sets of (M, d) and conversly.

Problem 6. Let (M, d) be a metric space. Define $\rho(x, y) = 2d(x, y)$. Then d and ρ are equivalent metrics.

Solution. We know that ρ is a metric on M.

We first prove that $B_d(a, r) = B_\rho(a, 2r)$

Let $x \in B_d(a, r)$

$\therefore d(a, x) < r$.

$\therefore 2d(a, x) < 2r$.

$\therefore \rho(a, x) < 2r$. Hence $x \in B_\rho(a, 2r)$

$\therefore B_d(a, r) \subseteq B_\rho(a, 2r)$ .... (1)

Now, let $x \in B_\rho(a, 2r)$

$\therefore \rho(a, x) < 2r$.

$\therefore \frac{1}{2}\rho(a, x) < r$.

$\therefore d(a, x) < r$. Hence $x \in B_d(a, r)$.

$\therefore B_\rho(a, 2r) \subseteq B_d(a, r)$ .... (2)

$\therefore$ By (1) and (2) we get $B_d(a, r) = B_\rho(a, 2r)$. .... (3)

Now, let G be any open subset in (M, d). Let $a \in G$. Hence there exists $r > 0$ such that $B_d(a, r) \subseteq G$.

$\therefore B_\rho(a, 2r) \subseteq G$ (using 3)

$\therefore G$ is open in (M, ρ).

Conversly suppose G is open in (M, ρ).

Let $a \in G$. Hence there exists $r > 0$ such that $B_\rho(a, r) \subseteq G$. Hence $B_d(a, \frac{1}{2}r) \subseteq G$ (using 3). Hence G is open in (M, d)

$\therefore d$ and ρ are equivalent metrics.

Problem 7. Let (M, d) be a metric space. Define $\rho(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$. Prove that d and ρ are equivalent metrics on M.

Solution. We know that ρ is a metric on M. (refer problem 5 in 2.1)

We first prove $B_\rho(a, r) = B_d\left(a, \dfrac{r}{1-r}\right)$ provided $0 < r < 1$.

Let $x \in B_\rho(a, r)$. Hence $\rho(a, x) < r$.

$\therefore \dfrac{d(a, x)}{1 + d(a, x)} < r.$

$\therefore d(a, x) < r[1 + d(a, x)].$

$\therefore d(a, x)[1 - r] < r.$

$\therefore d(a, x) < \dfrac{r}{1-r}$ (since $0 < r < 1$)

$\therefore x \in B_d\left(a, \dfrac{r}{1-r}\right)$

$\therefore B_\rho(a, r) \subseteq B_d\left(a, \dfrac{r}{1-r}\right).$ .... (1)

Now, let $x \in B_d\left(a, \dfrac{r}{1-r}\right)$ Hence $d(a, x) < \dfrac{r}{1-r}$

$\therefore d(a, x)(1 - r) < r.$

$\therefore d(a, x) < r[1 + d(a, x)]$

$\therefore \dfrac{d(a, x)}{1 + d(a, x)} < r.$

$\therefore \rho(a, x) < r.$

$\therefore x \in B_\rho(a, r).$

$\therefore B_d\left(a, \dfrac{r}{1-r}\right) \subseteq B_\rho(a, r)$ .... (2)

$\therefore$ By (1) and (2) we get $B_d\left(a, \frac{r}{1-r}\right) = B_\rho(a, r)$ ..... (3)

Now, let G be open in (M, ρ)

Let $a \in G$. Hence there exists $r > 0$ such that $B_\rho(a, r) \subseteq G$.

Without loss of generality we may assume that $r < 1$.

$\therefore$ $B_d\left(a, \frac{r}{1-r}\right) \subseteq G$ (by (3)).

$\therefore$ G is open in (M, d).

Conversely let G be open in (M, d).

$\therefore$ There exists $r > 0$ such that $B_d(a, r) \subseteq G$.

$\therefore$ $B_\rho\left(a, \frac{r}{1-r}\right) \subseteq G$ (using 3).

$\therefore$ G is open in (M, ρ).

Hence d and ρ are equivalent metrics.

Problem 8. If d and ρ are metrics on M and if there exists $k > 1$ such that $\frac{1}{k}\rho(x, y) \le d(x, y) \le k\rho(x, y)$ for all $x, y \in M$. Prove that d and ρ are equivalent metrics.

Solution. Suppose there exists $k > 1$ such that for all $x, y \in M$

$$\frac{1}{k}\rho(x, y) \le d(x, y) \le k\rho(x, y) \quad \text{..... (1)}$$

Let G be an open set in (M, d).

Let $a \in G$. Hence there exists $r > 0$ such that

$$B_d(a, r) \subseteq G. \quad \text{..... (2)}$$

We now claim that $B_\rho\left(a, \frac{r}{k}\right) \subseteq G$.

Let $x \in B_\rho(a, \frac{r}{k})$.

$\therefore \ \rho(a, x) < \frac{r}{k}$.

$\therefore \ k\rho(a, x) < r$.

$\therefore \ d(a, x) < r$ (using 1)

$\therefore \ x \in B_d(a, r) \subseteq G$ (by 2)

$\therefore \ x \in G$. Hence $B_\rho(a, \frac{r}{k}) \subseteq G$.

$\therefore \ G$ is open in (M, ρ)

Conversly let G be open in (M, ρ). Let $a \in G$.

$\therefore$ There exists $r > 0$ such that $B_\rho(a, r) \subseteq G$. ..... (3)

We claim $B_d(a, \frac{r}{k}) \subseteq G$.

Let $x \in B_d(a, \frac{r}{k})$.

$\therefore \ d(a, x) < \frac{r}{k}$.

$\therefore \ k\,d(a, x) < r$

$\therefore \ \rho(a, x) < r$ (using 1)

$\therefore \ x \in B_\rho(a, r) \subseteq G$ (by 3)

$\therefore \ x \in G$. Hence $B_d(a, \frac{r}{k}) \subseteq G$.

Hence G is open in (M, d).

$\therefore \ d$ and ρ are equivalent metrics.

2.5 SUBSPACE

Definition. Let (M, d) be a metric space. Let M_1 be a non-empty subset of M. Then M_1 is also a metric space with the same metric d. We say that (M_1, d) is a subspace of (M, d).

Note. If M_1 is a subspace of M a set which is open in M_1 need not be open in M.

For example, if $M = \mathbf{R}$ with usual metric and $M_1 = [0, 1]$ then $[0, \frac{1}{2})$ is open in M_1 but not open in M.

We now proceed to investigate the nature of open sets in a subspace M_1 of a metric space M.

Theorem 2.6. Let M be a metric space and M_1 a subspace of M. Let $A_1 \subseteq M_1$. Then A_1 is open in M_1 iff there exists an open set A in M such that $A_1 = A \cap M_1$.

Proof. Let M_1 be a subspace of M. Let $a \in M_1$.

We denote $B_1(a, r)$ the open ball in M_1 with centre a, radius r.

Then $B_1(a, r) = \{x \in M_1 / d(a, x) < r\}$.

Also, $B(a, r) = \{x \in M / d(a, x) < r\}$.

Hence, $B_1(a, r) = B(a, r) \cap M_1$. ..... (1)

Now, let A_1 be an open set in M_1.

$$A_1 = \bigcup_{x \in A_1} B_1(x, r(x)) \qquad \text{(by theorem 2.5.)}$$

$$= \bigcup_{x \in A_1} [B(x, r(x)) \cap M_1] \quad \text{(by (1))}$$

$$= \left[\bigcup_{x \in A_1} B(x, r(x))\right] \cap M_1.$$

$= A \cap M_1$ where $A = \bigcup_{x \in A_1} B(x, r(x))$ which is open M.

Conversely let $A_1 = A \cap M_1$ where A is open in M.

We claim that A_1 is open in M_1.

Let $x \in A_1$.

$\therefore$ $x \in A$ and $x \in M_1$.

Since A is open in M there exists a positive real number r such that $B(x, r) \subseteq A$.

$\therefore$ $M_1 \cap B(x, r) \subseteq M_1 \cap A$.

i.e. $B_1(x, r) \subseteq A_1$ (using (1))

$\therefore$ A_1 is open in M_1.

Example1. Let $M = \mathbf{R}$ and $M_1 = [0, 1]$. Let $A_1 = [0, \frac{1}{2})$

Now $A_1 = [0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \cap [0, 1]$ and $(-\frac{1}{2}, \frac{1}{2})$ is open in $\mathbf{R}$.

$\therefore$ $[0, \frac{1}{2})$ is open in $[0, 1]$.

Example 2. Let $M = \mathbf{R}$ and $M_1 = [1, 2] \cup [3, 4]$.

Let $A_1 = [1, 2]$. Then $A_1 = [1, 2] = (\frac{1}{2}, \frac{5}{2}) \cap M_1$.

$\therefore$ $[1, 2]$ is open in M_1.

Similarly $[3, 4]$ is open in M_1.

Solved Problems

Problem 1. Let M_1 be a subspace of a metric space M. Prove that every open set A_1 of M_1 is open in M iff M_1 itself is open in M.

Solution. Suppose every open set A_1 of M_1 is open in M.

Now, M_1 is open in M_1.

Hence M_1 is open in M.

Conversely, suppose M_1 is open in M.

Let A_1 be an open set in M_1.

Then by theorem 2.6. there exists an open set A in M such that $A_1 = A \cap M_1$.

Since A and M_1 are open in M_1 we get A_1 is open in M.

Exercises.

1. Give an example of a metric space M and a non-empty proper subspace M_1 of M such that every open set in M_1 is also an open set in M.

2. Let M_1 be a subspace of a metric space M. Let $A_1 \subseteq M_1$. If A_1 is open in M prove that it is open in M_1 also.

2.6 INTERIOR OF A SET

Definition. Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in A$. Then x is said to be an **interior point** of A if there exists a positive real number r such that $B(x, r) \subseteq A$.

The set of all interior points of A is called the **interior of** A and it is denoted by Int A.

Note. Int $A \subseteq A$.

Example 1. Consider **R** with usual metric.

(a) Let $A = [0, 1]$. Clearly 0 and 1 are not interior points of A and any point $x \in (0, 1)$ is an interior point of A. Hence Int $A = (0, 1)$.

(b) Let $A = \mathbf{Q}$. Let $x \in \mathbf{Q}$.

Then for any positive real number r, $B(x, r) = (x - r, x + r)$ contains irrational numbers.

$\therefore$ $B(x, r)$ is not a subset of $\mathbf{Q}$.

$\therefore$ x is not an interior point of $\mathbf{Q}$.

Since $x \in \mathbf{Q}$ is arbitrary, no point of $\mathbf{Q}$ is an interior point of $\mathbf{Q}$.

$\therefore$ $\text{Int } \mathbf{Q} = \Phi$

(c) Let A be a finite subset of $\mathbf{R}$. Then $\text{Int } A = \Phi$.

(d) Let $A = \{0, 1, \frac{1}{2}, \dots\dots\dots, \frac{1}{n}, \dots\dots\}$. Then $\text{Int } A = \Phi$.

Example 2. Consider $\mathbf{R}$ with discrete metric.

Let $A = [0, 1]$. Let $x \in [0, 1]$.

Then $B(x, \frac{1}{2}) = \{x\} \subseteq A$ (refer example 4 of 2.3.).

$\therefore$ x is an interior point of A.

Since $x \in [0, 1]$ is arbitrary $\text{Int } A = A$.

Example 3. In a discrete metric space M, $\text{Int } A = A$ for any subset A of M.

Basic properties of interior are given in the following theorem.

Theorem 2.7. Let (M, d) be a metric space. Let $A, B \subseteq M$.

(i) A is open iff $A = \text{Int } A$.

In particular $\text{Int } \Phi = \Phi$ and $\text{Int } M = M$.

(ii) $\text{Int } A$ = Union of all open sets contained in A.

(iii) $\text{Int } A$ is an open subset of A and if B is any other open set contained in A then $B \subseteq \text{Int } A$.

i.e. $\text{Int } A$ is the largest open set contained in A.

(iv) $A \subseteq B \Rightarrow \text{Int } A \subseteq \text{Int } B$.

(v) $\text{Int } (A \cap B) = \text{Int } A \cap \text{Int} B$.

(vi) $\text{Int } (A \cup B) \supseteq \text{Int } A \cup \text{Int } B$.

Proof. (i) follows from the definition of open set.

(ii) Let $G = \cup \{ B / B$ is an open subset of $A \}$.

To prove that Int $A = G$.

Let $x \in$ Int A.

$\therefore$ There exists a positive real number r such that $B(x, r) \subseteq A$.

Thus $B(x, r)$ is an open set contained in A.

$\therefore B(x, r) \subseteq G$.

$\therefore x \in G$.

$\therefore$ Int $A \subseteq G$. ..... (1)

Now, let $x \in G$.

Then there exists an open set B such that $x \in B$ and $B \subseteq A$.

Now, since B is open and $x \in B$ there exists a positive real number r such that $B(x, r) \subseteq B \subseteq A$.

$\therefore x$ is an interior point of A.

Hence $G \subseteq$ Int A. ...... (2)

From (1) and (2), we get $G =$ Int A.

(iii) Since union of any collection of open sets is open

(ii) $\Rightarrow$ Int A is an open set.

Trivially Int $A \subseteq A$.

Now, let B be any open set contained in A.

Then $B \subseteq G =$ Int A. (by 2)

$\therefore$ Int A is the largest open set contained in A.

(iv) Let $x \in$ Int A.

$\therefore$ There exists a real number $r > 0$ such that $B(x, r) \subseteq A$.

But $A \subseteq B$. Hence $B(x, r) \subseteq B$.

$\therefore x \in$ Int B. Hence Int $A \subseteq$ Int B.

Every finite set is closed.

2.7 CLOSED SETS

Definition. Let (M, d) be a metric space. Let $A \subseteq M$. Then A is said to be closed in M if the complement of A is open in M.

Example 1. In **R** with usual metric any closed interval $[a, b]$ is closed set.

Proof. $[a, b]^c = \mathbf{R} - [a, b] = (-\infty, a) \cup (b, \infty)$.

Also $(-\infty, a)$ and (b, ∞) are open in **R**.

i.e. $[a, b]^c$ is open in **R**.

$\therefore$ $[a, b]$ is closed in **R**.

Example 2. In **R** with usual metric $[a, b)$ is neither closed nor open.

Proof. $[a, b)$ is not open in **R** since a is not an interior point of $[a, b)$.

Now, $[a, b)^c = \mathbf{R} - [a, b) = (-\infty, a) \cup [b, \infty)$ and this set is not open since b is not an interior point.

$\therefore$ $[a, b)$ is not closed in **R**.

Hence $[a, b)$ is neither open nor closed in **R**.

Example 3. In **R** with usual metric $(a, b]$ is neither closed nor open.

Proof is similar to example 2.

Example 4. **Z** is closed.

Proof. $\mathbf{Z}^c = \bigcup_{n=-\infty}^{\infty} (n, n+1)$

The open interval $(n, n+1)$ is open and union of open sets is open.

$\mathbf{Z}^c$ is open. Hence **Z** is closed.

Example 5. **Q** is not closed in **R**.

Proof. $\mathbf{Q}^c$ = the set of irrationals which is not open in **R**.

$\therefore$ **Q** is not closed in **R**.

Example 6. The set of irrational numbers is not closed in **R**.

Proof is similar to that of example 5.

Example 7. In **R** with usual metric every singleton set is closed.

Proof. Let $a \in \mathbf{R}$.

Then $\{a\}^c = \mathbf{R} - \{a\} = (-\infty, a) \cup (a, \infty)$

Since $(-\infty, a)$ and (a, ∞) are both open sets $(-\infty, a) \cup (a, \infty)$ is open.

$\therefore$ $\{a\}^c$ is open **R**. Hence $\{a\}$ is closed in **R**.

Example 8. Every subset of a discrete metric space is closed.

Proof. Let (M, d) be a discrete metric space.

Let $A \subseteq M$.

Since every subset of a discrete metric space is open A^c is open. (refer example 10 of 2.4.)

$\therefore$ A is closed.

Definition. Let (M, d) be a metric space. Let $a \in M$. Let r be any positive real number. Then the **closed ball** or the **closed sphere** with centre a and radius r, denoted by $B_d[a, r]$, is defined by

$$B_d[a, r] = \{x \in M / d(a, x) \leq r\}$$

When the metric d under consideration is clear we write $B[a, r]$ instead of $B_d[a, r]$.

Example 1. In **R** with usual metric $B[a, r] = [a - r, a + r]$.

Example 2. In $\mathbf{R}^2$ with usual metric let $a = (a_1, a_2) \in \mathbf{R}^2$.

Then $B[a, r] = \{ (x, y) \in \mathbf{R}^2 / d((a_1, a_2), (x, y)) \leq r \}$

$= \{ (x, y) \in \mathbf{R}^2 / (x - a_1)^2 + (y - a_2)^2 \leq r^2 \}$.

Hence $B[a, r]$ is the set of all points which lie within and on the circumference of the circle with centre a and radius r.

Theorem 2.8. In any metric space every closed ball is a closed set.

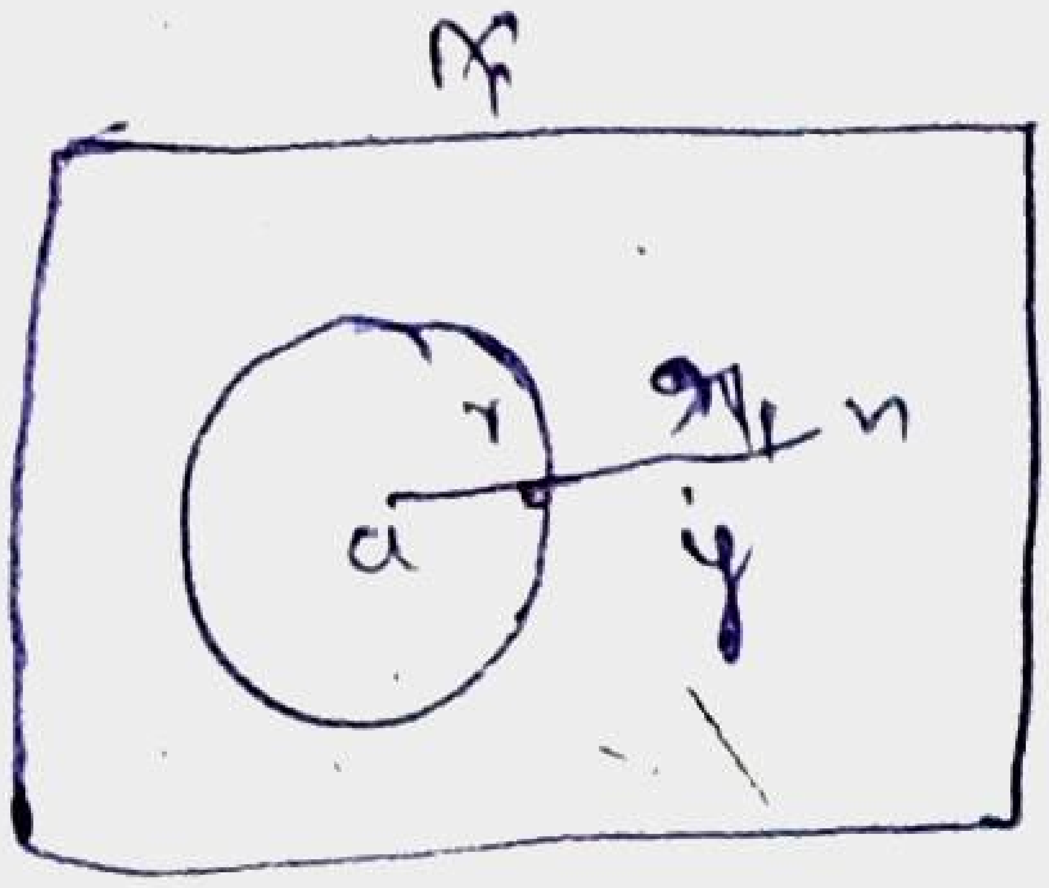
Proof. Let (M, d) be a metric space.

Let $B[a, r]$ be a closed ball in M.

Case (i) Suppose $B[a, r]^c = \Phi$

$\therefore$ $B[a, r]^c$ is open and hence $B[a, r]$ is closed.

Case (ii) Suppose $B[a, r]^c \neq \Phi$



Let $x \in B[a, r]^c$.

$\therefore$ $x \notin B[a, r]$.

$\therefore$ $d(a, x) > r$

$\therefore$ $d(a, x) - r > 0$.

Let $r_1 = d(a, x) - r$.

We claim that $B(x, r_1) \subseteq B[a, r]^c$.

Let $y \in B(x, r_1)$.

Then $d(x, y) < r_1 = d(a, x) - r$.

$\therefore$ $d(a, x) > d(x, y) + r$. ..... (1)

Now, $d(a, x) \leq d(a, y) + d(y, x)$.

$\therefore$ $d(a, y) \geq d(a, x) - d(y, x)$.

$> d(x, y) + r - d(y, x)$ (by 1)

$= r$.

Thus $d(a, y) > r$.

$\therefore$ $y \notin B[a, r]$.

Hence $y \in B[a, r]^c$.

$\therefore$ $B(x, r_1) \subseteq B[a, r]^c$.

$\therefore$ $B[a, r]^c$ is open in M.

$\therefore$ $B[a, r]$ is closed in M.

Theorem 2.9. In any metric space M, **(i)** Φ is closed, **(ii)** M is closed.

Proof. Since $M^c = \Phi$ is open. M is closed.

Similarly $\Phi^c = M$ is open and hence is Φ closed.

Note. We note that in any metric space M, Φ and M are both open and closed

Theorem 2.10. In any metric space *arbitrary intersection* of closed sets is closed.

Proof. Let (M, d) be a metric space.

Let $\{A_i \,/\, i \in I\}$ be a collection of closed sets.

We claim that $\bigcap_{i \in I} A_i$ is closed.

We have $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^{\,c}$ (by De Morgan's law)

1) Union<-> Intersection

Since A_i is closed $A_i^{\,c}$ is open.

Hence $\bigcup_{i \in I} A_i^{\,c}$ is open. (by theorem 2.3.)

$\therefore$ $\left(\bigcap_{i \in I} A_i\right)^c$ is open.

$\therefore$ $\bigcap_{i \in I} A_i$ is closed.

Theorem 2.11. In any metric space the union of a *finite* number of closed sets is closed.

Proof. Let (M, d) be a metric space.

Let $A_1, A_2, \ldots\ldots, A_n$ be closed sets in M.

By De-Morgan's law $(A_1 \cup A_2 \cup \ldots\ldots \cup A_n)^c = A_1^{\,c} \cap A_2^{\,c} \cap \ldots\ldots \cap A_n^{\,c}$.

Since each A_i is closed $A_i^{\,c}$ is open.

Hence $A_1^c \cap A_2^c \cap \ldots\ldots \cap A_n^c$ is open. (by theorem 2.4.)

$\therefore$ $(A_1 \cup A_2 \ldots\ldots \cup A_n)^c$ is open.

Hence $A_1 \cup A_2, \ldots\ldots \cup A_n$ is closed.

Note. The union of an infinite collection of closed sets need not be clos

For example, consider **R** with usual metric.

Let $A_n = \left[\frac{1}{n}, 1\right]$ where $n = 1, 2, 3, \ldots\ldots$

Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup \ldots\ldots$

$= (0, 1]$ which is not closed in **R** .

$\therefore$ $\bigcup_{n=1}^{\infty} A_n$ is not closed.

Theorem 2.12. Let M be a metric space and M_1 be a subspace of M . $F_1 \subseteq M_1$. Then F_1 is closed in M_1 iff there exists a set F which is close M such that $F_1 = F \cap M_1$.

Proof. Let F_1 be closed in M_1 .

$\therefore$ $M_1 - F_1$ is open in M_1 .

$\therefore$ $M_1 - F_1 = A \cap M_1$ where A is open in M . (by theorem 2

Now, $F_1 = M_1 - (A \cap M_1)$

$= M_1 - A = A^c \cap M_1$.

Also, since A is open in M , $A^{\,c}$ is closed in M .

$\therefore$ $F_1 = F \cap M_1$ where $F = A^{\,c}$ is closed in M !

Proof of the converse is similar.

M itself is closed in M.

2.8 CLOSURE

Let (M, d) be a metric space. Let $A \subseteq M$. Consider the collection of all closed sets which contain A. This collection is non empty since at least M is a member of this collection.

Definition. Let A be a subset of metric space (M, d). The **closure** of A, denoted by $\bar{A}$ is defined to be the intersection of all closed sets which contain A.

Thus $\bar{A} = \cup \{B/B$ is closed in M and $A \subseteq B\}$.

Note. Since intersection of any collection of closed sets is closed $\bar{A}$ *is a closed set.* Further $\bar{A} \supseteq A$. Also if B is any closed set containing A then $\bar{A} \subseteq B$. Thus $\bar{A}$ *is the smallest closed set containing* A.

Theorem 2.13. A is closed iff $A = \bar{A}$.

Proof. Suppose $A = \bar{A}$.

Since $\bar{A}$ is closed A is closed.

Conversely, suppose A is closed. Then the smallest closed set containint A is A itself.

$\therefore A = \bar{A}$.

Note. In particular (i) $\Phi = \bar{\Phi}$ (ii) $M = \bar{M}$ (iii) $\bar{\bar{A}} = \bar{A}$

Example 1. Consider **R** with usual metric.

(a) Let $A = [0, 1]$. We know that A is a closed set.

$\therefore \bar{A} = A = [0, 1]$.

(b) Let $A = (0, 1)$. Then $[0, 1]$ is a closed set containing $(0, 1)$. Obviously $[0, 1]$ is the smallest closed set containing $(0, 1)$.

$\therefore \bar{A} = [0, 1]$.

Example 2. In a discrete metric space (M, d) any subset A of M is closed. (refer example 8 of 2.7.) Hence $\bar{A} = A$.

Theorem 2.14. Let (M, d) be a metric space. Let $A, B \subseteq M$.

Then (i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

(ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(iii) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Proof. (i) Let $A \subseteq B$.

Now, $\bar{B} \supseteq B \supseteq A$.

$\therefore \bar{B}$ is a closed set containing A.

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subseteq \bar{B}$.

Exercises

1. Give an example to show that in a metric space closure of an open ball $B(x, r)$ need not be equal to the corresponding closed ball $B[x, r]$.

(*Hint.* Consider a ball of radius 1 in a discrete metric space).

Every finite set has no limit point.

2.9 LIMIT POINT

In this section we introduce the concept of limit point of a set. This concept can be used to characterise closed sets and describe the closure of a set.

Definition. Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in M$. Then x is called a **limit point** or a **cluster point** or an **accumulation point** of A if every open ball with centre x contains at least one point of A different from x.

(i.e.) $B(x, r) \cap (A - \{x\}) \neq \Phi$ for all $r > 0$.

The set of all limit points of A is called the **derived set of** A and is denoted by $D(A)$.

Note. x is not a limit point of A iff there exists an open ball $B(x, r)$ such that $B(x, r) \cap (A - \{x\}) = \Phi$.

Example 1. Consider **R** with usual metric.

(a) Let $A = [0, 1)$.

Any open ball with centre 0 is of the form $(-r, r)$ which contains a point of $[0, 1)$ other than 0.

Hence 0 is a limit point of $[0, 1)$.

Similarly 1 is a limit point of $[0, 1)$.

2 is not a limit point of A, since

$$\left(2 - \tfrac{1}{2},\ 2 + \tfrac{1}{2}\right) \cap [0, 1) = \left(\tfrac{3}{2}, \tfrac{5}{2}\right) \cap [0, 1) = \Phi.$$

2.10 DENSE SETS

Definition. A subset A of a metric space M is said to be **dense in M** or **every where dense** if $\overline{A} = M$.

Definition. A metric space M is said to be **separable** if there exists a countable dense subset in M.

Example 1. Let M be a metric space. Trivially, M is dense in M.

Hence any countable metric space is separable.

Example 2. In $\mathbf{R}$ with usual metric $\mathbf{Q}$ is dense in $\mathbf{R}$ since $\overline{\mathbf{Q}} = \mathbf{R}$.

Further $\mathbf{Q}$ is countable.

Hence $\mathbf{R}$ is separable.

Example 3. Let M be a discrete metric space.

Let $A \subset M$ and $A \ne M$.

Since A is closed, $\overline{A} = A$.

$\therefore$ A is not dense.

Hence any uncountable discrete metric space is not separable.

Example 4. In $\mathbf{R} \times \mathbf{R}$ with usual metric $\mathbf{Q} \times \mathbf{Q}$ is a dense set,

since $\overline{\mathbf{Q} \times \mathbf{Q}} = \mathbf{R} \times \mathbf{R}$.

Also $\mathbf{Q}$ is countable and hence $\mathbf{Q} \times \mathbf{Q}$ is countable

$\therefore$ $\mathbf{R} \times \mathbf{R}$ is separable.

Theorem 2.17. Let M be a metric space and $A \subseteq M$. Then the following are equivalent. (i) A is dense in M.

(ii) The only closed set which contains A is M.

(iii) The only open set disjoint from A is Φ.

(iv) A intersects every non-empty open set.

(v) A intersects every open ball.

Proof. (i) $\Rightarrow$ (ii).

Suppose A is dense in M.

Then $\overline{A} = M$. ..... (1)

Now, let $F \subseteq M$ be any closed set containing A.

Since $\overline{A}$ is the smallest closed set containing A, we have $\overline{A} \subseteq F$.

Hence $M \subseteq F$. (by (1)).

$\therefore M = F$.

$\therefore$ The only closed set which contains A is M.

(ii) $\Rightarrow$ (iii)

Suppose **(iii)** is not true.

Then there exists a non-empty open set B such that $B \cap A = \Phi$.

$\therefore B^c$ is a closed set and $B^c \supseteq A$.

Further, since $B \neq \Phi$ we have $B^c \neq M$ which is a contradiction to **(ii)**.

Hence **(ii)** $\Rightarrow$ **(iii)**.

Obviously **(iii)** $\Rightarrow$ **(iv)**.

(iv) $\Rightarrow$ **(v)**, since every open ball is an open set.

(v) $\Rightarrow$ (i)

Let $x \in M$. Suppose every open ball $B(x, r)$ intersects A.

Then by corollary (2) of theorem 2.16, $x \in \overline{A}$.

$\therefore M \subseteq \overline{A}$.

But trivially $\overline{A} \subseteq M$.

$\therefore \overline{A} = M$.

$\therefore A$ is dense in M.

UNIT - III

CHAPTER 3

COMPLETE METRICE SPACES

3.0 INTRODUCTION

The reader is familiar with the concept of convergent sequences and Cauchy sequences in **R**. In this chapter we generalise these concepts to sequences in any metric space.

3.1 COMPLETENESS

Definition. Let (M, d) be a metric space. Let $(x_n) = x_1, x_2, \ldots, x_n \ldots$ be a sequence of points in M. Let $x \in M$. We say that (x_n) **converges** to x if given $\varepsilon > 0$ there exists a positive integer n_o such that $d(x_n, x) < \varepsilon$ for all $n \geq n_o$. Also x is called a **limit** of (x_n).

If (x_n) converges to x we write $\lim_{n \to \infty} x_n = x$ or $(x_n) \to x$.

Note 1. $(x_n) \to x$ iff for each open ball $B(x, \varepsilon)$ with centre x there exists a positive integer n_o such that $x_n \in B(x, \varepsilon)$ for all $n \geq n_o$.

Thus the open ball $B(x, \varepsilon)$ contains all but a finite number of terms of the sequence.

Note 2. $(x_n) \to x$ iff the sequence of real numbers $(d(x_n, x)) \to 0$.

Theorem 3.1. For a convergent sequence (x_n) the limit is unique.

Proof. Suppose $(x_n) \to x$ and $(x_n) \to y$.

Let $\varepsilon > 0$ be given. Then there exist positive integers n_1 and n_2 such that

$\forall \varepsilon > 0, \exists N \ni d(x_n, x) < \varepsilon \ \forall n \geq N$

$x \neq y \quad x_n \to x \quad x_n \to y$

$\varepsilon/2 > 0 \quad \exists N_1 \ni d(x_n, x) < \varepsilon/2 \quad \forall n \geq N_1$

$\varepsilon/2 > 0 \quad \exists N_2, \ni d(x_n, y) < \varepsilon/2 \quad \forall n \geq N_2$

$N = \max\{n_1, n_2\}$,

$d(x,y) \leq d(x, x_n) + d(x_n, y)$

$< \varepsilon/2 + \varepsilon/2$

$d(x,y) < \varepsilon$

$$d(x_n, x) < \frac{1}{2}\varepsilon \text{ for all } n \geq n_1 \text{ and}$$

$$d(x_n, y) < \frac{1}{2}\varepsilon \text{ for all } n \geq n_2 .$$

Let m be a positive integer such that $m \geq n_1, n_2$.

Then $d(x, y) \leq d(x, x_m) + d(x_m, y)$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

$\therefore\ d(x, y) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary $d(x, y) = 0$.

$\therefore\ x = y$.

Note. In view of the above theorem if $(x_n) \to x$ then x is called *the limit* of the sequence (x_n).

The connection between the limit of a sequence and limit point of a set is given in the following theorem.

Theorem 3.2. Let M be a metric space and $A \subseteq M$. Then

(i) $x \in \overline{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \to x$.

(ii) x is a limit point of A iff there exists a sequence (x_n) of distinct points in A such that $(x_n) \to x$.

Proof. Let $x \in \overline{A}$.

Then $x \in A \cup D(A)$ (by theorem 2.16).

$\therefore\ x \in A$ or $x \in D(A)$

If $x \in A$, then the constant sequence $x, x, \ldots\ldots\ldots$ is a sequence in A converging to x.

If $x \in D(A)$ then the open ball $B(x, \frac{1}{n})$ contains infinite numbe

$\epsilon > 0$, $\exists N$ $|S_n - l| < \varepsilon$ $\forall$ $n \geq N$.

of points of A (by theorem 2.15)

$\therefore$ We can choose $x_n \in B(x, \frac{1}{n}) \cap A$ such that

$x_n \neq x_1, x_2, \ldots\ldots, x_{n-1}$ for each n

$\therefore$ (x_n) is a sequence of distinct points in A .

Also $d(x_n, x) < \frac{1}{n}$ for all n.

$\therefore \lim_{n \to \infty} d(x_n, x) = 0.$

$\therefore$ $(x_n) \to x$.

Conversely, suppose there exists a sequence (x_n) in A such that $(x_n) \to x$

Then for any $r > 0$ there exists a positive integer n_o such that $d(x_n, x) < r$ for all $n \geq n_o$.

$\therefore$ $x_n \in B(x, r)$ for all $n \geq n_o$. ..... (1)

$\therefore$ $B(x, r) \cap A \neq \Phi$

$\therefore$ $x \in \overline{A}$ (by corollary 2 of theorem 2.16).

Further if (x_n) is a sequece of distinct points, $B(x, r) \cap A$ is infinite.

$\therefore$ $x \in D(A)$.

$\therefore$ x is a limit point of A .

Definition. Let (M, d) be a metric space. Let (x_n) be a sequence of points in M . (x_n) is said to be a **Cauchy sequence** in M if given $\varepsilon > 0$ there exists a positive integer n_o such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_o$.

Theorem 3.3. Let (M, d) be a metric space. Then any convergent sequence in M is a Cauchy sequence.

Proof. Let (x_n) be a convergent sequence in M converging to $x \in M$.

Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_o such that

$$d(x_n, x) < \frac{1}{2}\varepsilon \quad \text{for all } n \geq n_o .$$

$$\therefore \ d(x_n, x_m) \leq (d(x_n, x) + d(x, x_m)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \quad \text{for all } n, m \geq n_o .$$

$$= \varepsilon \quad \text{for all } m, n \geq n_o .$$

Thus $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_o$.

$\therefore$ (x_n) is a Cauchy sequence.

Note. The converse of the above theorem is not true.

For example, consider the metric space (0, 1] with usual metric.

$(\frac{1}{n})$ is a Cauchy sequence in (0, 1].

But this sequence does not converge to any point in (0, 1].

Definition. A metric space M is said to be **complete** if every Cauchy sequence in M converges to a point in M.

Example 1. **R** with usual metric is complete. This is a fundamental fact of elementary analysis and a proof of this fact is given in section 6.3.

Note. The metric space (0,1] with usual metric is not complete (refer note given above)

Example 2. **C** with usual metric is complete.

Proof. Let (z_n) be a Cauchy sequence in **C**.

Let $z_n = x_n + iy_n$ where $x_n, y_n \in \mathbf{R}$.

We claim that (x_n) and (y_n) are Cauchy sequences in **R**.

Let $\varepsilon > 0$ be given.

Since (z_n) is a Cauchy sequence, there exists a positive integer n_o such that $|z_n - z_m| < \varepsilon$ for all $n, m \geq n_o$.

Now, $|x_n - x_m| \leq |z_n - z_m|$ and $|y_n - y_m| \leq |z_n - z_m|$.

Hence $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_o$ and

$|y_n - y_m| < \varepsilon$ for all $n, m \geq n_o$

$\therefore$ (x_n) and (y_n) are Cauchy sequences in **R**.

Since **R** is complete, there exist $x, y \in$ **R** such that $(x_n) \to x$ and $(y_n) \to y$.

Let $z = x + iy$. We claim that $(z_n) \to z$.

We have $|z_n - z| = |(x_n + iy_n) - (x + iy)|$

$= |(x_n - x) + i(y_n - y)|$

$\leq |x_n - x| + |y_n - y|$ ..... (1)

Now, let $\varepsilon > 0$ be given.

Since $(x_n) \to x$ and $(y_n) \to y$ there exist positive integers n_1 and n_2 such that $|x_n - x| < \frac{1}{2}\varepsilon$ for all $n \geq n_1$ and

$|y_n - y| < \frac{1}{2}\varepsilon$ for all $n \geq n_2$.

Let $n_3 = \max\{n_1, n_2\}$.

From (1) we get $|z_n - z| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ for all $n \geq n_3$

$\therefore$ $(z_n) \to z$

$\therefore$ **C** is complete.

Example 3. Any discrete metric space is complete.

Proof. Let (M, d) be a discrete metric space.

Let (x_n) be a Cauchy sequence in M.

Then there exists a positive integer n_0 such that

$$d(x_n, x_m) < \frac{1}{2} \quad \text{for all } n, m \geq n_0.$$

Since d is the discrete metric distance between any two points is either 0 or 1.

$\therefore$ $d(x_n, x_m) = 0$ for all $n, m \geq n_0$.

$\therefore$ $x_n = x_{n_0} = x$ (say) for all $n \geq n_0$.

$\therefore$ $d(x_n, x) = 0$ for all $n \geq n_0$.

$\therefore$ $(x_n) \to x$. Hence M is complete.

Example 4. $\mathbf{R}^n$ with usual metric is complete.

Proof. Let (x_p) be a Cauchy sequence in $\mathbf{R}^n$.

Let $x_p = (x_{p_1}, \ldots\ldots, x_{p_n})$. Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_0 such that

$$d(x_p, x_q) < \varepsilon \quad \text{for all } p, q \geq n_0.$$

$$\therefore \left[\sum_{k=1}^{n} (x_{p_k} - x_{q_k})^2\right]^{1/2} < \varepsilon \text{ for all } p, q \geq n_0$$

$$\therefore \sum_{k=1}^{n} (x_{p_k} - x_{q_k})^2 < \varepsilon^2 \text{ for all } p, q \geq n_0.$$

$\therefore$ For each $k = 1, 2, \ldots\ldots, n$ we have

$$|x_{p_k} - x_{q_k}| < \varepsilon \text{ for all } p, q \geq n_0.$$

$\therefore (x_{p_k})$ is a Cauchy sequence in **R** for each $k = 1, 2, \ldots, n$.

Since **R** is complete, there exists $y_k \in$ **R** such that $(x_{p_k}) \to y_k$.

Let $y = (y_1, y_2, \ldots, y_n)$. We claim that $(x_p) \to y$

Since $(x_{p_k}) \to y_k$ there exists a positive integer m_k such that $|x_{p_k} - y_k| < \frac{\varepsilon}{\sqrt{n}}$ for all $p \geq m_k$.

Let $m_o = \max\{m_1, m_2, \ldots, m_n\}$.

$$\text{Then } d(x_p, y) = \left[\sum_{k=1}^{n} (x_{p_k} - y_k)^2\right]^{1/2}$$

$$< [n(\varepsilon/\sqrt{n})^2]^{1/2} \text{ for all } p \geq m_o.$$

$$= \varepsilon \text{ for all } p \geq m_o.$$

Thus $d(x_p, y) < \varepsilon$ for all $p \geq m_o$.

$\therefore (x_p) \to y$ Hence $\mathbf{R^n}$ is complete.

Example 5. l_2 is complete.

Proof. Let (x_p) be a Cauchy sequence in l_2.

Let $x_p = (x_{p_1}, \ldots, x_{p_n}, \ldots)$

Let $\varepsilon > 0$ be given. Then there exists a positive integer n_o such that $d(x_p, x_q) < \varepsilon$ for all $p, q \geq n_o$.

$$\text{(i.e.)} \left[\sum_{n=1}^{\infty} (x_{p_n} - x_{q_n})^2\right]^{1/2} < \varepsilon \text{ for all } p, q \geq n_o$$

$$\therefore \sum_{n=1}^{\infty} \left(x_{p_n} - x_{q_n}\right)^2 < \varepsilon^2 \text{ for all } p, q \geq n_o \quad \ldots\ldots (1)$$

For each $n = 1, 2, 3, \ldots\ldots\ldots$ we have

$$|x_{p_n} - x_{q_n}| < \varepsilon \text{ for all } p, q \geq n_o.$$

$\therefore$ (x_{p_n}) is a Cauchy sequence in **R** for each n.

Since **R** is complete, there exists $y_n \in$ **R** such that

$$(x_{p_n}) \to y_n. \qquad \ldots\ldots (2)$$

Let $y = (y_1, y_2, \ldots\ldots, y_n, \ldots.)$.

We claim that $y \in l_2$ and $(x_p) \to y$.

For any fixed positive interger m, we have

$$\sum_{n=1}^{m} (x_{p_n} - x_{q_n})^2 < \varepsilon^2 \text{ for all } p, q \geq n_0 \text{ (using (1))}$$

Fixing q and allowing $p \to \infty$ in this finite sum we get

$$\sum_{n=1}^{m} (y_n - x_{q_n})^2 \leq \varepsilon^2 \text{ for all } q \geq n_o \text{ (using (2))}$$

Since this is true for every positive interger m

$$\sum_{n=1}^{\infty} (y_n - x_{q_n})^2 \leq \varepsilon^2 \text{ for all } q \geq n_o. \qquad \ldots\ldots (3)$$

Now, $$\left[\sum_{n=1}^{\infty} |y_n|^2\right]^{1/2} = \left[\sum_{n=1}^{\infty} |y_n - x_{q_n} + x_{q_n}|^2\right]^{1/2}$$

$$\leq \left[\sum_{n=1}^{\infty} |y_n - x_{q_n}|^2\right]^{1/2} + \left[\sum_{n=1}^{\infty} |x_{q_n}|^2\right]^{1/2}$$

(by Minkowski's inequality)

$$\leq \varepsilon + \left[\sum_{n=1}^{\infty} |x_{q_n}|^2\right]^{1/2} \quad \text{for all } q \geq n_o \text{ (by (3))}$$

Since $x_q \in l_2$ we have $\left[\sum_{n=1}^{\infty} |x_{q_n}|^2\right]^{1/2}$ converges.

$\therefore \left[\sum_{n=1}^{\infty} |y_n|^2\right]^{1/2}$ converges.

$\therefore y \in l_2$.

Also (3) gives $d(y, x_q) \leq \varepsilon$ for all $q \geq n_0$.

$\therefore (x_p) \to y$.

$\therefore l_2$ is complete.

Note. *A subspace of a complete metric space need not be complete.*

For example **R** with usual metric is complete. But the subspace (0, 1] is not complete. (refer example 1)

In the next theorem we give a necessary and sufficient condition for a subspace of a complete metric space to be complete.

Theorem 3.4. A subset A of a complete metric space M is complete iff A is closed.

Proof. Suppose A is complete.

To prove that A is closed, we shall prove that A contains all its limit points.

Let x be a limit point of A.

Then by theorem 3.2. there exists a sequence (x_n) in A such that $(x_n) \to x$.

Since A is complete $x \in A$.

$\therefore A$ contains all its limit points.

Hence A is closed.

Conversely, let A be a closed subset of M.

Let (x_n) be a Cauchy sequence in A.

Then (x_n) is a Cauchy sequence in M also and since M is complete there exists $x \in M$ such $(x_n) \to x$. Thus (x_n) is a sequence in A converging to x.

$\therefore\ x \in \overline{A}$. (by theorem 3.2.)

Now, since A is closed, $A = \overline{A}$.

$\therefore\ x \in A$.

Thus every Cauchy sequence (x_n) in A converges to a point in A.

$\therefore\ A$ is complete.

Note 1. $[0, 1]$ with usual metric is complete since it is a closed subset of th complete metric space **R**

Note 2. Consider **Q**. Since $\overline{\mathbf{Q}} = \mathbf{R}$, **Q** is not a closed subset of **R**.
Hence **Q** is not complete.

Solved problems

Problem 1 Let A, B be subsets of **R**. Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

Solution. Let $(x, y) \in \overline{A \times B}$

$\therefore$ There exists a sequence $((x_n, y_n)) \in A \times B$ such that

$$((x_n, y_n)) \to (x, y) \quad \text{(by theorem 3.2)}$$

$\therefore\ (x_n) \to x$ and $(y_n) \to y$.

Also (x_n) is a sequence in A and (y_n) is a sequence in B.

$\therefore\ x \in \overline{A}$ and $y \in \overline{B}$. (by theorem 3.2.)

$\therefore\ (x, y) \in \overline{A} \times \overline{B}$.

3. Determine which of the following subsets of **R** are complete.

(i) (a, b) (ii) $(a, b]$ (iii) $[a, b)$ (iv) $[a, b]$

(v) $\mathbf{R} - \mathbf{Q}$ (vi) $\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$

(vii) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ (viii) $[0, 1] \cup [2, 3]$

4. Let $(M_1, d_1), (M_2, d_2), \ldots, (M_n, d_n)$ be complete metric spaces. Let $M = M_1 \times M_2 \times \ldots \times M_n$. Let $x = (x_1, x_2, \ldots, x_n)$ and

$y = (y_1, y_2, \ldots, y_n) \in M$. Define $d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i)$ and

$d'(x, y) = \max \{d_i(x_i, y_i)\}$ Prove that (M, d) and (M, d') are complete metric spaces.

5. Let M be the subspace of l_2 consisting of all sequences (x_n) such that all but a finite number of terms are zero. Prove that M is not complete.

[Hint. Let $(a_n) = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots)$. Prove that $((a_n))$ is a Cauchy sequence in M but is not convergent in M .]

The following theorem provides a characterisation of complete metric spaces.

Theorem 3.5. (Cantor's Intersection Theorem)

Let M be a metric space. M is complete iff for every sequence (F_n) of non-empty closed subsets of M such that

$$F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots \text{ and } (d(F_n)) \to 0$$

$\bigcap_{n=1}^{\infty} F_n$ is nonempty.

Proof. Let M be a complete metric space.

Let (F_n) be a sequence of closed subsets of M such that

$$F_1 \supseteq F_2 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots \quad \ldots\ldots (1)$$

$$\text{and } (d(F_n)) \to 0. \quad \ldots\ldots (2)$$

We claim that $\bigcap_{n=1}^{\infty} F_n \neq \Phi$.

For each positive integer n, choose a point $x_n \in F_n$.

By (1), $x_n, x_{n+1}, x_{n+2}, \ldots\ldots$ all lie in F_n.

(i.e.) $x_m \in F_n$ for all $m \geq n$. ..... (3)

Since $(d(F_n)) \to 0$, given $\varepsilon > 0$, there exists a positive integer n_o, such that $d(F_n) < \varepsilon$ for all $n \geq n_o$.

In particular $d(F_{n_o}) < \varepsilon$. ..... (4)

$$\therefore\ d(x, y) < \varepsilon \text{ for all } x, y \in F_n.$$

Now, $x_m \in F_{n_o}$ for all $m \geq n_o$. (by (3))

$$\therefore\ m, n \geq n_o \Rightarrow x_m, x_n \in F_{n_o}.$$

$$\Rightarrow d(x_m, x_n) < \varepsilon. \text{ (by 4)}$$

$\therefore$ (x_n) is a Cauchy sequence in M.

Since M is complete there exists a point $x \in M$ such that

$$(x_n) \to x.$$

We claim that $x \in \bigcap_{n=1} F_n$.

Now, for any positive integer n, $x_n, x_{n+1}, \ldots\ldots$ is a sequence in F_n and this sequence converges to x.

$\therefore$ $x \in \overline{F}_n$ (by theorem 3.2)

But $\overline{F_n}$ is closed and hence $\overline{F_n} = F_n$.

$\therefore\ x \in F_n$.

$\therefore\ x \in \bigcap_{n=1}^{\infty} F_n$

Hence $\bigcap_{n=1}^{\infty} F_n \neq \Phi$.

To prove the converse let, (x_n) be any Cauchy sequence in M.

Let $F_1 = \{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\}$

$F_2 = \{x_2, x_3, \ldots\ldots, x_n, \ldots\ldots\}$

....

....

$F_n = \{x_n, x_{n+1}, \ldots\ldots\ldots\ldots\}$

Clearly $F_1 \supseteq F_2 \supseteq \ldots\ldots\ldots\ldots \supseteq F_n \supseteq \ldots\ldots$

$\therefore\ \overline{F_1} \supseteq \overline{F_2} \supseteq \ldots\ldots\ldots\ldots \supseteq \overline{F_n} \ldots\ldots\ldots$

$\therefore\ (\overline{F_n})$ is a decreasing sequence of closed sets.

Now, since (x_n) is a Cauchy sequence, given $\varepsilon > 0$ there exists a positive integer n_o, such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_o$.

$\therefore$ For any integer $n \geq n_o$, the distance between any two points of F_n is less than ε.

$\therefore\ d(F_n) < \varepsilon$ for all $n \geq n_o$

But $d(F_n) = d(\overline{F_n})$.

$\therefore\ d(\overline{F_n}) < \varepsilon$ for all $n \geq n_o$ ..... (5)

$\therefore\ (d(\overline{F_n})) \to 0.$

Hence $\bigcap_{n=1}^{\infty} \overline{F_n} \neq \Phi.$

Let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}$. Then x and $x_n \in \overline{F_n}$

$\therefore\ d(x_n, x) \leq d(\overline{F_n}).$

$\therefore\ d(x_n, x) < \varepsilon$ for all $n \geq n_o$ (by 5)

$\therefore\ (x_n) \to x.$

$\therefore\ M$ is complete.

Note 1. In the above theorem $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

For, suppose $\bigcap_{n=1}^{\infty} F_n$ contains two distinct points x and y.

Then $d(F_n) \geq d(x, y)$ for all n.

$\therefore\ (d(F_n))$ does not tend to zero which is a contradiction.

$\therefore\ \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Note 2. In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if each F_n is not closed

For example, consider $F_n = (0, \frac{1}{n})$ in **R**.

Clearly $F_1 \supseteq F_2 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ and

$$(d(F_n)) = \left(\frac{1}{n}\right) \to 0 \text{ as as } n \to \infty.$$

But $\bigcap_{n=1}^{\infty} F_n = \Phi.$

3.2 BAIRE'S CATEGORY THEOREM

In this section we prove a fundamental property of complete metric space called Baire's Category theorem.

Definition. A subset A of a metric space M is said to be **nowhere dense** in M if $\text{Int}\,\overline{A} = \Phi$.

Definition. A subset A of a metric space M is said to be of **first category** in M if A can be expressed as a countable union of nowhere dense sets.

A set which is not of first category is said to be of **second category.**

Note. If A is of first category then $A = \bigcup_{n=1}^{\infty} E_n$ where E_n is nowhere dense subsets in M.

Example 1. In $\mathbf{R}$ with usual metric $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots\ldots\}$ is nowhere dense.

For, $\overline{A} = A \cup D(A) = \{0, 1, \frac{1}{2}, \ldots\ldots, \frac{1}{n}, \ldots\}$

Clearly $\text{Int}\,\overline{A} = \Phi$.

$A' = D(A)$.

Proof is left ...

Theorem 3.7 (Baire's Category Theorem)

Any complete metric space is of second category.

Proof. Let M be a complete metric space.

We claim that M is not of first category.

- Let (A_n) be a sequence of nowhere dense sets in M.

We claim that $\bigcup_{n=1}^{\infty} A_n \neq M$.

Since M is open and A_1 is nowhere dense, there exists an open ball say B_1 of radius less than 1 such that B_1 is disjoint from A_1. (refer theorem 3.6)

Let F_1 denote the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_1.

Now Int F_1 is open and A_2 is nowhere dense.

$\therefore$ Int F_1 contains an open ball B_2 of radius less than $\frac{1}{2}$ such that B_2 is disjoint from A_2.

Let F_2 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_2. Now Int F_2 is open and A_2 is nowhere dense.

$\therefore$ Int F_2 contains an open ball B_3 of radius less than $\frac{1}{4}$ such that B_2 is disjoint from A_3.

Let F_3 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_3.

Proceeding like this we get a sequence of non-empty closed balls F_n such that $F_1 \supseteq F_2 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ and $d(F_n) < \frac{1}{2^n}$.

Hence $(d(F_n)) \to 0$ as $n \to \infty$.

Since M is complete, by Cantor's intersection theorem, there exists a point x in M such that $x \in \bigcap_{n=1}^{\infty} F_n$.

Also each F_n is disjoint from A_n.

Hence $x \notin A_n$ for all n.

$\therefore x \notin \bigcup_{n=1}^{\infty} A_n$.

$\therefore \bigcup_{n=1}^{\infty} A_n \neq M$. Hence M is of second category.

Corollary. **R** *is of second category.*

Proof. We know that **R** is a complete metric space. Hence **R** is of second category.

Note. The converse of the above theorem is not true.

(i.e.) A metric space which is of second category need not be complete.

For example, consider $M = \mathbf{R} - \mathbf{Q}$, the space of irrational numbers.

We know that $\mathbf{Q}$ is of of first category.

Suppose M is of first category. Then $M \cup \mathbf{Q} = \mathbf{R}$ is also of first category which is a contradiction.

$\therefore$ M is of second category.

Also M is not a closed subspace of $\mathbf{R}$ and hence M is not complete.

Solved problems

Problem 1. Prove that any nonempty open interval (a, b) in $\mathbf{R}$ is of second category.

Solution. Let (a, b) be a nonempty open interval in $\mathbf{R}$.

Suppose (a, b) is of first category.

Now, $[a, b] = (a, b) \cup \{a\} \cup \{b\}$.

$\therefore$ $[a, b]$ is of first category.

But $[a, b]$ is a complete metric space and hence is of second category which is a contradiction.

$\therefore$ (a, b) is of second category.

Problem 2. Prove that a closed set A in a meric space M is nowhere dense iff A^c is everywhere dense.

Solution. Let A be a closed set in M.

$\therefore$ $A = \overline{A}$ ..... (1)

Suppose A is nowhere dense in M.

$\therefore$ Int $\overline{A} = \Phi$.

$\therefore$ Int $A = \Phi$ (by 1). ..... (2)

Now we claim that $\overline{A^c} = M$.

Obviously $\overline{A^c} \subseteq M$. ..... (3)

Now, let $x \in M$. Let G be any open set such that $x \in G$.

Since Int $A = \Phi$, we have $G \not\subseteq A$.

$\therefore$ $G \cap A^c \neq \Phi$.

$\therefore$ $x \in \overline{A^c}$ (refer Corollary 3 of theorem 2.16)

$\therefore$ $M \subseteq \overline{A^c}$ ..... (4)

$\therefore$ By (3) and (4) we have $M = \overline{A^c}$.

$\therefore$ A^c is everywhere dense in M.

Conversely let A^c be everywhere dense in M.

$\therefore$ $\overline{A^c} = M$.

We claim that Int $A = \Phi$.

Let G be any nonempty open set in M.

Since $\overline{A} = M$, we have $G \cap A^c \neq \Phi$.

$\therefore$ $G \not\subseteq A$

$\therefore$ The only open set which is contained in A is the empty set.

$\therefore$ Int $A = \Phi$

$\therefore$ Int $\overline{A} = \Phi$. (by (1))

$\therefore$ A is nowhere dense in M.

CHAPTER 4

UNIT - IV

CONTINUITY

4.0 INTRODUCTION

In chapter 3 we discussed the concept of convergence of a sequence in any metric space. The definition of continuity for real valued functions depends on the usual metric of the real line. Hence the concept of continuity can be extended for functions defined from one metric space to another in a natural way.

4.1 CONTINUITY

Definition. Let (M_1, d_1) and (M_2, d_2) be metric spaces.

Let $f : M_1 \to M_2$ be a function. Let $a \in M_1$ and $l \in M_2$. The function f is said to have a **limit** as $x \to a$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < d_1(x, a) < \delta \Rightarrow d_2(f(x), l) < \varepsilon$.

We write $\lim\limits_{x \to a} f(x) = l$.

Definition. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A function $f : M_1 \to M_2$ is said to be **continuous at a** if given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$.

f is said to be **continuous** if it is continuous at every point of M_1.

Note 1. f is continuous at a iff $\lim\limits_{x \to a} f(x) = f(a)$.

Note 2. The condition $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$ can be rewritten as (i) $x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$ or

(ii) $f(B(a, \delta)) \subseteq B(f(a), \varepsilon)$.

Example 1. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then any constant function $f : M_1 \to M_2$ is continuous.

Proof. Let $f : M_1 \to M_2$ be given by $f(x) = a$ where $a \in M_2$ is a fixed element.

Let $x \in M_1$ and $\varepsilon > 0$ be given.

Then for any $\delta > 0$, $f(B(x, \delta)) = \{a\} \subseteq B(a, \varepsilon)$.

$\therefore$ f is continuous at x.

Since $x \in M_1$ is arbitrary, f is continuous.

Example 2. Let (M_1, d_1) be a discrete metric space and let (M_2, d_2) be any metric space. Then any function $f : M_1 \to M_2$ is continuous.

i.e any function whose domain is a discrete metric space is continuous.

Proof. Let $x \in M_1$. Let $\varepsilon > 0$ be given.

Since M_1 is discrete for any $\delta < 1$, $B(x, \delta) = \{x\}$.

$\therefore$ $f(B((x, \delta)) = \{f(x)\} \subseteq B(f(x), \varepsilon)$.

$\therefore$ f is continuous at x.

We now give a characterisation for continuity of a function at a point in terms of sequences converging to that point.

Theorem 4.1. Let (M_1, d_1) and (M_2, d_2) be two metric spaces.

Let $a \in M_1$. A function $f : M_1 \to M_2$ is continuous at a iff $(x_n) \to a$ $\Rightarrow (f(x_n)) \to f(a)$.

Proof. Suppose f is continuous at a.

Let (x_n) be a sequence in M_1 such that $(x_n) \to a$.

We claim that $(f(x_n)) \to f(a)$.

Let $\varepsilon > 0$ be given. By definition of continuity, there exists $\delta > 0$ such that $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$. .... (1)

Since $(x_n) \to a$, there exists a positive integer n_0 such that $d_1(x_n, a) < \delta$ for all $n \geq n_0$.

$\therefore$ $d_2(f(x_n), f(a)) < \varepsilon$ for all $n \geq n_o$ (by (1))

$\therefore$ $(f(x_n)) \to f(a)$.

Conversely, suppose $(x_n) \to a \Rightarrow (f(x_n)) \to f(a)$.

We claim that f is continuous at a.

Suppose f is not continuous at a.

Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$,

$$f(B(a, \delta) \not\subset B(f(a), \varepsilon)$$

In particular $f(B(a, \frac{1}{n}) \not\subset B(f(a), \varepsilon)$.

Choose x_n such that $x_n \in B(a, \frac{1}{n})$ and $f(x_n) \notin B(f(a), \varepsilon)$.

$\therefore$ $d_1(x_n, a) < \frac{1}{n}$, and $d_2(f(x_n), f(a)) \geq \varepsilon$.

$\therefore$ $(x_n) \to a$ and $(f(x_n))$ does not converge to $f(a)$ which is a contradiction to the hypothesis.

$\therefore$ f is continuous at a.

Corollary. A function $f : M_1 \to M_2$ is continuous iff

$$(x_n) \to x \Rightarrow (f(x_n)) \to f(x).$$

We now characterise continuous mappings in terms of open sets.

Theorem 4.2. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. $f : M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

(i.e.) *f is continuous iff inverse image of every open set is open.*

Proof. Suppose f is continuous.

Let G be an open set in M_2.

We claim that $f^{-1}(G)$ is open in M_1.

If $f^{-1}(G)$ is empty, then it is open.

Let $f^{-1}(G) \neq \Phi$.

Let $x \in f^{-1}(G)$. Hence $f(x) \in G$.

Since G is open, there exists an open ball $B(f(x), \varepsilon)$ such that $B(f(x), \varepsilon) \subseteq G$. ..... (1)

Now, by definition of continuity, there exists an open ball $B(x, \delta)$ such that $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$.

$\therefore\ f(B(x, \delta)) \subseteq G$ (by (1)).

$\therefore\ B(x, \delta) \subseteq f^{-1}(G)$.

Since $x \in f^{-1}(G)$ is arbitrary, $f^{-1}(G)$ is open.

Conversely, suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2. We claim that f is continuous.

Let $x \in M_1$.

Now, $B(f(x), \varepsilon)$ is an open set in M_2.

$\therefore\ f^{-1}(B(f(x), \varepsilon))$ is open in M_1 and $x \in f^{-1}(B(f(x), \varepsilon))$.

$\therefore$ There exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(f(x), \varepsilon))$.

$\therefore f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$.

$\therefore f$ is continuous at x.

Since $x \in M_1$ is arbitrary f is continuous.

Note 1. If $f : M_1 \to M_2$ is continuous and G is open in M_1, then it is not necesary that $f(G)$ is open in M_2.

(i.e.) Under a continuous map the image of an open set need not be an open set.

For example let $M_1 = \mathbf{R}$ with discrete metric and let $M_2 = \mathbf{R}$ with usual metric.

Let $f : M_1 \to M_2$ be defined by $f(x) = x$.

Since M_1 is discrete every subset of M_1 is open.

Hence for any open subset G of M_2, $f^{-1}(G)$ is open in M_1.

$\therefore f$ is continuous.

Now, $A = \{x\}$ is open in M_1.

But $f(A) = \{x\}$ is not open in M_2.

Note 2. In the above example f is a continuous bijection whereas

$f^{-1} : M_2 \to M_1$ is not continuous.

For, $\{x\}$ is an open set in M_1.

$(f^{-1})^{-1}(\{x\}) = \{x\}$ which is not open in M_2.

$\therefore f^{-1}$ is not continuous.

Thus *if f is a continuous bijection, f^{-1} need not be continuous.*

We now give yet another characterisation of continuous functions in terms of closed sets.

Theorem 4.3. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f : M_1 \to M_2$ is continuous iff $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

Proof. Suppose $f : M_1 \to M_2$ is continuous.

Let $F \subseteq M_2$ be closed in M_2.

$\therefore$ F^c is open in M_2.

$\therefore$ $f^{-1}(F^c)$ is open in M_1.

But $f^{-1}(F^c) = [f^{-1}(F)]^c$.

$f^{-1}(F)$ is closed in M_1.

Conversely, suppose $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

We claim that f is continuous.

Let G be an open set in M_2.

$\therefore$ G^c is closed in M_2.

$\therefore$ $f^{-1}(G^c)$ is closed in M_1.

$\therefore$ $[f^{-1}(G)]^c$ is closed in M_1.

$\therefore$ $f^{-1}(G)$ is open in M_1.

$\therefore$ f is continuous.

We give one more characterisation of continuous function in terms of closure of a set.

Theorem 4.4 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f : M_1 \to M_2$ is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Proof. Suppose f is continuous.

Let $A \subseteq M_1$. Then $f(A) \subseteq M_2$.

Since f is continuous, $f^{-1}\overline{(f(A))}$ is closed in M_1 .

Also $f^{-1}\overline{(f(A))} \supseteq A$ (since $\overline{f(A)} \supseteq f(A)$)

But $\overline{A}$ is the smallest closed set containing A .

$\therefore \overline{A} \subseteq f^{-1}\overline{(f(A))}$

$\therefore f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely, let $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

To prove that f is continuous, we shall show that if F is a closed set in M_2, then $f^{-1}(F)$ is closed in M_1 .

By hypothesis, $f\,(\overline{f^{-1}(F)}) \subseteq \overline{ff^{-1}(F)}$

$\subseteq \overline{F}$.

$= F$ (since F is closed.) .

Thus $f\,(\overline{f^{-1}(F)}) \subseteq F$.

$\therefore \overline{f^{-1}(F)} \subseteq f^{-1}(F)$

Also $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$.

$\therefore f^{-1}(F) = \overline{f^{-1}(F)}$.

Hence $f^{-1}(F)$ is closed.

$\therefore f$ is continuous.

$\therefore \overline{f(A)} = \overline{\{1\}} = \{1\}$.

$\therefore \overline{f(A)} \not\subset f(\overline{A})$

$\therefore f$ is not continuous.

Problem 3. Let M_1, M_2, M_3 be metric spaces. If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are continuous functions, prove that $g \circ f : M_1 \to M_3$ is also continuous.

(i.e.) *Composition of two continous functions is continuous.*

Solution. Let G be open in M_3.

Since g is continuous, $g^{-1}(G)$ is open in M_2.

Now, since f is continuous, $f^{-1}(g^{-1}(G))$ is open in M_1.

(i.e.) $(g \circ f)^{-1}(G)$ is open in M_1.

$\therefore g \circ f$ is continuous.

Problem 4. Let M be a metric space. Let $f : M \to \mathbf{R}$ and $g : M \to \mathbf{R}$ be two continuous functions. Prove that $f + g : M \to \mathbf{R}$ is continuous.

Solution. Let (x_n) be a sequence converging to x in M.

Since f and g are continous functions, $(f(x_n)) \to f(x)$ and

$(g(x_n)) \to g(x)$. (by theorem 4.1)

$\therefore (f(x_n) + g(x_n)) \to f(x) + g(x)$.

(i.e.) $((f+g)(x_n)) \to (f+g)(x)$

$\therefore f + g$ is continuous.

Problem 5. Let f, g be continuous real valued functions on a metric space M. Let $A = \{x \,/\, x \in M \text{ and } f(x) < g(x)\}$. Prove that A is open.

Solution. Since f and g are continuous real valued functions on M, $f - g$ is also a continuous real valued function on M.

Now $A = \{x \in M / f(x) < g(x)\}$

$= \{x \in M / f(x) - g(x) < 0\}$

$= \{x \in M / (f - g)(x) < 0\}$

$= \{x \in M / (f - g)x \in (-\infty, 0)\}$.

$= (f - g)^{-1} \{(-\infty, 0)\}$.

Now, $(-\infty, 0)$ is open in $\mathbf{R}$, and $f - g$ is continuous.

Hence $(f - g)^{-1}\{(-\infty, 0)\}$ is open in M.

$\therefore$ A is open in M.

Problem 6. If $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are both continuous functions on $\mathbf{R}$ and if $h : \mathbf{R}^2 \to \mathbf{R}^2$ is defined by $h(x, y) = (f(x), g(y))$ prove that h is continuous on $\mathbf{R}^2$.

Solution. Let (x_n, y_n) be a sequence in $\mathbf{R}^2$ converging to (x, y).

We claim that $(h(x_n, y_n))$ converges to $h(x, y)$.

Since $((x_n, y_n)) \to (x, y)$ in $\mathbf{R}^2$, $(x_n) \to x$ and $(y_n) \to y$ in $\mathbf{R}$.

Also f and g are continuous.

$\therefore$ $(f(x_n)) \to f(x)$ and $g(y_n) \to g(y)$.

$\therefore$ $((f(x_n), g(y_n)) \to (f(x), g(y))$.

$\therefore$ $(h(x_n, y_n)) \to h(x, y)$.

$\therefore$ h is continous on $\mathbf{R}^2$.

Problem 7. Let (M, d) be a metric space. Let $a \in M$. Show that the function $f : M \to \mathbf{R}$ defined by $f(x) = d(x, a)$ is continuous.

Solution. Let $x \in M$.

Let (x_n) be a sequence in M such that $(x_n) \to x$.

We claim that $(f(x_n)) \to f(x)$.

Let $\varepsilon > 0$ be given.

Now, $|f(x_n) - f(x)| = |d(x_n, a) - d(x, a)| \leq d(x_n, x)$.

Since $(x_n) \to x$, there exists a positive integer n_1 such that

$d(x_n, x) < \varepsilon$ for all $n \geq n_1$.

$\therefore$ $|f(x_n) - f(x)| < \varepsilon$ for all $n \geq n_1$.

$\therefore$ $(f(x_n)) \to f(x)$.

$\therefore$ f is continuous.

Problem 8. Let f be a function from $\mathbf{R}^2$ onto $\mathbf{R}$ defined by $f(x, y) = x$ for all $(x, y) \in R^2$. Show that f is continuous in $\mathbf{R}^2$.

Solution. Let $(x, y) \in \mathbf{R}^2$.

Let $((x_n, y_n))$ be a sequence in $\mathbf{R}^2$ converging to (x, y).

Then $(x_n) \to x$ and $(y_n) \to y$.

$\therefore$ $(f(x_n, y_n)) = (x_n) \to x = f(x, y)$.

$\therefore$ $(f(x_n, y_n)) \to f(x, y)$.

$\therefore$ f is continuous.

Problem 9. Define $f : l_2 \to l_2$ as follows. If $s \in l_2$ is the sequence $s_1, s_2, \ldots$ let $f(s)$ be the sequence $0, s_1, s_2, \ldots$ Show that f is continuous on l_2.

Solution. Let $y = (y_1, y_2, \ldots \ldots y_n, \ldots \ldots) \in l_2$.

Let (x_n) be a sequence in l_2 converging to y.

Let $x_n = (x_{n_1}, x_{n_2}, \ldots\ldots x_{n_k}, \ldots\ldots)$.

Then $(x_{n_1}) \to y_1; (x_{n_2}) \to y_2, \ldots. (x_{n_k}) \to y_k, \ldots\ldots$

$\therefore (f(x_n)) = ((0, x_{n_1}, x_{n_2}, \ldots\ldots, x_{n_k}, \ldots\ldots)) \to (0, y_1, y_2, \ldots\ldots, y_k, \ldots\ldots) = f(y)$

$\therefore (f(x_n)) \to f(y)$.

$\therefore f$ is continuous.

Problem 10. Let G be an open subset of $\mathbf{R}$. Prove that the characteristic function on G defined by $\chi_G(x) = \begin{cases} 1 \text{ if } & x \in G \\ 0 \text{ if } & x \notin G \end{cases}$ is continuous at every point of G.

Solution. Let $x \in G$ so that $\chi_G(x) = 1$.

Let $\varepsilon > 0$ be given.

Since G is open and $x \in G$, we can find a $\delta > 0$ such that $B(x, \delta) \subseteq G$.

$$\therefore \chi_G(B(x, \delta) \subseteq \chi_G(G)$$
$$= \{1\}.$$
$$\subseteq B(1, \varepsilon).$$

Thus $\chi_G(B(x, \delta)) \subseteq B(\chi_G(x), \varepsilon)$

$\therefore \chi_G$ is continuous at x.

Since $x \in G$ is arbitrary, χ_G is continuous on G.

Exercises

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \begin{cases} 1 \text{ if } & x < 0 \\ 2 \text{ if } & x \geq 0 \end{cases}$. Prove that f is not continuous by each of the following methods.

(i) By usual ε, δ method.

8. Let f and g be continuous real valued functions defined on a metric space M. Let $A = \{x \in M / f(x) < g(x)\}$. Prove that A is open.

[*Hint.* $A = (f - g)^{-1}((-\infty, 0))$].

9. Let M_1 and M_2 be two metric spaces and let $A \subseteq M_1$. If $f : M_1 \to M_2$ is continuous, show that $f|_A : A \to M_2$ is also continuous.

4.2 HOMEOMORPHISM

Definition. Let (M_1, d_1) and (M_2, d_2) be metric spaces. A function $f : M_1 \to M_2$ is called a **homeomorphism** if

(i) f is $1 - 1$ and onto

(ii) f is continuous.

(iii) f^{-1} is continuous.

M_1 and M_2 are said to be **homeomorphic** if there exists a homeomorphism $f : M_1 \to M_2$.

Definition. A function $f : M_1 \to M_2$ is said to be an **open map** if $f(G)$ is open in M_2 for every open set G in M_1.

(i.e.) f is an open map if the image of an open set in M_1 is an open set in M_2.

f is called a **closed map** if $f(F)$ is closed in M_2 for every closed set F in M_1.

Note 1. Let $f : M_1 \to M_2$ be a $1 - 1$ onto function. Then f^{-1} is continuous iff f is an open map.

For, f^{-1} is continuous iff for any open set G in M_1 $(f^{-1})^{-1}(G)$ is open in M_2.

But, $(f^{-1})^{-1}(G) = f(G)$.

$\therefore f^{-1}$ is continuous iff for every open set G in M_1, $f(G)$ is open in M_2.

$\therefore f^{-1}$ is continuous iff f is an open map.

Note 2. Similarly f^{-1} is continuous iff f is a closed map.

Note 3. Let $f : M_1 \to M_2$ be a 1– 1 onto map. Then the following are equivalent.

(i) f is a homeomorphism.

(ii) f is a continuous open map.

(iii) f is a continuous closed map.

Proof. (i) $\Leftrightarrow$ (ii) follows from Note 1 and the definition of homeomorphism.

(i) $\Leftrightarrow$ (iii) follows from Note 2 and the definition of homeomorphism.

Note 4. Let $f : M_1 \to M_2$ be a homeomorphism. $G \subseteq M_1$ is open in M_1 iff $f(G)$ is open in M_2.

For, since f is an open map G is open in $M_1 \Rightarrow f(G)$ is open in M_2.

Also since f is continuous, $f(G)$ is open in $M_2 \Rightarrow f^{-1}(f(G)) = G$ is open in M_1.

$\therefore G$ is open in M_1 iff $f(G)$ is open in M_2. ..... (1)

Conversely, if $f : M_1 \to M_2$ is a 1– 1 onto map satisfying (1) then f is a homeomorphism.

Thus a homeomorphism $f : M_1 \to M_2$ is simply a 1– 1 onto map between the points of the two spaces such that their open sets are also in 1– 1 correspondence with each other.

Hence f is not continuous.

Thus any bijection $f : M_1 \to M_2$ is not a homeomorphism.

Hence M_1 is not homeomorphic to M_2.

Definition. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $f : M_1 \to M_2$ be a 1– 1 onto map. f is said to be an **isometry** if $d_1(x, y) = d_2(f(x), f(y))$ for all $x, y \in M_1$. In other words, an *isometry is a distance preserving map.*

M_1 and M_2 are said to be **isometric** if there exists an isometry f from M_1 onto M_2.

Example 6. $\mathbf{R}^2$ with usual metric and $\mathbf{C}$ with usual metric are isometric and $f : \mathbf{R}^2 \to \mathbf{C}$ defined by $f(x, y) = x + iy$ is the required isometry.

Proof. Let d_1 denote the usual metric on $\mathbf{R}^2$ and d_2 denote the usual metric on $\mathbf{C}$.

Let $a = (x_1, y_1)$ and $b = (x_2, y_2) \in \mathbf{R}^2$

$$\text{Then } d_1(a, b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= |(x_1 - x_2) + i(y_1 - y_2)|$$

$$= |(x_1 + iy_1) - (x_2 + iy_2)|$$

$$= d_2(f(a), f(b)).$$

$\therefore$ f is an isometry.

Example 7. Let d_1 be the usual metric on $[0,1]$ and d_2 be the usual metric on $[0, 2]$.

The map $f : [0, 1] \to [0, 2]$ defined by $f(x) = 2x$ is not an isometry.

Proof. Let $x, y \in [0, 1]$.

4.3 UNIFORM CONTINUITY

Introduction. In this section we introduce the concept of uniform continuity.

Let (M_1, d_1) and (M_2, d_2) be two metric spaces.

Let $f : M_1 \to M_2$ be a continuous function. For each $a \in M_1$ the following is true.

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon.$$

In general the number δ depends on ε and the point a under consideration.

For example, consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$.

Let $a \in \mathbf{R}$. Let $\varepsilon > 0$ be given.

We want to find $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon. \quad \ldots\ldots (1)$$

Clearly, if $\delta > 0$ satisfies (1), then any δ_1 where $0 < \delta_1 < \delta$ also satisfies (1).

Hence if there exists a $\delta > 0$ satisfying (1) then we can find another δ_1 such that $0 < \delta_1 < 1$ and δ_1 also satisfies (1).

Hence we may restrict x such that $|x - a| < 1$.

$\therefore\ a - 1 < x < a + 1$.

$\therefore\ x + a < 2a + 1$.

$$\therefore\ |f(x) - f(a)| = |x^2 - a^2| = |x + a||x - a|$$
$$< |2a + 1|\ |x - a| \text{ if } |x - a| < 1.$$

Hence if we choose $\delta = \min\left\{1, \dfrac{\varepsilon}{|2a + 1|}\right\}$ then we have

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Thus, in this example we see that the number δ depends on both ε and the point a under consideration and if a becomes large, δ has to be chosen correspondingly small. In fact, there is no $\delta > 0$ such that (1) holds for all a.

For, suppose there exits $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon \text{ for all } a \in \mathbf{R}.$$

Take $x = a + \frac{1}{2}\delta$

Clearly, $|x - a| = \frac{1}{2}\delta < \delta$.

$\therefore\ |f(x) - f(a)| < \varepsilon$.

$\therefore\ |(a + \frac{1}{2}\delta)^2 - a^2| < \varepsilon$.

$\therefore\ \frac{1}{2}\delta\,|\frac{1}{2}\delta + 2a| < \varepsilon$.

However this inequality cannot be true for all $a \in \mathbf{R}$, since by taking a sufficiently large, we can make $\frac{1}{2}\delta\,|\frac{1}{2}\delta + 2a| > \varepsilon$.

Thus, there is no $\delta > 0$ such that (1) holds for all $a \in \mathbf{R}$.

We now consider another example.

Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 2x$.

Let $a \in \mathbf{R}$. Let $\varepsilon > 0$ be given.

Then $|f(x) - f(a)| = |2x - 2a| = 2|x - a|$.

$\therefore$ If we choose $\delta = \frac{1}{2}\varepsilon$ then we have

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Here δ depends on ε and not an a.

(i.e.) for a given $\varepsilon > 0$ we are able to find $\delta > 0$ such that δ works *uniformly* for all $a \in \mathbf{R}$.

Definition. Let (M_1, d_1) and (M_2, d_2) be metric spaces. A function $f : M_1 \to M_2$ is said to be **uniformly continuous** on M_1 if given $\varepsilon > 0$ there exitsts $\delta > 0$ such that $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.

Note 1. Uniform continuity is a global condition on the behaviour of a mapping on a set so that it is meaningless to ask whether a function is uniformly conti· ·ous at a point. Continuity is a local condition on the behaviour of a function at a point.

Note 2. If $f : M_1 \to M_2$ is uniformly continuous on M_1 then f is continuous at every point of M_1.

Moreover for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in M_1$ and $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.

Thus, uniform continuity is continuity plus the added condition that for a given $\varepsilon > 0$ we can find $\delta > 0$ which works *uniformly for all points of* M_1.

Note 3. A continuous function $f : M_1 \to M_2$ need not be uniformly continuous on M_1.

For example, $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ is continuous but not uniformly continuous on $\mathbf{R}$.

Solved problems

Problem 1. Prove that $f : [0, 1] \to \mathbf{R}$ defined by $f(x) = x^2$ is uniformly continuous on $[0, 1]$.

Solution. Let $\varepsilon > 0$ be given. Let $x, y \in [0, 1]$.

Then $|f(x) - f(y)| = |x^2 - y^2| = |x + y|\,|x - y|$
$\leq 2|x - y|$ (since $x \leq 1$ and $y \leq 1$)

$\therefore\ |x - y| < \frac{1}{2}\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$.

$\therefore\ f$ is uniformly continuous on $[0, 1]$.

Problem 2. Prove that the function $f : (0, 1) \to \mathbf{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Solution. Let $\varepsilon > 0$ be given. Suppose there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Take $x = y + \frac{1}{2}\delta$.

Clearly $|x - y| = \frac{1}{2}\delta < \delta$.

$\therefore\ |f(x) - f(y)| < \varepsilon$.

$$\therefore\ \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon.$$

$$\therefore\ \left|\frac{1}{y + \frac{1}{2}\delta} - \frac{1}{y}\right| < \varepsilon.$$

$$\therefore\ \left|\frac{\delta}{2\left(y + \frac{1}{2}\delta\right)y}\right| < \varepsilon.$$

$$\therefore\ \frac{\delta}{(2y + \delta)\,y} < \varepsilon.$$

This inequality cannot be true for all $y \in (0, 1)$ since $\frac{\delta}{(2y+\delta)y}$ becomes arbitrarily large as y approaches zero.

$\therefore$ f is not uniformly continuous.

Problem 3. Prove that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin x$ is uniformly continuous on $\mathbf{R}$.

Solution. Let $x, y \in \mathbf{R}$ and $x > y$.

$\sin x - \sin y = (x - y) \cos z$ where $x > z > y$ (by mean value theorem)

$$\therefore |\sin x - \sin y| = |x - y| \, |\cos z|$$

$$\leq |x - y| \ (\text{ since } |\cos z| \leq 1).$$

Hence for a given $\varepsilon > 0$, if we choose $\delta = \varepsilon$, we have

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| = |\sin x - \sin y| < \varepsilon.$$

$\therefore$ $f(x) = \sin x$ is uniformly continuous on $\mathbf{R}$.

Exercises

1. Determine which of the following functions are uniformly continuous.

(a) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = kx$ where $k \in \mathbf{R}$.

(b) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$.

(c) $f : [0, 1] \to \mathbf{R}$ defined by $f(x) = x^3$.

(d) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \cos x$.

(e) $f : (0, 1) \to \mathbf{R}$ defined by $f(x) = \dfrac{1}{1-x}$

(f) $f : [0, 1] \to \mathbf{R}$ defined by $f(x) = \sqrt{x}$.

2. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be two functions uniformly continuous on $\mathbf{R}$. Prove that $f + g$ is also uniformly continuous on $\mathbf{R}$.

3. Is the product of uniformly continuous real valued functions again uniformly continuous?

4.4 DISCONTINUOUS FUNCTIONS ON R

In this section we shall investigate the set of points at which a given function $f : \mathbf{R} \to \mathbf{R}$ is discontinuous. For this puruposc we introduce the concept of the left limit and the right limit of $f(x)$ at $x = a$ and classify the types of discontinuities for real functions. Throughout this section we deal with **R** with usual metric.

Definition. A function $f : \mathbf{R} \to \mathbf{R}$ is said to approach to a **limit** l as x tends to a if given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon \text{ and we write } \lim_{x \to a} f(x) = l.$$

It should be carefully noted that the condition $0 < |x - a| < \delta$ excludes the point $x = a$ from consideration. Hence the definition of limit employs only values of x in some interval $(a - \delta, a + \delta)$ other than a. Hence the value of $f(x)$ at $x = a$ is immaterial and in fact to consider $\lim_{x \to a} f(x)$ the function $f(x)$ need not even be defined at $x = a$. Even if $f(a)$ is defined it is not necessary that $\lim_{x \to a} f(x) = f(a)$.

When defining the limit of $f(x)$ as $x \to a$ we consider the behaviour of $f(x)$ at points which are near to a and these points can be either to the left of a or to the right of a. However it is often necessary to know the behaviour of $f(x)$ as $x \to a$ in such a way that x always remains greater than or less than a. This leads us to the concept of right and left limits of $f(x)$ at $x = a$.

Definition. A function f is that to have l as the **right limit** at $x = a$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x \to a+} f(x) = l$.

Also we denote the right limit l by $f(a+)$.

CHAPTER 5

CONNECTEDNESS

5.0 INTRODUCTION

In $\mathbb{R}$ consider the subsets $A = [1, 2]$ and $B = [1, 2] \cup [3, 4]$. The set A consists of a single 'piece' whereas B consists of 'two pieces'. We say that A is a connected set and B is not a connected set. This intuitive idea is made precise in the following definition.

5.1 DEFINITION AND EXAMPLES

Definition. Let (M, d) be a metric space. M is said to be **connected** if M cannot be represented as the union of two disjoint non-empty open sets.

If M is not connected it is said to be **disconnected.**

Example 1. Let $M = [1, 2] \cup [3, 4]$ with usual metric. Then M is disconnected.

Proof. $[1, 2]$ and $[3, 4]$ are open in M. (refer example 2 in 2.5)

Thus M is the union of two disjoint non-empty open sets namely $[1, 2]$ and $[3, 4]$.

Hence M is disconnected.

Example 2. Any discrete metric space M with more than one point is disconnected.

Proof. Let A be a proper non-empty subset of M. Since M has more than one point such a set exists.

Then A^c is also non-empty.

Since M is discrete every subset of M is open.

(separated
2 disjoint open

$\therefore$ A and A^c are open.

Thus $M = A \cup A^c$ where A and A^c are two disjoint non-empty open sets.

$\therefore$ M is not connected.

Theorem 5.1 Let (M, d) be a metric space. Then the following are equivalent.

(i) M is connected.

(ii) M cannot be written as the union of two disjoint non-empty closed sets.

(iii) M cannot be written as the union of two non-empty sets A and B such that $A \cap \overline{B} = \overline{A} \cap B = \Phi$.

(iv) M and Φ are the only sets which are both open and closed in M.

Proof. (i) $\Rightarrow$ (ii)

Suppose (ii) is not true.

$\therefore$ $M = A \cup B$ where A and B are closed $A \neq \Phi$, $B \neq \Phi$ and $A \cap B = \Phi$.

$\therefore$ $A^c = B$ and $B^c = A$.

Since A and B are closed, A^c and B^c are open.

$\therefore$ B and A are open.

Thus M is the union of two disjoint non-empty open sets.

$\therefore$ M is not connected which is a contradiction.

$\therefore$ (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii).

Suppose (iii) is not true.

Then $M = A \cup B$ where $A \neq \Phi$, $B \neq \Phi$ and $A \cap \overline{B} = \overline{A} \cap B = \Phi$.

We claim that A and B are closed.

Let $x \in \overline{A}$.

$\therefore\ x \notin B$ (since $\overline{A} \cap B = \Phi$)

$\therefore\ x \in A$ (since $A \cup B = M$)

$\therefore\ \overline{A} \subseteq A.$

But $A \subseteq \overline{A}.$

$\therefore\ A = \overline{A}$ and hence A is closed.

Similarly B is closed.

Now $A \cap B = \overline{A} \cap B$ (since $A = \overline{A}$).

$= \Phi$

Thus $M = A \cup B$ where $A \neq \Phi$, $B \neq \Phi$, A and B are closed and $A \cap B = \Phi$ which is a contradiction to (ii).

$\therefore$ (ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (iv).

Suppose (iv) is not true.

Then there exists $A \subseteq M$ such that $A \neq M$ and $A \neq \Phi$ and A is both open and closed.

Let $B = A^c$.

Then B is also both open and closed and $B \neq \Phi$.

Also $M = A \cup B$.

Further $\overline{A} \cap B = A \cap A^c$ (since $\overline{A} = A$ and $B = A^c$.)

$= \Phi.$

Similarly $A \cap \overline{B} = \Phi$.

$\therefore\ M = A \cup B$ where $A \cap \overline{B} = \Phi = \overline{A} \cap B$ which is a contradiction to (iii).

$\therefore$ (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i).

Suppose M is not connected.

$\therefore M = A \cup B$ where $A \neq \Phi$, $B \neq \Phi$, A and B are open and $A \cap B = \Phi$.

Then $B^c = A$.

Now, since B is open A is closed.

Also $A \neq \Phi$ and $A \neq M$ (since $B \neq \Phi$).

$\therefore A$ is a proper non-empty subset of M which is both open and closed which is a contradiction to (iv).

$\therefore$ (iv) $\Rightarrow$ (i).

The following theorem gives another equivalent characterisation for connectedness.

Theorem 5.2 A metric space M is connected iff there does not exist a continuous function f from M onto the discrete metric space $\{0, 1\}$.

Proof. Suppose there exists a continuous function f from M onto $\{0, 1\}$.

Since $\{0, 1\}$ is discrete, $\{0\}$ and $\{1\}$ are open.

$\therefore A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ are open in M.

Since f is onto, A and B are non-empty.

Clearly $A \cap B = \Phi$ and $A \cup B = M$.

Thus $M = A \cup B$ where A and B are disjoint non-empty open sets.

$\therefore M$ is not connected which is a contradiction.

Hence there does not exist a continuous function from onto the discrete metric space $\{0,1\}$.

Conversely, suppose M is not connected.

Then there exist disjoint non-empty open sets A and B in M such that $M = A \cup B$.

Now, define $f : M \to \{0, 1\}$ by $f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$

Clearly f is onto.

Also $f^{-1}(\Phi) = \Phi$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = B$ and $f^{-1}(\{0, 1\}) = M$.

Thus the inverse image of every open set in $\{0, 1\}$ is open in M.

Hence f is continuous. (refer theorem 4.2).

Thus there exists a continuous function f from M onto $\{0, 1\}$ which is a contradiction. Hence M is connected.

Note. The above theorem can be restated as follows.

M is connected iff every continuous function $f : M \to \{0, 1\}$ is not onto.

Solved problems

Problem 1. Let M be a metric space. Let A be a connected subset of M. If B is a subset of M such that $A \subseteq B \subseteq \overline{A}$ then B is connected. In particular $\overline{A}$ is connected.

Solution. Suppose B is not connected.

Then $B = B_1 \cup B_2$ where $B_1 \neq \Phi, B_2 \neq \Phi, B_1 \cap B_2 = \Phi$ and B_1 and B_2 are open in B.

Now, since B_1 and B_2 are open sets in B there exist open sets G_1 and G_2 in M such that $B_1 = G_1 \cap B$ and $B_2 = G_2 \cap B$.

$\therefore\ B = B_1 \cup B_2 = (G_1 \cap B) \cup (G_2 \cap B) = (G_1 \cup G_2) \cap B.$

$\therefore\ B \subseteq G_1 \cup G_2.$

$\therefore\ A \subseteq G_1 \cup G_2$ (since $A \subseteq B$).

$\therefore\ A = (G_1 \cup G_2) \cap A.$

$= (G_1 \cap A) \cup (G_2 \cap A)$.

Now, $G_1 \cap A$ and $G_2 \cap A$ are open in A.

Further, $(G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cap G_2) \cap A$.

$= (G_1 \cap G_2) \cap B$ (since $A \subseteq B$).

$= (G_1 \cap B) \cap (G_2 \cap B)$

$= B_1 \cap B_2$.

$= \Phi$.

$\therefore$ $(G_1 \cap A) \cap (G_2 \cap A) = \Phi$.

Now, since A is connected, either $G_1 \cap A = \Phi$ or $G_2 \cap A = \Phi$.

Without loss of generality let us assume that $G_1 \cap A = \Phi$.

Since G_1 is open in M, we have $G_1 \cap \bar{A} = \Phi$.

$\therefore$ $G_1 \cap B = \Phi$ (since $B \subseteq \bar{A}$)

$\therefore$ $B_1 = \Phi$ which is a contradiction.

$\therefore$ B is connected.

Problem 2. If A and B are connected subsets of a metric space M and if $A \cap B \neq \Phi$, prove that $A \cup B$ is connected.

Solution. Let $f : A \cup B \to \{0, 1\}$ be a continuous function.

Since $A \cap B \neq \Phi$, we can choose $x_0 \in A \cap B$.

Let $f(x_0) = 0$.

Since $f : A \cup B \to \{0, 1\}$ is continuous $f|_A : A \to \{0, 1\}$ is also continuous.

But A is connected.

Hence $f|_A$ is not onto. (by theorem 5.2)

$\therefore\ f(x) = 0$ for all $x \in A$ or $f(x) = 1$ for all $x \in A$.

But $f(x_o) = 0$ and $x_o \in A$.

$\therefore\ f(x) = 0$ for all $x \in A$.

Similarly $f(x) = 0$ for all $x \in B$.

$\therefore\ f(x) = 0$ for all $x \in A \cup B$.

Thus any continuous function $f : A \cup B \to \{0, 1\}$ is not onto.

$\therefore\ A \cup B$ is connected.

Exercises

1. Prove that $\{0, 1\}$ is not a connected subset of $\mathbf{R}$ with discrete metric.

2. Let $\{A_\alpha\}$ be a family of connected subsets of a metric space M such that $\cap A_\alpha \neq \Phi$. Then prove that $A = \cup A_\alpha$ is a connected subset of M.

3. Let $A_1, A_2, \ldots\ldots, A_n, \ldots\ldots$ be connected subsets of a metric space M each of which intersects its successor. Prove that $\bigcup_{n=1}^{\infty} A_n$ is connected.

4. Let M be a metric space and let $x_o \in M$. Let C be the union of all connected subsets of M which contains the point x_o. Show that C is a connected subset of M and is the largest connected subset of M which contains the point x_o. (C is called the **component** of the point x_o).

5. Prove that the set of all components of a metric space M forms a partition of M.

6. Prove that in a discrete metric space each component consists of a single point.

5.2 CONNECTED SUBSETS OF R

(10M)

Theorem 5.3 A subspace of **R** is connected iff it is an interval.

Proof. Let A be a connected subset of **R**.

Suppose A is not an interval.

Then there exist $a, b, c \in \mathbf{R}$ such that $a < b < c$ and $a, c \in A$ but $b \notin A$.

Let $A_1 = (-\infty, b) \cap A$ and $A_2 = (b, \infty) \cap A$.

Since $(-\infty, b)$ and (b, ∞) are open in $\mathbf{R}$, A_1 and A_2 are open sets in A.

Also $A_1 \cap A_2 = \Phi$ and $A_1 \cup A_2 = A$.

Further $a \in A_1$ and $c \in A_2$.

Hence $A_1 \neq \Phi$ and $A_2 \neq \Phi$.

Thus A is the union of two disjoint non-empty open sets A_1 and A_2.

Hence A is not connected which is a contradiction.

Hence A is an interval.

Conversely, let A be an interval.

We claim that A is connected.

Suppose A is not connected.

Let $A = A_1 \cup A_2$ where $A_1 \neq \Phi, A_2 \neq \Phi, A_1 \cap A_2 = \Phi$ and A_1 and A_2 are closed sets in A.

Choose $x \in A_1$ and $z \in A_2$.

Since $A_1 \cap A_2 = \Phi$ we have $x \neq z$.

Without loss of generality we assume that $x < z$.

Now, since A is an interval we have $[x, z] \subseteq A$.

(i.e.) $[x, z] \subseteq A_1 \cup A_2$.

$\therefore$ Every element of $[x, z]$ is either in A_1 or in A_2.

Now, let $y = \text{l.u.b.}\ \{[x, z] \cap A_1\}$.

Clearly $x \leq y \leq z$.

Hence $y \in A$.

Let $\varepsilon > 0$ be given. Then by the definition of l.u.b. there exists $t \in [x, z] \cap A_1$ such that $y - \varepsilon < t \leq y$.

$\therefore\ (y - \varepsilon, y + \varepsilon) \cap ([x, z] \cap A_1) \neq \Phi$.

$\therefore\ y \in \overline{[x, z] \cap A_1}$

$\therefore\ y \in [x, z] \cap A_1$ (since $[x, z] \cap A_1$ is closed in A).

$\therefore\ y \in A_1$. ..... (1)

Again by the definition of y, $y + \varepsilon \in A_2$ for all $\varepsilon > 0$ such that $y + \varepsilon \leq z$.

$\therefore\ y \in \overline{A_2}$.

$\therefore\ y \in A_2$ (since A_2 is closed). ..... (2)

$\therefore\ y \in A_1 \cap A_2$ [by (1) and (2)] which is a contradiction since $A_1 \cap A_2 = \Phi$.

Hence A is connected.

Theorem 5.4 **R** is connected.

Proof. $\mathbf{R} = (-\infty, \infty)$ is an interval.

$\therefore$ **R** is connected.

Solved problems.

Problem 1. Give an example to show that a subspace of a connected metric space need not be connected.

Solution. We know that **R** is connected.

$A = [1, 2] \cup [3, 4]$ is a subspace of $\mathbf{R}$ which is not connected. (refer example 1 of 5.1).

Problem 2. Prove or disporve *if A and C are connected subsets of a metric space M and if $A \subseteq B \subseteq C$, then B is connected.*

Solution. We disporve this statement by giving a counter example.

Let $A = [1, 2]$; $B = [1, 2] \cup [3, 4]$; $C = \mathbf{R}$.

Clearly $A \subset B \subset C$.

Here A and C are connected. But B is not connected.

Exercises

1. Prove that any connected subset of $\mathbf{R}$ containing more than one point is uncountable. (*Hint.* Any interval containing more than one point is uncountable).

2. Give an example to show that union of two connected sets need not be connected.

3. Determine which of the following are connected subsets of $\mathbf{R}$.

(i) (a, b)	(ii) $(a, b]$	(iii) $[a, b]$	(iv) $[a, b)$
(v) $[4, 6] \cup [8, 10]$	(vi) $[4, 6] \cup [5, 7]$	(vii) $[4, 6] \cap [5, 7]$	
(viii) $[4, 6] \cap [5, 7]$	(ix) $\{0\}$	(x) $(0, \infty)$	(xi) $(-\infty, 0)$
(xii) $\mathbf{Q}$	(xiii) $\mathbf{Z}$	(xiv) $\mathbf{R} - \{0\}$.	

5.3 CONNECTEDNESS AND CONTINUITY

Theorem 5.5 Let M_1 be a connected metric space. Let M_2 be any metric space. Let $f : M_1 \to M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of M_2

(i.e.) Any continuous image of a connected set is connected.

Proof. Let $f(M_1) = A$ so that f is a function from M_1 onto A.

We claim that A is connected.

Suppose A is not connected. Then there exists a proper non-empty subset B of A which is both open and closed in A.

$\therefore$ $f^{-1}(B)$ is a proper non-empty subset of M_1 which is both open and closed in M_1. Hence M_1 is not connected which is a contradiction.

Hence A is connected.

Theorem 5.6 Let f be a real valued continuous function defined on an interval I. Then f takes every value between any two values it assumes. (This is known as **the intermediate value theorem**).

Proof. Let $a, b \in I$ and let $f(a) \neq f(b)$.

Without loss of generality we assume that $f(a) < f(b)$.

Let c be such that $f(a) < c < f(b)$.

The interval I is a connected subset of **R**.

$\therefore$ $f(I)$ is a connected subset of **R**. (by theorem 5.5)

$\therefore$ $f(I)$ is an interval. (by theorem 5.3)

Also $f(a), f(b) \in f(I)$. Hence $[f(a), f(b)] \subseteq f(I)$.

$\therefore$ $c \in f(I)$ [since $f(a) < c < f(b)$]

$\therefore$ $c = f(x)$ for some $x \in I$.

Solved problem

Problem 1. Prove that if f is a non-constant real valued continuous function on **R** then the range of f is uncountable.

Solution. We know that **R** is connected.

Since f is a continuous function on $\mathbf{R}$, $f(\mathbf{R})$ is a connected subset of $\mathbf{R}$.

$\therefore$ $f(\mathbf{R})$ is an interval in $\mathbf{R}$.

Also, since f is a non-constant function the interval. $f(\mathbf{R})$ contains more than one point.

$\therefore$ $f(\mathbf{R})$ is uncountable. (ie) The range of f is uncountable.

Exercises

1. Prove that if $f : \mathbf{R} \to \mathbf{R}$ is a continuous function which assumes only rational values then f is a constant function. (*Hint.* Use intermediate value theorem).

2. Prove that $A = \{(x, y) / x^2 + y^2 = 1\}$ is a connected subset of $\mathbf{R}^2$.
[*Hint:* Consider $f : [0, 2\pi] \to A$ given by $f(x) = (\cos x, \sin x)$.

Revision questions on chapter 5

Determine which of the following statements are true and which are false.

1. $\mathbf{R}$ is connected.

2. Q is connected.

3. A subspace of a connected space is connected.

4. If A and B are connected subsets of a metric space M then $A \cup B$ is connected.

5. If A and B are connected subsets of M and $A \cap B \neq \Phi$ then $A \cup B$ is connected.

6. Any discrete metric space having more than one point is disconnected.

7. If M is a metric space and $x \in M$ then $\{x\}$ is a connected subset of M.

8. A subset of a discrete metric space is connected iff it is a singleton set.

9. Continuous image of a connected set is connected.

Answers. 1, 5, 6, 7, 8 and 9 are true.

COMPACTNESS

6.0 INTRODUCTION

UNIT - V ↳ closed & bounded

We have seen that the concept of completeness is the abstraction of a property of the real number system. The concept of compactness is also an abstraction of an important property possessed by subsets of **R** which are closed and bounded. This property is known as Heine Borel theorem which states that if $I \subseteq \mathbf{R}$ is a closed interval, any family of open intervals in **R** whose union contains I has a finite subfamily whose union contains I. We now introduce the class of *compact metric spaces* in which the conclusion of Heine Borel theorem is valid.

6.1 COMPACT METRIC SPACES

Definition. Let M be a metric space. A family of open sets $\{G_\alpha\}$ in M is called an **open cover** for M if $\cup G_\alpha = M$.

A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a **subcover.**

A metric space M is said to be **compact** if every open cover for M has finite subcover.

(i.e) for each family of open sets $\{G_\alpha\}$ such that $\cup G_\alpha = M$, there exist a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}\}$ such that $\bigcup\limits_{i=1}^{n} G_{\alpha_i} = M$.

Example 1. **R** with usual metric is not compact.

Proof. Consider the family of open intervals $\{(-n, n)/n \in \mathbf{N}\}$.

This is a family of open sets in **R**.

Every finite set is compact & closed.

Clearly $\bigcup_{n=1}^{\infty} (-n, n) = \mathbf{R}$.

$\therefore$ $\{(-n, n)/n \in \mathbf{N}\}$ is an open cover for $\mathbf{R}$ and this open cover has no finite subcover.

$\therefore$ $\mathbf{R}$ is not compact.

Example 2. (0, 1) with usual metric is not compact.

Proof. Consider the family of open intervals $\left\{\left(\frac{1}{n}, 1\right)/n = 2, 3, \ldots\right\}$

Clearly $\bigcup_{n=2}^{\infty} \left(\frac{1}{n}, 1\right) = (0, 1)$.

$\therefore$ $\left\{\left(\frac{1}{n}, 1\right)/n = 2, 3, \ldots\right\}$ is an open cover for (0, 1) and this open cover has no finite subcover.

Hence (0, 1) is not compact.

Example 3. $[0, \infty)$ with usual metric is not compact.

Proof. Consider the family of intervals $\{[0, n)/n \in \mathbf{N}\}$.

$[0, n)$ is open in $[0, \infty)$ for each $n \in \mathbf{N}$.

Also $\bigcup_{n=1}^{\infty} [0, n) = [0, \infty)$.

$\therefore$ $\{[0, n)/n \in \mathbf{N}\}$ is an open cover for $[0, \infty)$ and this open cover has no finite subcover.

Hence $[0, \infty)$ is not compact.

Example 4. Let M be an infinite set with discrete metric. Then M is not compact.

Proof. Let $x \in M$. Since M is a discrete metric space $\{x\}$ is open in M.

Also $\bigcup_{x \in M} \{x\} = M$.

Hence $\{\{x\}/x \in M\}$ is an open cover for M and since M is infinite, this open cover has no finite subcover.

Hence M is not compact.

Example 5. We will prove in 6.2 that any closed interval $[a, b]$ with usual metric is compact.

Theorem 6.1 Let M be a metric space. Let $A \subseteq M$. A is compact iff given a family of open sets $\{G_\alpha\}$ in M such that $\cup G_\alpha \supseteq A$ there exists a subfamily

$G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Proof. Let A be a compact subset of M.

Let $\{G_\alpha\}$ be a family of open sets in M such that $\cup G_\alpha \supseteq A$.

Then $(\cup G_\alpha) \cap A = A$.

$\therefore \cup (G_\alpha \cap A) = A$.

Also $G_\alpha \cap A$ is open in A. (refer theorem 2.6)

$\therefore$ The family $\{G_\alpha \cap A\}$ is an open cover for A.

Since A is compact this open cover has a finite subcover, say, $G_{\alpha_1} \cap A, G_{\alpha_2} \cap A, \ldots\ldots, G_{\alpha_n} \cap A$.

$$\therefore \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A.$$

$$\therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap A = A.$$

$$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$$

Conversely let $\{H_\alpha\}$ be an open cover for A.

$\therefore$ Each H_α is open in A.

$\therefore\ H_\alpha = G_\alpha \cap A$ where G_α is open in M.

Now, $\cup H_\alpha = A$.

$\therefore\ \cup (G_\alpha \cap A) = A$.

$\therefore\ (\cup G_\alpha) \cap A = A$

$\therefore\ \cup G_\alpha \supseteq A$.

Hence by hypothesis there exists a finite subfamily

$$G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n} \text{ such that } \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$$

$$\therefore\ \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap A = A.$$

$$\therefore\ \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A.$$

$$\therefore\ \bigcup_{i=1}^{n} H_{\alpha_i} = A.$$

Thus $\{H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}\}$ is a finite subover of the open cover $\{H_\alpha\}$.

$\therefore\ A$ is compact.

Theorem 6.2 Any compact subset A of a mectric space M is bounded.

Proof. Let $x_o \in A$.

Consider $\{B(x_o, n) / n \in \mathbf{N}\}$.

Clearly $\bigcup_{n=1}^{\infty} B(x_o, n) = M$.

$$\therefore\ \bigcup_{n=1}^{\infty} B(x_o, n) \supseteq A.$$

Since A is compact there exists a finite subfamily say,

$B(x_o, n_1)$, $B(x_o, n_2)$, ..., $B(x_o, n_k)$ such that $\bigcup_{i=1}^{k} B(x_{0,} n_1) \supseteq A$.

Let $n_o = \max\{n_1, n_2, \ldots, n_k\}$.

Then $\bigcup_{i=1}^{k} B(x_o, n_i) = B(x_o, n_0)$.

$\therefore\ B(x_o, n_o) \supseteq A$.

We know that $B(x_o, n_o)$ is a bounded set and a subset of a bounded set is bounded. Hence A is bounded.

Note. The converse of the above theorem is not true.

For example, $(0, 1)$ is a bounded subset of $\mathbf{R}$.

But it is not compact. (refer example 2 of 6.1)

Theorem 6.3 Any compact subset A of a metric space (M, d) is closed.

Proof. To prove that A is closed we shall prove that A^c is open.

Let $y \in A^c$ and let $x \in A$. Then $x \neq y$.

$\therefore\ d(x, y) = r_x > 0$.

It can be easily verified that $B(x, \frac{1}{2} r_x) \cap B(y, \frac{1}{2} r_x) = \Phi$.

Now consider the collection $\{B(x, \frac{1}{2} r_x) / x \in A\}$.

Clearly $\bigcup_{x \in A} B(x, \frac{1}{2} r_x) \supseteq A$.

Since A is compact there exists a finite number of such open balls say,

$B(x_1, \frac{1}{2} r_{x_1})$, ..., $B(x_n, \frac{1}{2} r_{x_n})$ such that $\bigcup_{i=1}^{n} B(x_i, \frac{1}{2} r_{x_i}) \supseteq A$.

..... (1)

Now, let $V_y = \bigcap_{i=1}^{n} B(y, \frac{1}{2}r_{x_i})$.

Clearly V_y is an open set containing y.

Since $B(y, \frac{1}{2}r_y) \cap B(x, \frac{1}{2}r_x) = \Phi$, we have $V_y \cap B(x, \frac{1}{2}r_{x_i}) = \Phi$ for each $i = 1, 2, \ldots, n$.

$\therefore V_y \cap \left[\bigcup_{i=1}^{n} B(x, \frac{1}{2}r_{x_i}) \right] = \Phi.$

$\therefore V_y \cap A = \Phi$ [by (1)].

$\therefore V_y \subseteq A^c$.

$\therefore \bigcup_{y \in A^c} V_y = A^c$ and each V_y is open.

$\therefore A^c$ is open. Hence A is closed.

Note 1. The converse of the above theorem is not true.

For example, $[0, \infty)$ is a closed subset of **R**. But it is not compact (refer eg. 3 of 6.1)

Note 2. It follows from theorems 6.2 and 6.3 that any compact subset of a metric space is closed and bounded.

Theorem 6.4 A closed subspace of a compact metric space is compact.

Proof. Let M be a compact metric space. Let A be a non-empty closed subset of M.

We claim that A is compact.

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in M such that

$$\bigcup_{\alpha \in I} G_\alpha \supseteq A.$$

$$\therefore\ A^c \cup \left[\bigcup_{\alpha \in I} G_\alpha\right] = M.$$

Also A^c is open. (since A is closed).

$\therefore\ \{G_\alpha / \alpha \in I\} \cup \{A^c\}$ is an open cover for M.

Since M is compact it has a finite subcover say

$$G_{\alpha_1},\ G_{\alpha_2}, \ldots, G_{\alpha_n}, A^c.$$

$$\therefore\ \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cup A^c = M.$$

$$\therefore\ \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$$

$\therefore\ A$ is compact.

Exercises

1. Give an example of an open cover which has no finite subcover for the following subsets of **R**.

(i) $(5, 6)$ (ii) $(5, \infty)$ (iii) $[5, \infty)$ (iv) $[7, 9)$

2. Show that every finite metric space is compact.

3. Give an example of a connected subset of **R** which is not compact. (Hint. Any interval in **R** is connected)

4. A and B are two compact subsets of a metric space M. Prove that $A \cup B$ is also compact.

6.2 COMPACT SUBSETS OF R.

We have already proved that every compact subset of a metric space is closed and bounded.

However the converse is not true.

For example, consider an infinite discrete metric space (M, d).

Let A be an infinite subset of M.

Then A is bounded since $d(x, y) \leq 1$ for all $x, y \in A$.

Also A is closed since any subset of a discrete metric space is closed.

Hence A is closed and bounded.

However A is not compact (refer example 4 of 6.1)

In this section we shall prove that for **R** with usual metric the converse is also true.

Theorem 6.5 (Heine Borel theorem)

Any closed interval $[a, b]$ *is a compact subset of* **R**.

Proof. Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in **R** such that $\bigcup_{\alpha \in I} G_\alpha \supseteq [a, b]$.

Let $S = \{x / x \in [a, b]$ and $[a, x]$ can be covered by a finite number of G_α's.

Clearly $a \in S$ and hence $S \neq \Phi$.

Also S is bounded above by b.

Let c denote the $l.u.b.$ of S.

Clearly $c \in [a, b]$.

$\therefore$ $c \in G_{\alpha_1}$ for some $\alpha_1 \in I$.

Since G_{α_1} is open, there exists $\varepsilon > 0$ such that

$$(c - \varepsilon, c + \varepsilon) \subseteq G_{\alpha_1}.$$

Choose $x_1 \in [a, b]$ such that $x_1 < c$ and $[x_1, c] \subseteq G_{\alpha_1}$.

Now, since $x_1 < c$, $[a, x_1]$ can be covered by a finite number of G_α's.

These finite number of G_α's together with G_{α_1} covers $[a, c]$.

$\therefore$ By definition of S, $c \in S$.

Now, we claim that $c = b$.

Suppose $c \neq b$.

Then choose $x_2 \in [a, b]$ such that $x_2 > c$ and $[c, x_2] \subseteq G_{\alpha_1}$.

As before, $[a, x_2]$ can be covered by a finite number of G_α's.

Hence $x_2 \in S$.

But $x_2 > c$ which is a contradiction, since c is the *l. u. b.* of S.

$\therefore$ $c = b$.

$\therefore$ $[a, b]$ can be covered by a finite number of G_α's.

$\therefore$ $[a, b]$ is a compact subset of **R**.

Theorem 6.6 A subset A of **R** is compact iff A is closed and bounded.

Proof. If A is compact then A is closed and bounded.

Conversely, let A be subset of **R** which is closed and bounded.

Since A is bounded we can find a closed interval $[a, b]$ such that $A \subseteq [a, b]$.

Since A is closed in **R**, A is closed in $[a, b]$ also.

Thus A is a closed subset of the compact space $[a, b]$.

Hence A is compact. (by theorem 6.4)

Exercises

1. Determine which of the following subset of **R** are comapct.

   (i) **Z** (ii) **Q** (iii) $[1, 2]$

   (iv) $(3, 4)$ (v) $(0, \infty)$ (vi) $[1, 2] \cup [3, 4]$

(vii) $[1, 3] \cap [3, 4]$ (viii) $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$

(ix) $\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$.

2. If A and B are compact subsets of $\mathbb{R}$ prove that $A \cap B$ is also a compact subset of $\mathbb{R}$.

6.3 EQUIVALENT CHARACTERISATIONS FOR COMPACTNESS

In this section we obtain several equivalent characterisation for compactness in a metric space.

Definition. A family $\mathfrak{I}$ of subsets of a set M is said to have the **finite intersection property** if any finite members of $\mathfrak{I}$ have non-empty intersection.

Example. In $\mathbb{R}$ the family of closed intervals $\mathfrak{I} = \{[-n, n] / n \in \mathbb{N}\}$ has finite intersection property.

Theorem 6.7 A metric space M is compact iff any family of closed sets with finite intersection property has non-empty intersection.

Proof. Suppose M is compact.

Let $\{A_\alpha\}$ be a family of closed subsets of M with finite intersection property.

We claim that $\cap A_\alpha \neq \Phi$.

Suppose $\cap A_\alpha = \Phi$ then $(\cap A_\alpha)^c = \Phi^c$.

$\therefore \cup A_\alpha^c = M$.

Also, since each A_α is closed, A_α^c is open.

$\therefore \{A_\alpha^c\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say,

$A_1{}^c$, $A_2{}^c$, ... , $A_n{}^c$.

$$\therefore \bigcup_{i=1}^{n} A_i^c = M.$$

$$\therefore \left(\bigcap_{i=1}^{n} A_i\right)^c = M.$$

$\therefore \bigcap_{i=1}^{n} A_i = \Phi$ which is a contradiction to the finite intersection property.

$$\therefore \cap A_\alpha \neq \Phi.$$

Conversely, suppose that each family of closed sets in M with finite intersection property has non empty intersection.

To prove that M is compact, let $\{G_\alpha / \alpha \in I\}$ be an open cover for M.

$$\therefore \bigcup_{\alpha \in I} G_\alpha = M.$$

$$\therefore \left(\bigcup_{\alpha \in I} G_\alpha\right)^c = M^c.$$

$$\therefore \bigcap_{\alpha \in I} G_\alpha^c = \Phi.$$

Since G_α is open , G_α^c is closed for each α.

$\therefore \Im = \{G_\alpha^c / \alpha \in I\}$ is a family of closed sets whose intersection is empty.

Hence by hypothesis this family of closed sets does not have the finite intersection property.

Hence there exists a finite sub-collection of $\Im$ say,

$$\{G_1^c , G_2^c , ... G_n^c\} \text{ such that } \bigcap_{i=1}^{n} G_i^c = \Phi.$$

$$\therefore \left(\bigcup_{i=1}^{n} G_i\right)^c = \Phi.$$

$$\therefore \bigcup_{i=1}^{n} G_i = M.$$

$\therefore \{G_1, G_2, \ldots\ldots\ldots, G_n\}$ is a finite subcover of the given open cover.

Hence M is compact.

Definition. A metric space M is said to be **totally bounded** if for every $\varepsilon > 0$ there exists a finite number of elements $x_1, x_2, \ldots\ldots, x_n \in M$ such that $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup B(x_n, \varepsilon) = M$.

A non-empty subset A of a metric space M is said to be totally bounded if the subspace A is a totally bounded metric space.

Theorem 6.8 Any compact metric space is totally bounded.

Proof. Let M be a compact metric space.

Then $\{B(x, \varepsilon)/x \in M\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say, $B(x_1, \varepsilon), B(x_2, \varepsilon), \ldots\ldots, B(x_n, \varepsilon)$.

$\therefore\ M = B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup (B(x_n, \varepsilon)$.

$\therefore\ M$ is totally bounded.

Theorem 6.9 Let A be a subset of a metric space M. If A is totally bounded then A is bounded.

Proof. Let A be a totally bounded subset of M. Let $\varepsilon > 0$ be given.

Then there exists a finite number of points $x_1, x_2, \ldots\ldots, x_n \in A$, such that $B_1(x_1, \varepsilon) \cup B_1(x_2, \varepsilon) \cup \ldots \cup B_1(x_n, \varepsilon) = A$, where $B_1(x_i, \varepsilon)$ is an open ball in A.

Further we know that an open ball is a bounded set.

Thus A is the union of a finite number of bounded sets and hence A is bounded.

Note. The converse of the above theorem is not true.

For, let M be an infinite set with discrete metric.

Clearly M is bounded.

Now, $B(x, \frac{1}{2}) = \{x\}$.

Since M is infinite, M cannot be written as the union of a finite number of open balls $B(x, \frac{1}{2})$.

$\therefore$ M is not totally bounded.

Definition. Let (x_n) be sequence in a metric space M.
Let $n_1 < n_2 < \ldots\ldots < n_k < \ldots\ldots$ be an increasing sequence of positive integers. Then (x_{n_k}) is called a subsequence of (x_n).

Theorem 6.10 A metric space (M, d) is totally bounded iff every sequence in M has a Cauchy subsequence.

Proof. Suppose every sequence in M has a Cauchy subsequence.

We claim that M is totally bounded.

Let $\varepsilon > 0$ be given. Choose $x_1 \in M$.

If $B(x_1, \varepsilon) = M$ then obviously M is totally bounded.

If $B(x_1, \varepsilon) \neq M$, choose $x_2 \in M - B(x_1, \varepsilon)$ so that $d(x_1, x_2) \geq \varepsilon$.

Now, if $B(x_1, \varepsilon) \cup (Bx_2, \varepsilon) = M$ the proof is complete.

If not choose $x_3 = M - [B(x_1, \varepsilon) \cup B(x_2, \varepsilon)]$ and so on.

Suppose this process does not stop at a finite stage.

Then we obtain a sequence $x_1, x_2, \ldots\ldots, x_n, \ldots\ldots$ such that $d(x_n, x_m) \geq \varepsilon$ if $n \neq m$.

Clearly this sequence (x_n) cannot have a Cauchy subsequence which is a contradiction.

Hence the above process stops at a finite stage and we get a finite set of points $\{x_1, x_2, \ldots\ldots, x_n\}$ such that

$$M = B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup B(x_n, \varepsilon).$$

$\therefore$ M is totally bounded.

Conversely suppose M is totally bounded.

Let $S_1 = \{x_{i_1}, x_{i_2}, \ldots\ldots, x_{i_n} \ldots\ldots\}$ be a sequence in M.

If one term of the sequence is infinitely repeated then S_1 contains a constant subsequence which is obviously a Cauchy subsequence.

Hence we assume that no term of S_1 is infinitely repeated so that the range of S is infinite.

Now, since M is totally bounded M can be covered by a finite number of open balls of radius $\frac{1}{2}$.

Hence atleast one of these balls must contain an infinite number of terms of the sequence S_1.

$\therefore$ S_1 contains a subsequence $S_2 = (x_{2_1}, x_{2_2}, \ldots\ldots, x_{2_n}, \ldots\ldots)$ all terms of which lie within an open ball of radius $\frac{1}{2}$.

Similarly S_2 contains a sub sequence $S_3 = (x_{3_1}, \ldots\ldots, x_{3_n}, \ldots\ldots)$ all terms of which lie within an open ball of radius $\frac{1}{3}$.

We repeat this process of forming successive subsequences and finally we take the diagonal sequence.

$S = (x_{1_1}, x_{2_2}, \ldots\ldots x_{n_n} \ldots\ldots)$.

We claim that S is a Cauchy subsequence of S_1.

If $m > n$ both x_{m_m} and x_{n_n} lie within an open ball of radius $\frac{1}{n}$.

$\therefore\ d(x_{m_m}, x_{n_n}) < \frac{2}{n}$.

Hence $d(x_{m_m}, x_{n_n}) < \varepsilon$ if $n, m > \frac{2}{\varepsilon}$.

This shows that S is a Cauchy subsequence of S_1.

Thus every sequence in M contains a Cauchy subsequence.

Corollary. A non-empty subset of a totally bounded set is totally bounded.

Proof. Let A be a totally bounded subset of a metric space M.

Let B be a non-empty subset of A.

Let (x_n) be a sequence in B.

$\therefore$ (x_n) is a sequence in A.

Since A is totally bounded (x_n) has a Cauchy subsequence.

Thus every sequence in B has a Cauchy subsequence.

$\therefore$ B is totally bounded.

Definition. A metric space M is said to be **sequentially compact** if every sequence in M has a convergent sub-sequence.

Theorem 6.11. Let (x_n) be a Cauchy sequence in a metric space M. If (x_n) has a subsequence $\left(x_{n_k}\right)$ converging to x, then (x_n) converges to x.

Proof. Let $\varepsilon > 0$ be given. Since (x_n) is a Cauchy sequence, there exists a positive integer m_1 such that $d(x_n, x_m) < \frac{1}{2}\varepsilon$ for all $n, m \geq m_1$ ..... (1)

Also, since $(x_{n_k}) \to x$, there exists a positive integer m_2 such that

$$d(x_{n_k}, x) < \frac{1}{2}\varepsilon \quad \text{for all } n_k \geq m_2 \qquad \ldots\ldots (2)$$

Let $m_o = \max\{m_1, m_2\}$ and fix $n_k \geq m_o$.

Then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ for all } n \geq m_o \qquad [\text{ by (1) and (2) }]$$

$$= \varepsilon \text{ for all } n \geq m_o.$$

Hence $(x_n) \to x$..

Theorem 6.12 In a metric space M the following are equivalent.

(i) M is compact.

(ii) Any infinite subset of M has a limit point.

(iii) M is sequentially compact.

(iv) M is totally bounded and complete.

Proof. (i) $\Rightarrow$ (ii)

Let A be an infinite subset of M.

Suppose A has no limit point in M.

Let $x \in M$.

Since x is not a limit point of A there exists an open ball $B(x, r_x)$ such that $B(x, r_x) \cap (A - \{x\}) = \Phi$.

$$\therefore\ B(x, r_x) \cap A = \begin{cases} \{x\} & \text{if } x \in A \\ \Phi & \text{if } x \notin A \end{cases}$$

Now, $\{B(x, r_x) / x \in M\}$ is open cover for M.

Also each $B(x, r_x)$ covers atmost one point of the infinite set A.

Hence this open cover cannot have a finite sub cover which is a contradiction to (i). Hence A has atleast one limit point.

(ii) ⇒ (iii).

Let (x_n) be a sequence in M.

If one term of the sequence is infinitely repeated then (x_n) contains a constant subsequence which is convergent.

Otherwise (x_n) has an infinite number of terms.

By hypothesis this infinite set has a limit point, say x.

By theorem 2.15 for any $r > 0$ the open ball $B(x, r)$ contains infinite number of terms of the sequence (x_n).

Now, choose a positive integer n_1 such that $x_{n_1} \in B(x, 1)$.

Then choose $n_2 > n_1$ such that $x_{n_2} \in B(x, \frac{1}{2})$.

In general for each positive integer k choose n_k such that $n_k > n_{k-1}$ and $x_{n_k} \in B(x, \frac{1}{k})$.

Clearly (x_{n_k}) is a subsequence of (x_n).

Also $d(x_{n_k}, x) < \frac{1}{k}$.

$\therefore$ $(x_{n_k}) \to x$.

Thus (x_{n_k}) is a convergent subsequence of (x_n).

Hence M is sequentially compact.

(iii) ⇒ (iv).

By hypothesis every sequence in M has a convergent subsequence. But every convergent sequence is a Cauchy sequence.

Thus every sequence in M has a Cauchy subsequence.

$\therefore$ By theorem 6.10, M is totally bounded.

Now we prove that M is complete.

Let (x_n) be a Cauchy sequence in M.

By hypothesis (x_n) contains a convergent subsequence (x_{n_k}).

Let $(x_{n_k}) \to x$. (say)

Then by theorem 6.11, $(x_n) \to x$.

$\therefore$ M is complete.

(iv) $\Rightarrow$ (i)

Suppose M is not compact.

Then there exists an open cover $\{G_\alpha\}$ for M which has no finite subcover.

Let $r_n = \frac{1}{2^n}$.

Since, M is totally bounded. M can be covered by a finite number of open balls of radius r_1.

Since M cannot be covered by a finite number of G_α's atleast one of these open balls, say $B(x_1, r_1)$ cannot be covered by a finite number of G_α's

Now, $B(x_1, r_1)$ is totally bounded.

Hence as before we can find $x_2 \in B(x_1, r_1)$ such that $B(x_2, r_2)$ cannot be covered by a finite number of G_α's.

Proceeding like this we obtain a sequence (x_n) in M such that $B(x_n, r_n)$ cannot be covered by a finite number of G_α's and

$$x_{n+1} \in B(x_n, r_n) \text{ for all } n.$$

Now, $d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \text{------} + d(x_{n+p-1}, x_{n+p})$

$$< r_n + r_{n+1} + \text{-----} + r_{n+p-1}$$

$$= \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots\dots + \frac{1}{2^{n+p-1}}$$

$$= \frac{1}{2^{n-1}}\left(\frac{1}{2} + \frac{1}{2^2} + \dots\dots + \frac{1}{2^p}\right)$$

$$< \frac{1}{2^{n-1}}.$$

$\therefore$ (x_n) is a Cauchy sequence in M.

Since M is complete there exists $x \in M$ such that $(x_n) \to x$.

Now, $x \in G_\alpha$ for some α.

Since G_α is open we can find $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq G_\alpha$. ..... (1)

We have $(x_n) \to x$ and $(r_n) = \left(\frac{1}{2^n}\right) \to 0$.

Hence we can find a positive integer n_1 such that $d(x_n, x) < \frac{1}{2}\varepsilon$ and $r_n < \frac{1}{2}\varepsilon$ for all $n \geq n_1$.

Now, fix $n \geq n_1$.

We claim that $B(x_n, r_n) \subseteq B(x, \varepsilon)$.

Let $y \in B(x_n, r_n)$

$\therefore$ $d(y, x_n) < r_n < \frac{1}{2}\varepsilon$ (since $n \geq n_1$)

Now, $d(y, x) \leq d(y, x_n) + d(x_n, x)$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

$\therefore$ $y \in B(x, \varepsilon)$.

$\therefore$ $B(x_n, r_n) \subseteq B(x, \varepsilon) \subseteq G_\alpha$ (by (1))

Thus $B(x_n, r_n)$ is covered by the single set G_α which is a contradiction since $B(x_n, r_n)$ cannot be covered by a finite number of G_α 's.

Hence M is compact.

Theorem 6.13. $\mathbf{R}$ with usual metric is complete.

Proof. Let (x_n) be a Cauchy sequence in $\mathbf{R}$.

Then (x_n) is a bounded sequence and hence is contained in a closed interval $[a, b]$.

Now, $[a, b]$ is compact and hence is complete.

Hence (x_n) converges to some point $x \in [a, b]$.

Thus every Cauchy sequence (x_n) in $\mathbf{R}$ converges to some point x in $\mathbf{R}$ and hence $\mathbf{R}$ is complete.

Solved problems

Problem 1. Give an example of a closed and bounded subset of l_2 which is not compact.

Solution. Consider $\mathbf{0} = (0, 0, 0, \ldots\ldots) \in l_2$.

Consider the closed ball $B[\mathbf{0}, 1]$.

Clearly, $B[\mathbf{0}, 1]$ is bounded.

Also, $B[\mathbf{0}, 1]$ is a closed set.

We claim that $B[\mathbf{0}, 1]$ is not compact.

Consider $e_1 = (1, 0, 0, \ldots)$; $e_2 = (0, 1, 0, \ldots)$; ......

.... $e_n = (0, 0, 0, \ldots, 1, 0, \ldots)$.

Now, $d(\mathbf{0}, e_n) = 1$ and hence $e_n \in B[0, 1]$ for all n.

Thus (e_n) is a sequence in $B[\mathbf{0}, 1]$.

Also $d(e_n, e_m) = \sqrt{2}$ if $n \neq m$.

Hence the sequence (e_n) does not contain a Cauchy subsequence.

$\therefore$ $B[0, 1]$ is not totally bounded.

$\therefore$ $B[0, 1]$ is not compact.

oblem 2. Prove that any totally bounded metric space is separable.

lution. Let M be a totally bounded metric space.

For each natural number n let $A_n = \{x_{n_1}, x_{n_2}, \ldots, x_{n_k}\}$ be a bset of M such that $\bigcup_{i=1}^{k} B\left(x_{n_i}, \frac{1}{n}\right) = M$. ..... (1)

Let $A = \bigcup_{n=1}^{\infty} A_n$.

Since each A_n is finite, A is a countable subset of M.

We claim that A is dense in M.

Let $B(x, \varepsilon)$ be any open ball.

Choose a natural number n such that $\frac{1}{n} < \varepsilon$.

Now, $x \in B\left(x_{n_i}, \frac{1}{n}\right)$ for some i. (by (1)).

$\therefore$ $d(x_{n_i}, x) < \frac{1}{n} < \varepsilon$.

$\therefore$ $(x_{n_i}) \in B(x, \varepsilon)$.

$\therefore$ $B(x, \varepsilon) \cap A \neq \Phi$.

Thus every open ball in M has non-empty intersection with A. lence by theorem 2.17, A is dense in M.

Thus A is a countable dense subset of M.

Hence M is separable.

Problem 3. Prove that any bounded sequence in **R** has a convergent sub-sequence.

Solution. Let (x_n) be a bounded sequence in **R**.

Then there exists a closed interval $[a, b]$ such that $x_n \in [a, b]$ for all n.

Thus (x_n) is a sequence in the compact metric space $[a, b]$.

Hence by theorem 6.12, (x_n) has a convergent sub-sequence.

Problem 4. Prove that the closure of a totally bounded set is totally bounded.

Solution. Let A be a totally bounded subset of a metric space M.

We claim that $\overline{A}$ is totally bounded.

We shall show that every sequence in $\overline{A}$ contains a Cauchy subsequence.

Let (x_n) be a sequence in $\overline{A}$.

Let $\varepsilon > 0$ be given.

Then since $x_n \in \overline{A}$, $B(x_n, \frac{1}{3}\varepsilon) \cap A \neq \Phi$.

Choose $y_n \in B(x_n, \frac{1}{3}\varepsilon) \cap A$.

$\therefore \; d(y_n, x_n) < \frac{1}{3}\varepsilon.$ ..... (1)

Now, (y_n) is a sequence in A. Since A is totally bounded (y_n) contains a Cauchy sequence say (y_{n_k}).

Hence there exists a natural number m such that

$$d(y_{n_i}, y_{n_j}) < \frac{1}{3}\varepsilon \text{ for all } n_i, n_j \geq m \qquad \text{..... (2)}$$

$$\therefore \; d(x_{n_i}, x_{n_j}) \leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, y_{n_j}) + d(y_{n_j}, x_{n_j})$$

$$< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon \text{ for all } n_i, n_j \geq m. \text{ (by (1) and (2))}$$

Hence (x_{n_k}) is a Cauchy subsequence of (x_n).

$\therefore \overline{A}$ is totally bounded.

Problem 5. Let A be a totally bounded subset of $\mathbf{R}$. Prove that $\overline{A}$ is compact.

Solution. Since A is totally bounded $\overline{A}$ is also totally bounded.

Also, since $\overline{A}$ is a closed subset of $\mathbf{R}$ and $\mathbf{R}$ is complete $\overline{A}$ is complete.

(refer theorem 3.4)

Hence $\overline{A}$ is totally bounded and complete.

$\therefore \overline{A}$ is compact. (refer theorem 6.12)

Exercises

1. Let M be a complete metric space. Prove that a closed subset A of M is compact iff A is totally bounded.

2. Give an example of a complete metric space which is not compact.

3. Prove that a connected subset of a discrete metric space M is compact. (*Hint.* Any connected subset of M is a singleton set.)

4. Prove that a compact metric space is separable.

5. Prove that any bounded infinite subset of $\mathbf{R}$ has a limit point.

6. Prove that any Cauchy sequence in a metric space is totally bounded. (*Hint:* Use theorem 6.10)

6.4 COMPACTNESS AND CONTINUITY

In this section we prove some results about continuous functions defined on a compact metric space. These are generalizations of the corresponding results for continuous real valued functions defined on any closed interval $[a, b]$.

Theorem 6.14. Let f be a continuous mapping from a compact metric space M_1 to any metric space M_2. Then $f(M_1)$ is compact.

(i.e.,) *Continuous image of a compact metric space is compact.*

Proof. Without loss of generality we assume that $f(M_1) = M_2$.

Let $\{G_\alpha\}$ be a family of open sets in M_2 such that $\cup G_\alpha = M_2$.

$\therefore \cup G_\alpha = f(M_1)$

$\therefore f^{-1}(\cup G_\alpha) = M_1.$

$\therefore \cup f^{-1}(G_\alpha) = M_1.$

Also since f is continuous $f^{-1}(G_\alpha)$ is open in M_1 for each α.

$\therefore \{f^{-1}(G_\alpha)\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover, say, $f^{-1}(G_{\alpha_1}), \ldots\ldots, f^{-1}(G_{\alpha_n})$.

$$\therefore f^{-1}(G_{\alpha_1}) \cup f^{-1}(G_{\alpha_2}) \cup \ldots\ldots \cup f^{-1}(G_{\alpha_n}) = M_1.$$

$$\therefore f^{-1}\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) = M_1.$$

$$\therefore \bigcup_{i=1}^{n} G_\alpha = f(M_1) = M_2.$$

$\therefore G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}$ is a cover for M_2.

Thus the given open cover $\{G_\alpha\}$ for M_2 has a finite subcover.

$\therefore$ M_2 is compact.

Corollary 1. Let f be a continuous map from a compact metric space M_1 into any metric M_2. Then $f(M_1)$ is closed and bounded.

Proof. $f(M_1)$ is compact and hence is closed and bounded.

Corollary 2. Any continuous real valued function f defined on a compact metric space is bounded and attains its bounds.

Proof. Let M be a compact metric space.

Let $f : M \to \mathbf{R}$ be a continuous real valued function.

Then $f(M)$ is a compact subset of $\mathbf{R}$

$\therefore$ $f(M)$ is a closed and bounded subset of $\mathbf{R}$.

Since $f(M)$ is bounded f is a bounded function.

Now, let $a = l.u.b.$ of $f(M)$ and $b = g.l.b.$ of $f(M)$.

By definition of $l.u.b.$ and $g.l.b.$ $a, b \in \overline{f(M)}$.

But $f(M)$ is closed. Hence $f(M) = \overline{f(M)}$.

$\therefore$ $a, b \in f(M)$

$\therefore$ There exist $x, y \in M$ such that $f(x) = a$ and $f(y) = b$.

Hence f attains its bounds.

Note. Corollary (2) is not true if M is not compact.

The function $f : (0, 1) \to \mathbf{R}$ defined by $f(x) = 1/x$ is continuous but not bounded.

The function $g : (0, 1) \to \mathbf{R}$ defined by $g(x) = x$ is bounded having $l.u.b. = 1$ and $g.l.b. = 0$. However this function never attains these bounds at any point in $(0, 1)$.

Theorem 6.15. Any continuous mapping f defined on a compact metric space (M_1, d_1) into any other metric space (M_2, d_2) is uniformly continuous on M_1.

Proof. Let $\varepsilon > 0$ be given. Let $x \in M_1$.

Since f is continuous at x there exists $\delta_x > 0$ such that

$$d_1(y, x) < \delta_x \Rightarrow d_2(f(y), f(x)) < \frac{1}{2}\varepsilon. \qquad \ldots\ldots (1)$$

Now, the family of open balls $\{B(x, \frac{1}{2}\delta_x) / x \in M_1\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover say $B(x_1, \frac{1}{2}\delta_{x_1}), \ldots\ldots B(x_n, \frac{1}{2}\delta_{x_n})$.

Let $\delta = \min\{\frac{1}{2}\delta_{x_1}, \ldots\ldots \frac{1}{2}, \delta_{x_n}\}$

We claim that $d_1(p, q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon$.

Let $p \in B(x_i, \frac{1}{2}\delta_{x_i})$ for some i where $1 \leq i \leq n$.

$\therefore\ d_1(p, x_i) < \frac{1}{2}\delta_{x_i}$.

$\therefore\ d_2(f(p), f(x_i)) < \frac{1}{2}\varepsilon$ (by (1)) ..... (2)

Now, $d_1(q, x_i) \leq d_1(q, p) + d_1(p, x_i)$

$$\leq \delta + \frac{1}{2}\delta_{x_i}$$

$$\leq \frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{x_i} = \delta_{x_i}.$$

Thus $d_1(q, x_i) < \delta_{x_i}$

$\therefore\ d_2(f(q), f(x_i)) < \frac{1}{2}\varepsilon$ (by (1)) ..... (3)

$$\text{Now, } d_2(f(p), f(q)) \le d_2(f(p), f(x_i)) + d_2(f(x_i), f(q))$$

$$< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \quad (\text{by (2) and (3)})$$

Thus $d_1(p, q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon$.

This proves that f is uniformly continuous on M_1.

Note. The above theorem is not true if M_1 is not compact.

We have seen that if f is a continuous bijection then f^{-1} need not be continuous. Now we shall prove that if f is a continuous bijection defined on a compact metric space, then f^{-1} is also continuous.

Theorem 6.16 Let f be a 1 - 1 continuous function from a compact metric space M_1 onto any metric space M_2. Then f^{-1} is continuous on M_2. Hence f is a homeomorphism from M_1 onto M_2.

Proof. We shall show that f^{-1} is continuous by proving that

F is a closed set in $M_1 \Rightarrow (f^{-1})^{-1}(F) = f(F)$ is a closed set in M_2.

Let F be a closed set in M_1.

Since M_1 is compact F is compact. (by theorem 6.4).

Since f is continuous $f(F)$ is a compact subset of M_2.

$\therefore$ $f(F)$ is a closed subset of M_2.

$\therefore$ f^{-1} is continuous on M_2.

Solved problems

Problem 1. Prove that the range of a continuous real valued function f on a compact connected metric space M must be either a single point or a closed and bounded interval.