number.

①

Complex Numbers

1. The algebra of Complex Numbers :-

1.1 Arithmatic operations

A Complex Number z is of the form $x+iy$ where x and y are real Numbers and i is an imaginary unit with the property $i^2=-1$.

x and y are called the real and imaginary parts of z. and we write $x = \text{Re } z$, $y = \text{Im } z$. The complex number z is called purely imaginary. if $x=0$, If $y=0$ z is called purely real.

Two Complex numbers and said to be equal iff they have the same real and imaginary parts.

Let C denote the set of all Complex numbers then $C = \{x+iy / x, y \in R\}$

we defined addition and Multiplication is as follows

$$(\alpha+i\beta).(\gamma+i\delta) = (\alpha\gamma-\beta\delta)+i(\alpha\delta+\beta\gamma))$$

By division $\dfrac{\alpha+i\beta}{\gamma+i\delta} = \dfrac{(\alpha+i\beta)(\gamma-i\delta)}{(\gamma+i\delta)(\gamma-i\delta)}$

$$= \frac{(\alpha\gamma+\beta\delta)+i(\beta\gamma-\alpha\delta)}{\gamma^2+\delta^2}$$

provided $\gamma+i\delta \neq 0$

Conjugation and modulus

Let $z = x + iy$ be a Complex Number then the Complex number $x - iy$ is called the Conjugate of z and it is denoted by $\bar{z}$.

The mapping $f : C \to C$ defined $f(z) = \bar{z}$ is called the Complex Conjugation.

Results

(i) $z = \bar{z}$ iff z is real.

(ii) $\bar{\bar{z}} = z$

(iii) $x = Re\, z = \dfrac{z + \bar{z}}{2}$

$y = Im z = \dfrac{z - \bar{z}}{2}$

Since $z = x + iy$

$\& \; \bar{z} = x - iy$

$z + \bar{z} = 2x$

$x = \dfrac{z + \bar{z}}{2}$

III^{ly} $y = \dfrac{z - \bar{z}}{2i}$

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Proof: Let $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in R$

Then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2)$

$= (x_1 - iy_1) + (x_2 - iy_2)$

(v) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

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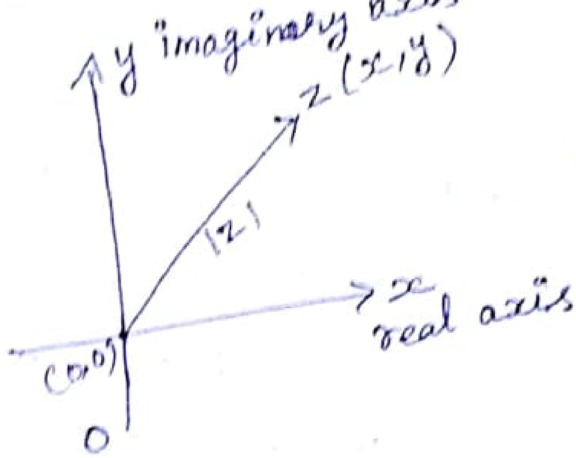
Let $z = x+iy$ is a Complex number. The modulus or the absolute value of z denoted by $|z|$ and it is defined by $|z| = \sqrt{x^2+y^2}$ and $|z|$ represents the distance between $z(x,y)$ and the origin.



$z = x+iy$
$= (x,y)$
$Oz = \sqrt{(x-0)^2+(y-0)^2}$
$= \sqrt{x^2+y^2}$

Results:-

(i) $|z| \geq 0$.

(ii) $z\bar{z} = |z|^2$

(iii) $|z_1 z_2| = |z_1||z_2|$

$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ Provided $z_2 \neq 0$

For any three complex numbers z, z_1 and z_2

(i) $-|z| \leq Re\, z \leq |z|$

(ii) $-|z| \leq Im z \leq |z|$

(iii) $|z_1+z_2| \leq |z_1|+|z_2|$ } Traingle is equality

(iv) $|z_1-z_2| \geq ||z_1|-|z_2||$ } Traiangle in equality.

Proff: Let $z = x+iy$

Then $|z| = \sqrt{x^2+y^2}$

$\therefore -\sqrt{x^2+y^2} \leq x \leq \sqrt{x^2+y^2}$

& $-\sqrt{x^2+y^2} \leq y \leq \sqrt{x^2+y^2}$

(ie) $-|z| \leq Re\,z \leq |z|$

$-|z| \leq Im\,z \leq |z|$.

(iii) $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$

$\leq (z_1+z_2)(\bar{z_1}+\bar{z_2})$

$\leq z_1\bar{z_1} + z_1\bar{z_2} + z_2\bar{z_1} + z_2\bar{z_2}$

$\leq |z_1|^2 + 2Re(z_1\bar{z_2}) + |z_2|^2$

$\leq |z_1|^2 + 2|z_1\bar{z_2}| + |z_2|^2$ by (1)

$\leq |z_1|^2 + 2|z_1||\bar{z_2}| + |z_2|^2$

$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ ($\because |\bar{z}| = |z|$)

$= (|z_1|+|z_2|)^2$

$\therefore |z_1+z_2| \leq |z_1| + |z_2|$

(iv) $z_1 = z_1 - z_2 + z_2$

$|z_1| = |z_1 - z_2 + z_2|$

$\leq |z_1 - z_2| + |z_2|$ by (iii)

$|z_1| - |z_2| \leq |z_1 - z_2| \longrightarrow (1)$

$z_2 = (z_2 - z_1) + z_1$

$|z_2| - |z_1| \leq |z_1 - z_2|$

$|z_1| - |z_2| \geq -|z_1 - z_2| \longrightarrow (2)$

From (1) & (2)

$-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2|$

is $||z_1| - |z_2|| \leq |z_1 - z_2|$

Results:

i) $z = \bar{z}$ iff, z is a real.

ii) $\bar{\bar{z}} = z$.

iii) $x = \operatorname{Re} z$.

$$x = \frac{z+\bar{z}}{2}$$

$$z = x+iy.$$

$$\bar{z} = x-iy.$$

$$z+\bar{z} = 2x.$$

(ie) $2x = z+\bar{z}$

then $\boxed{x = \frac{z+\bar{z}}{2}}$ (*)(*)

III^{ly}.

$\boxed{y = \frac{z-\bar{z}}{2i}}$ (*)(*)

iv) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$.

Proof: $z_1 = x_1+iy_1$

$z_2 = x_2+iy_2$

where $x_1, x_2, y_1 \& y_2 \in \mathbb{R}$

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Let $z = x + iy$

then $|z| = \sqrt{x^2+y^2}$

(i) $-|z| \le x \le |z|$

(ii) $-\sqrt{x^2+y^2} \le x \le \sqrt{x^2+y^2}$

(iii) $-\sqrt{x^2+y^2} \le y \le \sqrt{x^2+y^2}$

$$\boxed{\begin{aligned}(ii) \Rightarrow \quad & -|z| \le Re\, z \le |z| \\ & -|z| \le Im \le |z|.\end{aligned}}$$

— x —

$$\boxed{|z|^2 = z\bar{z}}$$

$$|z_1 + z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$$

$$\boxed{z = z_1 + z_2}$$

$$|z|^2 = z\bar{z}$$

$$\le (z_1+z_2)(\bar{z_1}+\bar{z_2})$$

$$\le z_1\bar{z_1} + z_1\bar{z_2} + z_2\bar{z_1} + z_2\bar{z_2}$$

$$\le |z_1|^2 + |z_2|^2 + z_1\bar{z_2} + z_2\bar{z_1}$$

$$\le |z_1|^2 + |z_2|^2 + 2\,Re(z_1\bar{z_2})$$

— x —

$$\leq |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2$$

$$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$$

$$\leq |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$$

$$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$\boxed{\text{ie } |\bar{z}| = |z|}$

$$|z_1 + z_2| \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|.$$

ie $|z_1 + z_2| \leq (|z_1| + |z_2|)^2$

— x —

(iv) $z_1 = z_1 - z_2 + z_2$

$$|z_1| = |z_1 - z_2| + |z_2|$$

$$\leq |z_1 - z_2| + |z_2|$$

ie $|z_1| - |z_2| \leq |z_1 - z_2|$. — (1)

$$z_2 = z_2 - z_1 + z_1$$

$$= (z_2 - z_1) + z_1$$

$$|z_2| \leq |z_2 - z_1| + |z_1|$$

$$|z_2| - |z_1| \leq |z_2 - z_1|$$

ie $|z_2| - |z_1| \leq |z_2 - z_1|$

$|z_1| - |z_2| \geq -|z_1 - z_2|$ — (2)

from (1) & (2),

$$-|z_1 - z_2| \le |z_1| - |z_2| \le |z_1 - z_2|$$

———x———

V). $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

[Parallelogram law]

Soln. :

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 =$$
$$(z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)}$$
$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$
$$= z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{z_2\bar{z}_1} + z_2\bar{z}_2 + z_1\bar{z}_1 - \cancel{z_1\bar{z}_2} - \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$$
$$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_1 + z_2\bar{z}_2$$
$$= |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2$$
$$= 2|z_1|^2 + 2|z_2|^2$$
$$= 2\left[|z_1|^2 + |z_2|^2\right]$$

———x———

Results

1. Zero is the only number which is at once real and purely Imaginary.

2. The reciprocal of a complex number $\neq 0$ is given by

$$\frac{1}{\alpha+i\beta} = \frac{\alpha-i\beta}{\alpha^2+\beta^2}$$

3. i^n. $n=0,1,2,\ldots$

$$\boxed{\begin{aligned} i^n &= i^0 = 1 \\ i^n &= i^1 = i \\ i^n &= i^2 = -1 \\ i^n &= i^3 = i^2 \times i = -1 \times i = -i \end{aligned}}$$

$i^n = i^4 = i^2 \times i^2 = -1 \times -1 = 1$

$i^n = i^5 = i^4 \times i = 1 \times i = i$

$i^n = i^6 = i^3 \times i^3 = -i \times -i = i$

$i^n = i^7 = i^4 \times i^3 =$

i^n has only four values.

(ie) $1, i, -1, -i$.

They corresponds to the values of n which divided by 4, leave the remainder 0,1,2,3

iii) i^n has 4 values
(i.e.) $i^0 : i^1 : i^2 : i^3$

(11).

Find the values of

① $(1+2i)^3$

Soln: $(1+2i)^3 = (1+2i)^2 (1+2i)$

$= (1+(4i^2)+2\times1\times2i)(1+2i)$

$= (1-4+4i)(1+2i)$

$= (4i-3)(1+2i)$

$= 4i+8i^2-3-6i$

$= 4i-8-3-6i$

$= -11-2i$

$$\boxed{\therefore (1+2i)^3 = -11-2i}$$

— x —

② $\dfrac{5}{-3+4i}$

Soln: $\dfrac{5}{-3+4i} = \dfrac{5}{-3+4i} * \dfrac{-3-4i}{-3-4i}$

$= \dfrac{5(-3-4i)}{3^2+4^2}$

$= \dfrac{-15-20i}{9+16} = \dfrac{-15-20i}{25}$

$= \dfrac{-15}{25} - \dfrac{20}{25}i = \dfrac{-3}{5} - \dfrac{4i}{5}$

(i.e.) $\boxed{5/-3+4i = -3/5 - 4i/5}$

iii) $\left(\frac{2+i}{3-2i}\right)^2$

Soln:

$$\left(\frac{2+i}{3-2i}\right)^2 = \left(\frac{2+i}{3-2i}\right)\left(\frac{2+i}{3-2i}\right)$$

$$= \frac{4+2i+2i+i^2}{9-6i-6i+4i^2}$$

$$= \frac{4+4i-1}{9-12i+4(-1)}$$

$$= \frac{4i+3}{-12i+9-4}$$

$$= \frac{3+4i}{5-12i}$$

$$\Rightarrow \frac{3+4i}{5-12i} \times \frac{5+12i}{5+12i}$$

$$= \frac{(3+4i)(5+12i)}{5^2+12^2}$$

$$= \frac{15+36i+20i+48i^2}{25+144.}$$

$$= \frac{15+56i-48}{169}$$

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i has 4 values
0 . 1 . 2 . 3

$$= \frac{56i - 33}{169}$$

$$= \frac{-33}{169} + \frac{56i}{169}$$

$$\boxed{\therefore \left(\frac{2+i}{3-2i}\right)^2 = \frac{-33}{169} + \frac{56i}{169}}$$

—— x ——

$Z = x + iy$, $(x, y \in R)$ find the Real & Imaginary parts of $1/z$.

Soln: $z = x + iy$.

$$1/z = \frac{1}{x+iy}$$

$$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{x - iy}{x^2+y^2}$$

$$\text{Real } \frac{1}{z} = \frac{x}{x^2+y^2}$$

$$\text{Ima. } 1/z = -y/x^2+y^2.$$

—— x ——

Assignment : I

Find the real & Im. Parts of i) $\frac{z-1}{z+1}$. (ii) $\frac{1}{z^2}$

Show that $\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1$ for all combination of signs.

Soln:

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = \left(\frac{-1 \pm i\sqrt{3}}{2}\right)\left(\frac{-1 \pm i\sqrt{3}}{2}\right)\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

$$= \left[\left(\frac{-1 \pm i\sqrt{3}}{2}\right)\left(\frac{-1 \pm i\sqrt{3}}{2}\right)\right]\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

$$= \frac{1}{4}\left(1 - i\sqrt{3} - i\sqrt{3} + (-3)\right)\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

$$= \frac{1}{4}\left(-2 - 2i\sqrt{3}\right)\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

$$= \frac{2}{4}\left(\left(-1 - i\sqrt{3}\right)\right)\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left(-1 - i\sqrt{3}\right)\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

$$= \frac{1}{4}\left(1 - i\sqrt{3} + i\sqrt{3} + 3\right)$$

$$= \frac{1}{4} \times 4 = 1.$$

iii) i^n has 4 values

(ie) i^0, i^1, i^2, i^3

$\Rightarrow 1, i, -1, -i$

They corresponds to values of n, which divided by 4, leave the remainder 0, 1, 2, 3.

Find the values of

(i) $(1+2i)^3$

Sol: $(1+2i)^3 = (1+2i)^2 (1+2i)$

$\Rightarrow [1+(2i)^2+(2\times 2i\times 1)]\ [(1+2i)]$

$\Rightarrow (4i-3)\ (1+2i)$

$\Rightarrow 4i + 8i^2 - 3 - 6i$

$\Rightarrow 4i+8(-1)-3-6i$

$\Rightarrow 4i-8-3-6i$

$\boxed{(1+2i)^3 = -11-2i}$

ii) $\frac{5}{-3+4i}$

Soln: $\frac{5}{-3+4i} = \frac{5}{-3+4i} \times \frac{-3-4i}{-3-4i}$

$\Rightarrow \frac{5\times(-3-4i)}{3^2+4^2} \Rightarrow \frac{5(-3-4i)}{9+16}$

$\Rightarrow \frac{5(-3-4i)}{25}$

$\Rightarrow \frac{-15-20i}{25}$

$\Rightarrow \frac{-15}{25} - \frac{20i}{25}$

$$\Rightarrow \frac{-3}{5} - \frac{4i}{5}$$

$$\boxed{\therefore \frac{5}{-3+4i} = -\frac{3}{5} - \frac{4i}{5}}$$

— x —

iii) $\left(\frac{2+i}{3-2i}\right)^2$

Soln:

$$\left(\frac{2+i}{3-2i}\right)\left(\frac{2+i}{3-2i}\right)$$

$$\Rightarrow \frac{4+2i+2i+i^2}{9-6i-6i+4i^2}$$

$$\Rightarrow \frac{4+4i+i^2}{9-12i+4i^2}$$

$$\Rightarrow \frac{4+4i+(-1)}{9-12i+4(-1)}$$

$$\Rightarrow \frac{4+4i-1}{9-12i-4}$$

$$\Rightarrow \frac{3+4i}{5-12i}$$

$$\Rightarrow \frac{3+4i}{5-12i} * \frac{5+12i}{5+12i}$$

$$\Rightarrow \frac{-33+56i}{169} = \frac{-33}{169} + \frac{56}{169}i$$

$$\boxed{\therefore \left(\frac{2+i}{3-2i}\right)^2 = \frac{-33}{169} + \frac{56}{169} i}$$

— x —

Find the real & Imaginary Parts of the following:

i) Let $z = x + iy$.

To Prove $\frac{1}{z} = \frac{1}{x+iy}$.

$$\Rightarrow \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$\therefore$ The Real Part of $z = \frac{x}{x^2+y^2}$

Im. Part of $z = \frac{-y}{x^2+y^2}$

ii) $\frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1}$

$$\Rightarrow \frac{(x+iy)-1}{(x+iy)+1} \times \frac{(x+iy)-1}{(x+iy)-1}$$

$$\boxed{\text{Real } \frac{x^2+y^2-1}{(x+1)^2+y^2}} \quad \boxed{\text{Im}: \frac{2y}{(x+1)^2+y^2}}$$

iii) $\dfrac{1}{z^2} = \dfrac{1}{(x+iy)^2}$

$\Rightarrow \dfrac{1}{x^2+2xy+y^2}$

$\Rightarrow \dfrac{1}{(x^2-y^2)+2ixy}$

$\Rightarrow \dfrac{1}{(x^2-y^2)+2ixy} \times \dfrac{(x^2-y^2)-2ixy}{(x^2-y^2)-2ixy}$

$\Rightarrow \dfrac{(x^2-y^2)-2ixy}{(x^2-y^2)^2+4x^2y^2}$

$\therefore$ Real of $\dfrac{1}{z^2} = \dfrac{x^2-y^2}{(x^2-y^2)^2+4x^2y^2}$

Imaginary of $\dfrac{1}{z^2} = \dfrac{-2xy}{(x^2-y^2)^2+4x^2y^2}$

Show that $\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1$

Soln :

$\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)^3 = \left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)$

$= \left[\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)\right]\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)$

$= \dfrac{1}{4}\left[1 - i\sqrt{3} - i\sqrt{3} + (-3)\right]\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)$

$= \dfrac{1}{4}\left(-2 - 2i\sqrt{3}\right)\left(\dfrac{-1 \pm i\sqrt{3}}{2}\right)$

$\Rightarrow \frac{2}{4} \left[-1 - i\sqrt{3} \right] \left(\frac{-1 \pm i\sqrt{3}}{2} \right)$

$\Rightarrow \frac{1}{2} \left(-1 - i\sqrt{3} \right) \left(\frac{-1 \pm i\sqrt{3}}{2} \right)$

$\Rightarrow \frac{1}{4} \left(1 - i\sqrt{3} + i\sqrt{3} + 3 \right)$

$\Rightarrow \frac{1}{4} (4)$

$\Rightarrow 4/4 = 1$

$$\boxed{\therefore \frac{-1 \pm i\sqrt{3}}{2} = 1}$$

— x —

Prove that the absolute value of the Product = the product of the absolute value.

Proof:

Let $|z_1 z_1|^2 = (z_1 z_2) (\overline{z_1 z_2})$

$\Rightarrow (z_1 z_2) (\bar{z_1} \bar{z_2})$

$\Rightarrow z_1 \bar{z_1} \cdot z_2 \bar{z_2}$

$\Rightarrow |z_1|^2 |z_2|^2$

$\therefore$ The absolute value of the product = The product of the absolute value

— x —

Prove that the absolute value of the sum is atmost equal to the sum of the absolute value.

Proof:

Let $z_1 = \frac{z_1}{z_2} \cdot z_2$

(ie) $|z_1| = \left|\frac{z_1}{z_2} \cdot z_2\right|$

$\Rightarrow |z_1| = \left|\frac{z_1}{z_2}\right| \cdot |z_2|$.

Now,

$z_1 + z_2 + z_3 + \cdots + z_n = z_1\left(1 + \frac{z_2}{z_1} + \frac{z_3}{z_1} + \cdots + \frac{z_n}{z_1}\right)$

$\therefore |z_1 + z_2 + \cdots + z_n| = |z_1| \left[1 + \frac{z_2}{z_1} + \frac{z_3}{z_1} + \cdots + \frac{z_n}{z_1}\right]$

$\leq |z_1| \left[1 + \frac{|z_2|}{|z_1|} + \frac{|z_3|}{|z_1|} + \cdots + \frac{|z_n|}{|z_1|}\right]$

$= |z_1| + |z_2| + \cdots + |z_n|$

(ie) $|z_1 + z_2 \cdots + z_n| = |z_1| + |z_2| + \cdots + |z_n|$

$\therefore$ The absolute value of the sum is atmost equal to the sum of the absolute value of the terms.

— x —

Square root of any complex number:-

Let $a+ib$ be given any complex number.

Let us assign $x+iy$ be the square root of $a+ib$.

(i.e) $\sqrt{a+ib} = x+iy$.

(ie) $a+ib = (x+iy)^2$

$$= x^2 + y^2 i^2 + 2(x)(iy)$$

$$= x^2 - y^2 + 2ixy.$$

Now, we equating the real & Imaginary parts of the above equation

(ie) $\boxed{a = x^2 - y^2}$ — ①

$\boxed{b = 2xy}$ — ②

$$(x^2+y^2)^2 = x^4 + y^4 + 2x^2y^2 \boxed{\begin{matrix} -2x^2y^2 \\ +2x^2y^2 \end{matrix}}$$

$$\Rightarrow x^4 + y^4 - 2x^2y^2 + 4x^2y^2$$

$$\Rightarrow (x^2 - y^2) + 4x^2y^2$$

$$\therefore (x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$= a^2 + b^2$ — ③ $\boxed{\because \text{from ① \& ②}}$

$$\Rightarrow \boxed{x^2+y^2 = \sqrt{a^2+b^2}}$$

from ① & ③.

① ⇒ $x^2 - y^2 = a$

③ ⇒ $x^2 + y^2 = \sqrt{a^2+b^2}$

(+)

$2x^2 = a + \sqrt{a^2+b^2}$

$\therefore 2x^2 = a + \sqrt{a^2+b^2}$

$x^2 = \dfrac{a+\sqrt{a^2+b^2}}{2}$

$x = \pm\left(\dfrac{a+\sqrt{a^2+b^2}}{2}\right)^{1/2}$

again from ① & ③.

① ⇒ $x^2 - y^2 = a$

③ ⇒ $x^2 + y^2 = \sqrt{a^2+b^2}$

(−)

$-2y^2 = a - \sqrt{a^2+b^2}$

$\therefore 2y^2 = -a + \sqrt{a^2+b^2}$

$y^2 = \dfrac{-a+\sqrt{a^2+b^2}}{2}$

$\therefore y = \pm\left(\dfrac{-a+\sqrt{a^2+b^2}}{2}\right)^{1/2}$

$\therefore$ $x+iy$ be a square root of $a+ib$.

where $x = \pm\left(\dfrac{a+\sqrt{a^2+b^2}}{2}\right)^{1/2}$

& $y = \pm\left(\dfrac{-a+\sqrt{a^2+b^2}}{2}\right)^{1/2}$

From (2)

if $b=0$, z is a real

if $b>0$, x & y have same sign

if $b<0$, x & y are in opposite sign.

Geometrical representation of a complex number:

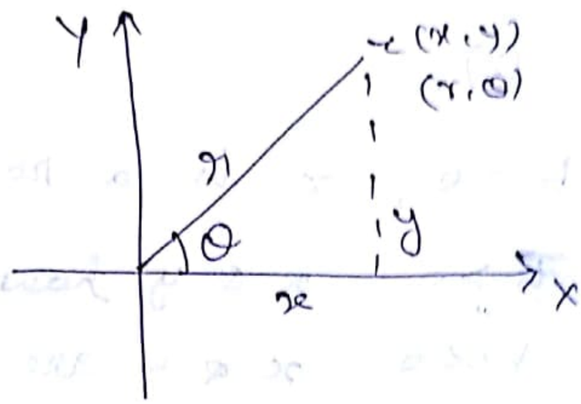
We represent any complex number $x+iy$ by a point (x,y) in $R\times R$.

The plane $R\times R$ representing the complex numbers is called the complex plane.

Polar form of a Complex Number:

Consider any non zero complex number $z = x + iy$.

Let (r, θ) denote the polar co-ordinates of the point (x, y)



Complex plane.

From the above figure

$\therefore \frac{x}{r} = \cos\theta$ (ie) $\boxed{x = r\cos\theta}$ | $\frac{y}{r} = \sin\theta$ (ie) $\boxed{y = r\sin\theta}$

Now, w.r.t

$$z = x + iy$$

$$z = r\cos\theta + i\, r\sin\theta.$$

$$= r(\cos\theta + i\sin\theta)$$

Now, $|z| = \sqrt{x^2 + y^2}$

$$= \sqrt{r^2(\cos\theta + i\sin\theta)}$$

$$= \sqrt{r^2} = r.$$

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

which is the magnitude of the complex number and θ is called the amplitude (or) argument and it is denoted by ~~arg z~~

$\boxed{\arg z = \text{amp } z.}$

The value of arg z lies between $-\pi$ to π and it's called the principal value of arg z.

—x—

Theorems:

1. If z_1 & z_2 are any two non zero complex numbers then

i) $\arg z_1 = -\arg \bar{z}_1$

ii) $\arg z_1 z_2 = \arg z_1 + \arg z_2$

iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Proof: $\arg z_1 = -\arg \bar{z}_1$

i) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\therefore \bar{z}_1 = r_1(\cos\theta_1 - i\sin\theta_1)$

$= r_1[\cos(-\theta_1) + i\sin(-\theta_1)]$

hence $\arg z_1 = r_1[(\cos\theta_1 + i\sin\theta_1)]$

$\therefore \arg \bar{z}_1 = -\theta_1$

$= -\arg z_1$

(ie) $\arg \bar{z}_1 = -\arg z_1$

ii) $\arg z_1 z_2 = \arg z_1 + \arg z_2$

Proof:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Now $z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 [\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\cos\theta_2\sin\theta_1 + i^2\sin\theta_1\sin\theta_2]$

$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i((\sin\theta_1\cos\theta_2) + \cos\theta_1\sin\theta_2)]$

$= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$$\boxed{\Rightarrow \arg z_1 z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2}$$

$\cos a+b = \cos a\cos b - \sin a\sin b$

$\sin a+b = \sin a\cos b + \cos a\sin b$

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iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Soln:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \left(\frac{r_1}{r_2}\right)\left(\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}\right)$$

$$= \left(\frac{r_1}{r_2}\right)\left(\frac{(\cos\theta_1 + i\sin\theta_1)\,(\cos(-\theta_2) + i\sin(-\theta_2))}{\cos^2\theta_2 + \sin^2\theta_2}\right)$$

$$= \frac{r_1}{r_2}\,\frac{\left[\cos\theta_1\cos(-\theta_2) + i(\sin\theta_1\sin(-\theta_2))\right]}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

by (ii)

$\therefore \frac{z_1}{z_2} = \theta_1 - \theta_2$

$\therefore \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$

Hence the theorem.

Theorem.

Let $z = r(\cos\theta + i\sin\theta)$ be any non zero complex number and n be any integer, then $z^n = r^n(\cos n\theta + i\sin n\theta)$

Proof:

Given: z be any complex number

To prove: $z^n = r^n(\cos n\theta + i\sin \theta)$

First we have to prove this result for positive integers by induction on 'n'.

The result is obviously true when $n=1$.

Suppose, the result is true for $n=m$.

Hence $z^m = r^m(\cos m\theta + i\sin m\theta)$

Now, $z^{m+1} = z^m . z = r^m(\cos m\theta + i\sin m\theta)$
$r(\cos\theta + i\sin\theta)$

$\Rightarrow r^{m+1}(\cos(m+1)\theta + i\sin(m+1)\theta)$

Hence the result is true for $n=m+1$.

Hence, $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all positive integers n.

The result is obviously true if $n=0$.

How, $z^{-1} = 1/z = \dfrac{1}{r(\cos\theta + i\sin\theta)}$] <0.

$$= \frac{1}{r}\,\frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= r^{-1}\left[\cos(-\theta) + i\sin(-\theta)\right]$$

$\therefore$ The result is also true for $n=-1$.

Hence it follows that the result is true for all negative integers.

Hence $z^n = r^n(\cos n\theta + i\sin n\theta)\ \forall\ n \in \mathbb{Z}$.

Note:

The corollary of the above theorem is De-Moivre's theorem.

(ie) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Prove the following:

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1 \text{ if either } |a| = 1$$

or $|b| = 1$.

Proof:

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \left(\frac{a-b}{1-\bar{a}b}\right)\overline{\left(\frac{a-b}{1-\bar{a}b}\right)}$$

$$= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\frac{\bar{a}-\bar{b}}{1-a\bar{b}}\right)$$

$$= \frac{a\bar{a}-a\bar{b}-\bar{a}b+b\bar{b}}{1-a\bar{b}-\bar{a}b+a\bar{a}b\bar{b}}$$

$$= \frac{|a|^2-a\bar{b}-b\bar{a}+|b|^2}{1-a\bar{b}-\bar{a}b+|a|^2|b|^2}$$

$$= 1.$$

If $|a|=1$, then

$$\left|\frac{a-b}{1-\bar{a}b}\right| = \frac{|a|^2-a\bar{b}-b\bar{a}+|b|^2}{1-a\bar{b}-\bar{a}b+|b|^2|a|^2}$$

$$= 1.$$

If $|b|=1$, then

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{|a|^2-a\bar{b}-b\bar{a}+|b|^2}{1-a\bar{b}-\bar{a}b+|b|^2|a|^2}$$

$\therefore \left|\frac{a-b}{1-\bar{a}b}\right| = 1$ if either $|a|=1$ (or) $|b|=1$.

— x —

Prove that $\left|\frac{a-b}{1-\bar{a}b}\right| < 1$, if $|a|<1$ & $|b|<1$.

Solution:

we already proved,

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{|a|^2 - a\bar{b} - b\bar{a} + |b|^2}{1 - a\bar{b} - \bar{a}b + |a|^2|b|^2}$$

$$< \frac{2 - a\bar{b} - b\bar{a}}{2 - a\bar{b} - \bar{a}b}$$

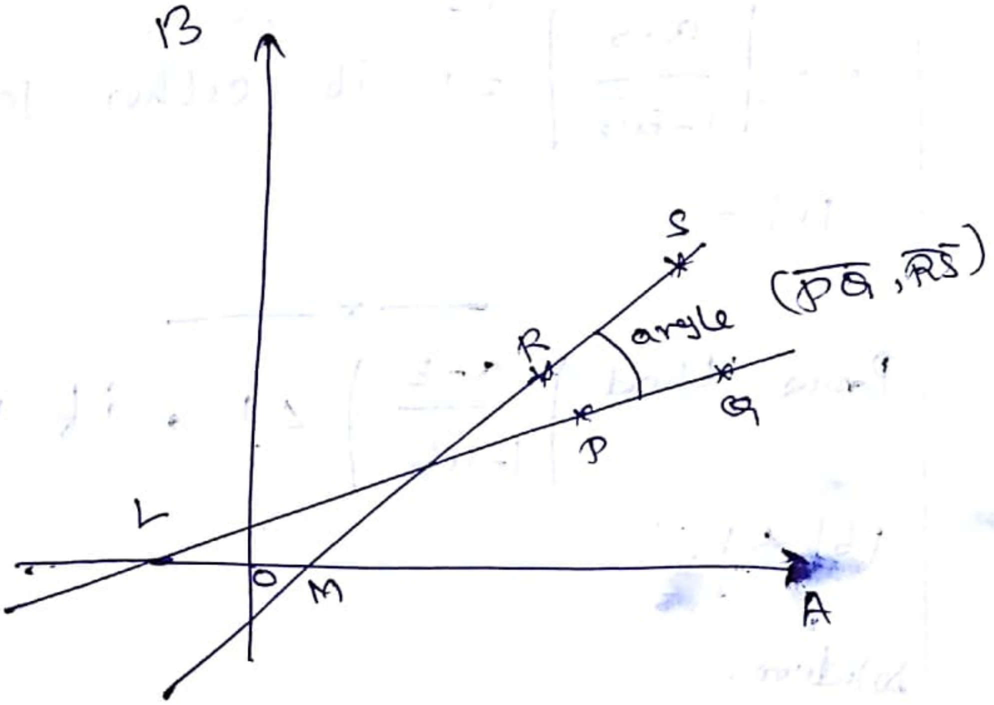
$$= 1$$

$\therefore \left|\frac{a-b}{1-\bar{a}b}\right| < 1$ if $|a|<1$ & $|b|<1$.

— x —

Angle between two rays:-

Let $P(z_1)$, $Q(z_2)$, $R(z_3)$ & $S(z_4)$ be the four points in the complex plane



The angle measured from the vector PQ to the vector RS is denoted by the notation angle $(\overline{PQ}, \overline{RS})$.

It is positive if the sense of measurement is anticlockwise and otherwise it is negative.

Now,

$$\text{angle}\,(\overline{PQ}, \overline{RS}) = \text{angle } RMA - \text{angle } PLA$$

$$= \arg(z_4 - z_3) - \arg(z_2 - z_1)$$

$$= \arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$$

$$= \arg\left(\frac{z_3 - z_4}{z_1 - z_2}\right).$$

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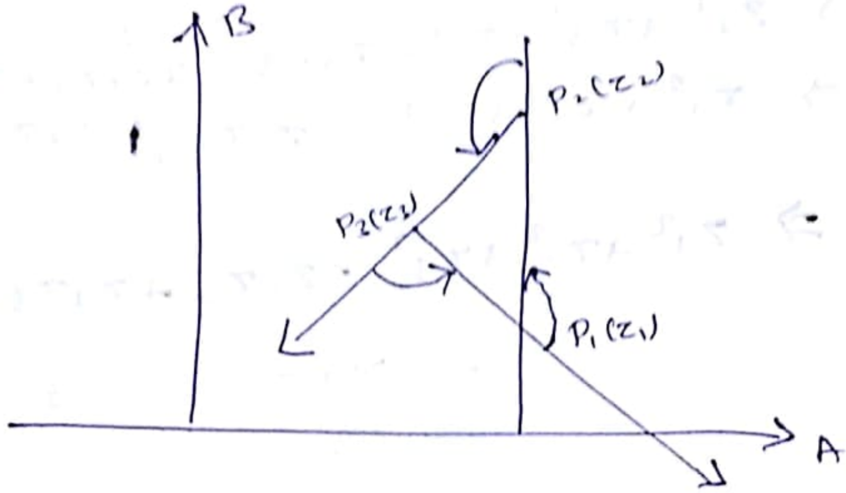
Let z_1, z_2 and z_3 be the vertices of an ~~equilateral~~ equilateral triangle, so that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$



From the figure,

$$\boxed{\begin{aligned} &\text{angle}\,(\overline{P_1P_2}, \overline{P_2P_3}) \\ &= \text{angle}\,(\overline{P_2P_3}, \overline{P_3P_1}) \end{aligned}}$$

(i.e) $\arg\left(\dfrac{z_2 - z_3}{z_1 - z_2}\right) = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_3}\right)$ — (1)

& also $\left|\dfrac{z_2 - z_3}{z_1 - z_2}\right| = \left|\dfrac{z_3 - z_1}{z_2 - z_3}\right|$ — (2)

Since $|z_1 - z_2|, |z_2 - z_3|$ & $|z_3 - z_1|$ are equal.

From (1) & (2), $\dfrac{z_2 - z_3}{z_1 - z_2} = \dfrac{z_3 - z_1}{z_2 - z_3}$

(ii) $(z_2 - z_3)(z_2 - z_3) = (z_1 - z_2)(z_3 - z_1)$

$\Rightarrow (z_2 - z_3)^2 = z_1 z_3 - z_1^2 - z_2 z_3 + z_1 z_2$ ... (1).

$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 - z_1z_3 + z_1^2 + z_2z_3 - z_1z_2$

$\Rightarrow z_2^2 + z_3^2 + z_1^2 - z_1z_3 - 2z_2z_3 + z_2z_3 - z_1z_3$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_2z_3$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

—— x ——

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$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

$$= ax^3 - 3d\,x^2y - 3a\,xy^2 + dy^3$$

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + k}$$

$$= 2\left[a\left(\frac{z}{2}\right)^3 - 3d\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right) - 3a\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2 + d\left(\frac{z}{2i}\right)^3\right] + k$$

$$= 2\left[\frac{az^3}{8} - 3\frac{dz^3}{8i} + \frac{3az^3}{8} + \frac{dz^3}{8}i\right] + k$$

$$= 2\left[\frac{az^3}{8} + \frac{3dz^3}{8}i + \frac{3az^3}{8} + \frac{dz^3}{8}i\right] + k$$

$$= \frac{2}{8}\left[z^3\left[a + 3di + 3a + di\right]\right] + k$$

$$= \frac{1}{4}\left[z^3(4di + 4a)\right] + k$$

$$= \frac{4z^3}{4}(a + di) + k$$

$$= z^3(a + di) + k$$

$$\therefore f(z) = (a + id)\,z^3 + k.$$

which is the req. of analytic

Polynomials:

The sum and product of few analytic functions are analytic, it follows that every Poly.

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n \quad -(1)$$

is an analytic function.

Now,

$$p'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \dots + n a_n z^{n-1}$$

In eqn (1), if $a_n \neq 0$, then the degree of $P(z) = n$

For, $n > 0$, the equation $P(z) = 0$ has atleast one root.

If α_1 is the root of $P(z) = 0$ then, $P(\alpha_1) = 0$

(ie) $P(z) = (z - \alpha_1) P_1(z)$, where $P_1(z)$ is a Poly. of degree $(n-1)$.

III^{ly}. If α_2 is the root of $P_1(z) = 0$ then

$$P(z) = (z - \alpha_1)(z - \alpha_2) \cdot P_2(z)$$

$P_2(z)$ is a Poly. of deg. $(n-2)$.

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Repitation of this process finally leads to a complete factorization.

(a) $\boxed{P(z) = a_n (z-\alpha_1)(z-\alpha_2) \ldots (z-\alpha_n)}$

where the roots $\alpha_1, \alpha_2, \ldots \alpha_n$ are not necessarily distinct.

LUCAS THEOREM

Stat: If all zeros of a polynomial $P(z)$ lie in a half plane then all the zeros of the derivative $p'(z)$ lie in the same half plane.

Proof: Let $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, $a_n \neq 0$.

be the polynomial of degree n.

If $\alpha_1, \alpha_2, \ldots \alpha_n$ are the roots of the equation $P(z) = 0$.

then,

$$P(z) = a_n (z-\alpha_1)(z-\alpha_2) \ldots (z-\alpha_n)$$

$$p'(z) = a_n (z-\alpha_2)(z-\alpha_3) \ldots (z-\alpha_n) + a_n (z-\alpha_1)(z-\alpha_3) \ldots (z-\alpha_n) + a_n (z-\alpha_1)(z-\alpha_2)(z-\alpha_4) \ldots (z-\alpha_n) + \ldots + a_n (z-\alpha_1)(z-\alpha_2) \ldots (z-\alpha_{n-1})$$

Now, $\frac{p'(z)}{p(z)} = \frac{1}{(z-\alpha_1)} + \frac{1}{(z-\alpha_2)} + \cdots + \frac{1}{(z-\alpha_n)}$ → (2).

An inequality describes the inside of the circle forms a line $z = a + bt$, determines a right half plane consisting of a point z with $\operatorname{Im}\left(\frac{z-a}{b}\right) < 0$ and left plane (half) with $\operatorname{Im}\left(\frac{z-a}{b}\right) > 0$.

Suppose, that the half plane, H is defined as the part of the plane where $\operatorname{Im}\left(\frac{z-a}{b}\right) < 0$.

If α_k is in H and z is not, we have

$$\operatorname{Im}\left(\frac{z-\alpha_k}{b}\right) = \operatorname{Im}\left(\frac{z-a}{b}\right) - \operatorname{Im}\left(\frac{\alpha_k - a}{b}\right) > 0$$

But, the im. parts of reciprocal numbers have some opposite sign.

$$\therefore \operatorname{Im}\left(\frac{b}{z-\alpha_k}\right) < 0.$$

If this is true for all b
we conclude from ①,
that $\operatorname{Im}\left(\frac{b\,p'(z)}{p(z)}\right) = \sum_{k=1}^{n} \operatorname{Im}\left(\frac{b}{z-\alpha_k}\right) < 0.$

and consequently $p'(z) \neq 0$.

~~The said ...~~

— ✳ —

Rational functions:

$R(z) = \frac{p(z)}{Q(z)}$ is the quotient

of two polynomials.

If $p(z)$ & $Q(z)$ have no common factors, then they have no common zeros.

The zeros of $p(z)$ are called poles of $R(z)$ & order of the pole is the order of the corresponding zeros of $Q(z)$.

∴ The derivative

$$R'(z) = \frac{p'(z).Q(z) - Q'(z)\,p(z)}{Q(z)^2}$$

exists, only when $Q(z) \neq 0$

$R'(z)$ has the same poles as $R(z)$. the order of each pole is increased by 1.

write $R\left(\frac{1}{z}\right) = R_1(z)$

and let $R(\infty) = R_1(0)$.

$$R(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

Then
$$\boxed{R_1(z) = z^{m-n} \frac{(a_0 z^n + a_1 z^{n-1} + \cdots + a_n)}{(b_0 z^m + b_1 z^{m-1} + \cdots + b_m)}}$$

Roots, Pole, order

A rational func. $R(z)$ is of order p has p zeros and poles and every equation $R(z) = 0$, has exactly "p" roots.

A rational function of order 1 is a linear function;

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \text{ with } \alpha\delta - \beta\gamma \neq 0$$

Such function are called as "linear transformation".

Ex:

$u(x,y) = x^3 - 3xy^2$.

Prove that u is a harmonic also find the corresponding analytic function.

Proof:

$u(x,y) = x^3 - 3xy^2$.

$u(x,y) = 2\cdot\left[\left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2\right] - 0 + ic.$

$= 2\left[\frac{z^3}{8} - 3\left(\frac{z}{2}\right)\left(\frac{z^2}{4i^2}\right)\right] - 0 + ic$

$= 2\left[\frac{z^3}{8} + \frac{3z^3}{8}\right] - 0 + ic$

$= \cancel{2}\left[\frac{\cancel{4}z^3}{\cancel{8}}\right] - 0 + ic$

$= z^3 - 0 + ic$

$\therefore$ The required analytic function $\boxed{f(z) = z^3 + ic}$

Assignment $\rightarrow$

harmonic

$$\boxed{\begin{array}{l} u_{xx} + u_{yy} = 0 \\ u_{yy} + u_{xx} = 0. \end{array}}$$

Find the most general harmonic polynomial of the form,

$$ax^3 + bx^2y + cxy^2 + dy^3.$$

determine an analytic function which is as it's real part.

Proof:

Given.

$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3.$$

$$u_x = 3ax^2 + 2bxy + cy^2.$$

$$u_{xx} = 6ax + 2by$$

$$u_y = bx^2 + 2cxy + 3dy^2$$

$$u_{yy} = 2cx + 6dy.$$

$$u_{xx} + u_{yy} = 0 \Rightarrow$$

$$6ax + 2by + 2cx + 6dy = 0.$$

$$\Rightarrow x(6a+2c) + y(2b+6d) = 0$$

$$\Rightarrow x(6a+2c) = 0 \ \&$$

$$y(2b+6d) = 0.$$

$$\boxed{\because x \neq 0 \quad y \neq 0}$$

$$6a + 2c = 0$$

$$2c = -6a$$

$$\boxed{c = -3a}$$

$$\text{uly} \quad 2b + 6d = 0$$

$$2b = -6d$$

$$\boxed{b = -3d}$$

Thus, the most general harmonic polynomial is given by,

$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3.$

$u(x,y) = ax^3 - 3dx^2y \cancel{+} \cdot 3axy^2 + dy^3.$

$f(z) =$ Observation

Results:

(1) The absolute value of the Product = The Product of the absolute value.

(2) The absolute value of the Sum is almost equal to the sum of the absolue value of the terms.

Square root of any Complex Numbers:

Let a+ib be given any complex number.

Let x+iy be the Square root of a+ib.

(ie) $\sqrt{a+ib} = x+iy$

(ie) $a+ib = (x+iy)^2$

$= x^2 + 2ixy - y^2$

(ie) $\boxed{a+ib = x^2 - y^2 + 2ixy}$ — (1)

Equating the real & Im. parts in (1)

$$\boxed{a + ib = x^2 - y^2 + 2ixy} \quad -(1)$$

$$a = x^2 - y^2 \quad —(2).$$
$$b = 2xy \quad —(3)$$

—x—

Now, $(x^2+y^2)^2 = x^4 + y^4 + 2x^2y^2$

$$\Rightarrow x^4 + y^4 + 2x^2y^2 - 2x^2y^2 + 2x^2y^2$$
$$= x^4 + y^4 - 2x^2y^2 + 4x^2y^2$$
$$= (x^2-y^2)^2 + 4x^2y^2$$

$\therefore (x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$$(x^2+y^2)^2 = a^2 + b^2$$

then $x^2+y^2 = \sqrt{a^2+b^2} \quad —(4).$

from (2) & (4).

$$x^2 + \cancel{y^2} = \sqrt{a^2+b^2}$$
$$x^2 - \cancel{y^2} = a$$

$$2x^2 = a + \sqrt{a^2+b^2}$$
$$x^2 = \frac{a + \sqrt{a^2+b^2}}{2}$$

$$x^2 = \frac{a + \sqrt{a^2+b^2}}{2}$$

$$\boxed{x = \pm\left(\frac{a+\sqrt{a^2+b^2}}{2}\right)^{1/2}}$$

IIIly,

$$2y^2 = -a + \sqrt{a^2+b^2}$$

$$y^2 = \frac{-a+\sqrt{a^2+b^2}}{2}$$

$$\boxed{y = \pm\left(\frac{-a+\sqrt{a^2+b^2}}{2}\right)^{1/2}}$$

$\therefore$ $x+iy$ be a Square root of $a+ib$.

where $x = \pm\left(\frac{a+\sqrt{a^2+b^2}}{2}\right)^{1/2}$

$y = \pm\left(\frac{-a+\sqrt{a^2+b^2}}{2}\right)^{1/2}$

From, $b = 2xy$

if $b>0$ x & y have Same sign

if $b<0$, x & y have some opposite sign

if $b=0$, z is Real.

Geometrical representation of a Complex No.s

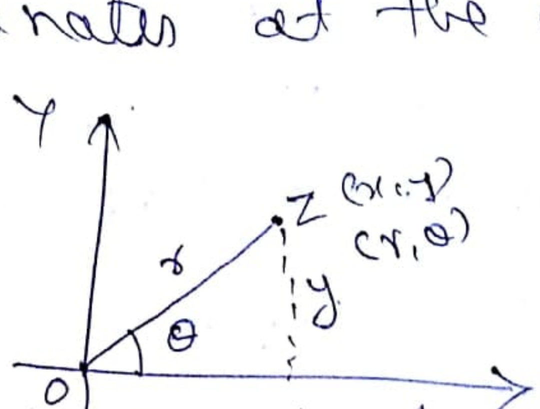
we represent any complex No. $x+iy$ by a point (x,y) in $R \times R$.

The plane $R \times R$ representing the complex numbers is called the complex plane.

Polar form of a complex Number.

Consider any non zero complex number $z = x+iy$.

Let (r, θ) denote the polar co-ordinates at the point (x,y)



Complex plane.

$\frac{x}{r} = \sin\theta \Rightarrow x = r\sin\theta.$

$\frac{y}{r} = \cos\theta \Rightarrow y = r\cos\theta.$

$z = x + iy$
$= r\sin\theta + i\, r\cos\theta.$
$= r(\sin\theta + i\cos\theta).$

$|z| = \sqrt{x^2+y^2}$
$= \sqrt{(r^2\sin^2\theta) + (r^2\cos^2\theta)}$
$= \sqrt{r^2(\sin^2\theta + \cos^2\theta)}$
$= \sqrt{r^2}$
$= r.$

which is the magnitude of the complex number and $\underline{\underline{\theta}}$ is called the amplitude or argument and it is denoted by

$\boxed{\arg z = \operatorname{amp} z.}$

The value of arg z is lies between $-\pi$ to π. is called the principal value of arg z.

Theorem: Let $z = r(\cos\theta + i\sin\theta)$ be any non zero complex no. and n be any integer then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Proof: We first prove this result for positive integer by induction on n.

The result is true when $n=1$.

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$z = r(\cos\theta + i\sin\theta)$$

Suppose the result is true for $n = m$.

Hence $z^m = r^m(\cos m\theta + i\sin m\theta)$

Now, $z^{m+1} = z^m \cdot z$

$\Rightarrow r^m(\cos m\theta + i\sin m\theta) \cdot$
$r(\cos\theta + i\sin\theta)$

$\Rightarrow r^{m+1}\left[\cos(m+1)\theta + i\sin(m+1)\theta\right]$

Hence, the result true for $\underline{n = m+1}$

Hence, $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all positive integers n.

The result is true if $n = 0$.

Now, $z^{-1} = \frac{1}{z}$

$$= \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{r}(\cos\theta + i\sin\theta)$$

$$= \frac{1}{r}\frac{(\cos\theta - i\sin\theta)}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= r^{-1}\left[\cos(-\theta) + i\sin(-\theta)\right]$$

$\therefore$ The result is true for $n = -1$.

Hence, it follows that the result is true for all integers. $\forall n \in \mathbb{Z}$. Hence $\underline{z^n = r^n}$ $(\cos n\theta + i\sin n\theta)$

Proof:

(20)

Corollary: De-Movire's theorem

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Problems: P.T $\left|\frac{a-b}{1-\bar{a}b}\right| = 1$ if either $|a|=1$ or $|b|=1$

Proof:

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \left(\frac{a-b}{1-\bar{a}b}\right)\overline{\left(\frac{a-b}{1-\bar{a}b}\right)}$$

$$= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\frac{\bar{a}-\bar{b}}{1-a\bar{b}}\right)$$

$$= \frac{a\bar{a} - a\bar{b} - \bar{a}b + b\bar{b}}{1 - a\bar{b} - \bar{a}b + \bar{a}ba\bar{b}}$$

$$= \frac{|a|^2 - a\bar{b} - \bar{a}b + |b|^2}{1 - a\bar{b} - \bar{a}b + a\bar{a}b\bar{b}}$$

$$\boxed{\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{|a|^2 - a\bar{b} - \bar{a}b + |b|^2}{1 - a\bar{b} - \bar{a}b + |a|^2|b|^2}}$$

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If $\underline{|a| = 1}$. $\Rightarrow$

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{|a|^2 - a\bar{b} - b\bar{a} + |b|^2}{1 - a\bar{b} - \bar{a}b + |a|^2|b|^2}$$

$$= \frac{1 - a\bar{b} - b\bar{a} + |b|^2}{1 - a\bar{b} - \bar{a}b + |b|^2}$$

$$= 1$$

If $\underline{|b| = 1}$ $\Rightarrow$

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{|a|^2 - a\bar{b} - b\bar{a} + 1}{1 - a\bar{b} - \bar{a}b + |a|^2}$$

$$= 1$$

$\therefore \left|\dfrac{a-b}{1-\bar{a}b}\right| = 1$ if either $|a| = 1$ or $|b| = 1$

—x—

Theorem: If z_1 & z_2 are any two non zero complex no's then

i) $\arg z_1 = -\arg \bar{z_1}$

ii) $\arg z_1 z_2 = \arg z_1 + \arg z_2$

iii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Proof:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\bar{z}_1 = r_1(\cos\theta_1 - i\sin\theta_1)$

$= r_1[\cos(-\theta_1) + i\sin(-\theta_1)]$

Hence, $\arg \bar{z}_1 = -\theta_1$

$= \arg z.$

Con

$$\boxed{\arg \bar{z}_1 = -\theta_1 = -\arg z_1}$$

ii) $\arg z_1 z_2 = \arg z_1 + \arg z_2$

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ &

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Now $z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow r_1 r_2[\cos\theta_1\cos\theta_2 + \cos\theta_1\, i\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2]$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$$= r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$\because \cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\boxed{\therefore \arg z_1 z_2 = \theta_1+\theta_2 = \arg z_1 + \arg z_2.}$$

$$\boxed{(iii)\ \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2}$$

to Exercise

2/8/20

When does $az + b\bar{z} + c = 0$ represent a line?

Soln: Let $z = x+iy$

$a = \alpha_1 + i\alpha_2$

$b = \beta_1 + i\beta_2$

$c = \gamma_1 + i\gamma_2$

$\therefore$ The given equation

$$(\alpha_1 + i\alpha_2)(x+iy) + (\beta_1+i\beta_2)(x-iy) + (\gamma_1+i\gamma_2) = 0$$

ii

$(\alpha_1 + i\alpha_2)(x+iy) + (\beta_1 + i\beta_2)(x-iy) + (\gamma_1 + i\gamma_2) = 0$

$\Rightarrow (\alpha_1 x + \alpha_1 iy + i\alpha_2 x + i^2\alpha_2 y) +$
$(\beta_1 x - \beta_1 iy + i\beta_2 x - i^2\beta_2 y) +$
$+ (\gamma_1 + i\gamma_2) = 0.$

$\Rightarrow \alpha_1 x + i\alpha_1 y + i\alpha_2 x - \alpha_2 y +$
$\beta_1 x - i\beta_1 y + i\beta_2 x + \beta_2 y +$
$\gamma_1 + i\gamma_2 = 0$

Real
$\Rightarrow \alpha_1 x - \alpha_2 y + \beta_1 x + \beta_2 y + \gamma_1 = 0$

Imag:
$x(\beta_2 + \alpha_2) + y(\alpha_1 - \beta_1) + \gamma_2 = 0$

$\therefore$ The given equation is

$az + b\bar{z} + c = 0.$

After equating the real & Im. parts

$(\alpha_1 + \beta_1)x + (-\alpha_2 + \beta_2)y + \gamma_1 = 0.$

III^{ly}, $(\alpha_2 + \beta_2)x + (\alpha_1 - \beta_1)y + \gamma_2 = 0.$

The given ~~line~~ equation represent a line if z has a unique value.

$\therefore$ A unique solution for x & y exists iff.

$$\frac{\alpha_1+\beta_1}{\alpha_2+\beta_2} \neq \frac{-\alpha_2+\beta_2}{\alpha_1-\beta_1}$$

which leads to the results.

(ie) $(\alpha_1+\beta_1)(\alpha_1-\beta_1) \neq (-\alpha_2+\beta_2)(\alpha_2+\beta_2)$

$\Rightarrow \alpha_1^2 - \cancel{\alpha_1\beta_1} + \cancel{\beta_1\alpha_1} - \beta_1^2 \neq -\alpha_2^2 - \cancel{\alpha_2\beta_2} + \cancel{\beta_2\alpha_2} + \beta_2^2$

$\Rightarrow (\alpha_1^2 - \beta_1^2) \neq (\beta_2^2 - \alpha_2^2)$

$\Rightarrow \alpha_1^2 + \alpha_2^2 \neq \beta_1^2 + \beta_2^2$

$\Rightarrow |a| \neq |b|$

$\therefore az + b\bar{z} + c = 0$ represents a line when $|a| \neq |b|$

— x —

Theorem: Set of all complex no's $\mathbb{C}$ is a field under addition & multiplication.

Proof:

Obviously $z_1 + z_2$ & $z_1 z_2 \in \mathbb{C}$

Since addition of real no's is associative and commutative it follows that addition in $\mathbb{C}$ is also associative and commutative.

$0 = 0 + i0$ is the additive identity and the additive inverse of $z = x + iy$ is $(-x + i(-y))$.

Hence, $(\mathbb{C}, +)$ is an abelian group.

Let $z_1, z_2 \in \mathbb{C}^*$, the set of non-zero complex numbers.

Then $z_1 = x_1 + iy_1$ where x_1 & y_1 are not simultaneously zero. and $z_2 = x_2 + iy_2$ where x_2 & y_2 are not simultaneously zero.

we claim that $z_1 z_2 \in C^*$

we have $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$\Rightarrow x_1 x_2 + x_1 i y_2 + i y_1 x_2 + i^2 y_1 y_2$

$\Rightarrow (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Suppose $z_1 z_2 = 0$

Then, $x_1 x_2 - y_1 y_2 = 0$ — (1)

and $x_1 y_2 + x_2 y_1 = 0$ — (2)

Multiply Equation (1) by "y_2" and

Equation (2) by x_2 and subtracting

we get

$$(x_1 x_2) y_2 - (y_1 y_2) y_2 = 0 * y_2$$

$$(x_1 x_2) \cancel{y_2} - (y_1 y_2) y_2 = 0$$

$$(x_1 y_2) x_2 + (x_2 y_1) x_2 = 0$$

$(-) \quad (-)$

$$y_1 (y_2^2 + x_2^2) = 0$$

$\therefore$ either $y_1 = 0$ or $y_2^2 + x_2^2 = 0$

$\therefore$ either $y_1 = 0$ or $(y_2 = 0$ and $x_2 = 0)$

$||^{ly}$, $\therefore$ either $x_1 = 0$ or $(y_2 = 0$ and $x_2 = 0)$

Thus, ($x_1 = 0$ & $y_1 = 0$) (or)

($x_2 = 0$ & $y_2 = 0$)

$\therefore$ $z_1 = 0$ (or) $z_2 = 0$ which is a $\Rightarrow\Leftarrow$

Hence $z_1 z_2 \in C^*$.

It can be easily verified that multiplication is associative & commutative.

$1 + i0 \in C$ is the multiplicative identity element.

Let $z = x+iy$ be a non zero Complex no.

$\therefore$ Either $x \neq 0$ (or) $y \neq 0$.

Hence $x^2+y^2 > 0$

Now, $$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

Thus $\frac{1}{z} \in C^*$ and it is the multiplicative inverse of z.

Further it can be easily verified that $z_1(z_2+z_3) = z_1z_2 + z_1z_3$ for all complex nos. $z_1, z_2, z_3 \in C$.

Hence $(C, +, \cdot)$ is a field.

—×—

Remark: $(z_1+z_2) \otimes (z_3+z_4)$

① It is important that to note there is no order structure in the complex no. system so that we cannot compare two complex no's.

—×—

② The complex no. $a+ib$ can also be represented by the ordered pair of real no's (a,b)

$z = a+ib$
$z = x+iy$
$z = (a,b)$
$z = (x,y)$

—×—

2:[8]90

Limits & Continuity

Defn: A function $f(x)$ is said to have the limit A as x tends to a.

$$\lim_{x \to a} f(x) = A \quad \text{iff}$$

the following is true.

For any $\epsilon > 0$, $\exists$ a no. $\delta > 0$ with the following property that,

$|f(x) - A| < \epsilon$ for all values of x such that $|x-a| < \delta$ and $\boxed{\delta \neq 0}$, (or) $\delta \neq a$. (x)

If $\lim_{x \to a} f(x) = A$

then $\lim_{x \to a} \overline{f(x)} = \overline{A}$

then, $\lim_{x \to a} \operatorname{Re} f(x) = \operatorname{Re} A$

$\lim_{x \to a} \operatorname{Im} f(x) = \operatorname{Im} A.$

Defn: A function $f(x)$ (or $f(z)$) is said to be continuous at a iff $\lim_{x \to a} f(x) = f(a)$

Theorem: If the functions $f(z)$ & $g(z)$ are defined in a region R then (i) $f+g$ (ii) fg iii) f/g, $g \neq 0$, at z_0 are continuous at z_0.

Proof: Given f & g are continuous at z_0.

$\therefore$ for every $\epsilon > 0$, $\exists$ $\delta > 0$

Such that,

$$|z-z_0| < \delta_1 \Rightarrow |f(z)-f(z_0)| < \epsilon_1$$

& $$|z-z_0| < \delta_2 \Rightarrow |g(z)-g(z_0)| < \epsilon_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

$$|(f+g)(z) - (f+g)(z_0)| \le$$

$$|f(z)-f(z_0)| + |g(z)-g(z_0)|$$

$$< \epsilon_1 + \epsilon_2 = \epsilon.$$

when $|z-z_0| < \delta$.

$\therefore (f+g)$ is continuous at z_0.

Now, $|(fg)(z) - (fg)(z_0)|$

$$= |f(z)\,g(z) - f(z_0)\,g(z_0)|$$

$$= |f(z)\,g(z) - f(z_0)\,g(z) + f(z_0)\,g(z) - f(z_0)\,g(z_0)|$$

$$\le |g(z)|\,|f(z)-f(z_0)| + |f(z_0)|\,|g(z)-g(z_0)|$$

(1) $\longleftarrow$

Since, g is continuous, therefore g is bounded

$\therefore |g(z)| \leq M \;\forall z \in R$

Sim, $|f(z_0)| \leq r, \;\forall z_0 \in R$. for any $r \in M$.

(ii) $$\left| \frac{f(z)}{g(z)} - \frac{f(z_0)}{g(z_0)} \right| = \left| \frac{f(z)\,g(z_0) - f(z_0)\,g(z)}{g(z) \cdot g(z_0)} \right|$$

$$= \left| \frac{f(z) \cdot g(z_0) - f(z_0)\,g(z_0) + f(z_0)\,g(z_0) - f(z_0)\,g(z)}{g(z) \cdot g(z_0)} \right|$$

Choose $\epsilon_2 = \frac{|g(z_0)|}{2}$ as

Rough work $$\boxed{\begin{aligned} &|g(z) - g(z_0)| \geq |g(z_0)| - |g(z)| \\ &\epsilon_2 \geq |g(z_0)| - |g(z_0)| \end{aligned}}$$

$$\epsilon_2 \geq |g(z_0)| + |g(z)|.$$

(i) $-|g(z)| + |g(z_0)| < \epsilon_2$

$$-|g(z)| < \frac{|g(z_0)|}{2} - |g(z_0)|$$

$$-|g(z)| < -\frac{|g(z_0)|}{2}$$

(ii) $|g(z)| > \dfrac{|g(z_0)|}{2}$ ——— (2)

$$\left|\left(\frac{f}{g}\right)(z) - \left(\frac{f}{g}\right)(z_0)\right| = \frac{|g(z_0)[f(z) - f(z_0)] + f(z_0)[g(z) - g(z_0)]|}{|g(z)||g(z_0)|}$$

$$\leq \frac{\epsilon_1 |g(z_0)| + \epsilon_2 |f(z_0)|}{|g(z)||g(z_0)|}$$

$$\leq \left[\epsilon_1 |g(z_0)| + \epsilon_2 |f(z_0)|\right] \cdot \frac{2}{|g(z_0)|^2}$$

[by (2)]

for any $\epsilon > 0$, choosing ϵ_1 & ϵ_2 so that

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$$2\left[\frac{\epsilon_1 |g(z_0)| + \epsilon_2 |f(z_0)|}{|g(z_0)|^2}\right] = \epsilon$$

$\therefore$ $0 < |z - z_0| < \delta \Rightarrow$

$$\left|\frac{f(z)}{g(z)} - \frac{f(z_0)}{g(z_0)}\right| < \epsilon$$

$\therefore$ f/g is a continuous at z_0.

—x—

Differentiability

Let f be a complex function defined in a region D. and let $z \in D$. Then f is said to be differentiable at z, if $\boxed{\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}}$ exists and is finite.

This limit is denoted by $f'(z)$ or $\frac{df}{dz}$ and is called the derivative of $f(z)$ at z.

The function is said to be differentiable in D, if it is differentiable at all points of D.

Ex: The function $f(z) = z^2$ is differentiable at every point and $f'(z) = 2z$.

$$\Rightarrow \frac{f(z+h) - f(z)}{h} = \frac{(z+h)^2 + z^2}{h}$$

$$= \frac{z^2 + h^2 + 2zh - z^2}{h}$$

$$= \frac{h^2 + 2zh}{h}$$

$$= 2z + h.$$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} 2z + h$$

$\Rightarrow \lim_{h \to 0} 2z+h$

$= 2z.$ $\qquad\qquad f(z) = z^2$

$f'(z) = 2z.$

$\therefore f'(z) = 2z.$

---x---

Ex: The function $f(z) = \bar{z}$ is nowhere differentiable.

$$\Rightarrow \frac{f(z+h) - f(z)}{h} = \frac{\overline{(z+h)} - \bar{z}}{h}$$

$$= \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \frac{\bar{h}}{h}$$

$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}$ doesn't exist.

$\therefore f(z) = \bar{z}$ is nowhere differen.

---x---

Remark: If $f(z)$ is differentiable at a point then it is continuous at a point.

---x---

But, continuity does not imply differentiation. ---x---

9/9/20

C-R. equations in polar co-ordinates

If $z = re^{i\theta} (= x+iy)$ then

$x = r\cos\theta$ & $y = r\sin\theta$

where $\theta \in (-\pi, \pi]$, $r \neq 0$.

W.K.T

$$z = x+iy.$$

then $\bar{z} = x - iy$.

$$z + \bar{z} = 2x$$

i.e. $2x = z + \bar{z}$

$$\Rightarrow \boxed{x = \frac{z+\bar{z}}{2}}$$

IIIly

$$\boxed{y = \frac{z-\bar{z}}{2i}}$$

$x^2 + y^2 = r^2$ —— (1)

& $\frac{y}{x} = \tan\theta.$

i.e. $y = x\tan\theta$ —— (2)

Diff.

$x^2+y^2=r^2$ Partially w.r.to x

$$2x = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= \frac{r\cos\theta}{r}$$

$$\boxed{\frac{\partial r}{\partial x} = \cos\theta}$$

Diff. (1), w.r. to y

$$2y = 2r.\frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r}$$

$$\boxed{\frac{\partial r}{\partial y} = \sin\theta}.$$

Diff. (2), w.r. to θ.

$$y = x\tan\theta \quad \text{--- (2)}.$$

$$0 = x\sec^2\theta + \tan\theta\frac{\partial x}{\partial\theta}$$

$$= r\cos\theta.\sec^2\theta + \tan\theta.\frac{\partial x}{\partial\theta}.$$

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$$0 = r\cos\theta \cdot \sec^2\theta + \tan\theta \cdot \frac{\partial x}{\partial \theta}.$$

$$= r\cos\theta \cdot \frac{1}{\cos^2\theta} + \frac{\sin\theta}{\cos\theta} \cdot \frac{\partial x}{\partial \theta}.$$

$$= \frac{r}{\cos\theta} + \tan\theta \frac{\partial x}{\partial \theta}$$

$$\therefore \tan\theta \frac{\partial x}{\partial \theta} = \frac{-r}{\cos\theta}$$

$$\therefore \frac{\partial x}{\partial \theta} = \frac{-r}{\cos\theta \cdot \tan\theta}$$

$$\frac{\partial \theta}{\partial x} = -\frac{\tan\theta \cdot \cos\theta}{r}$$

$$= -\frac{\sin\theta \cancel{\cos\theta}}{\cancel{\cos\theta}\, r}$$

$$\boxed{\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}.}$$

$$w = f(z) = u(r,\theta) + iv(r,\theta)$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= u_r \cdot \cos\theta + u_\theta\left(-\frac{\sin\theta}{r}\right)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= u_r \sin\theta + \frac{1}{r} u_\theta \cdot \sin\theta.$$

$$v_x = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= v_r \cos\theta - \frac{1}{r} v_\theta \sin\theta$$

$$v_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= v_r \sin\theta + v_\theta \frac{\cos\theta}{r}.$$

———x———

Remark:

If $f(z) = u + iv$ is analytic at z, it is necessary that the four of partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ exists and satisfy the CR equation.

Hence, if the CR equations are not satisfied for a complex function ~~is not diff~~ at any point then we say that the function is not differentiable. (analytic)

Ex: $f(z) = \bar{z}$

Given $f(z) = \bar{z} = x - iy$.

Here, $u(x,y) = x$, $v(x,y) = -y$

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = -1$

$\Rightarrow u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$

Remarks:

$$f'(z) = u_x + i v_x$$

(by CR equation) $\Rightarrow v_y - i u_y$

$$\therefore |f'(z)|^2 = u_x^2 + v_x^2$$

$$\Rightarrow u_y^2 + v_y^2$$

$\Rightarrow$ further,

$$|f'(z)|^2 \Rightarrow u_x^2 + v_x^2$$

$$\Rightarrow u_x u_x + v_x v_x$$

$$\Rightarrow u_x v_y - u_y v_x \text{ (by CR)}$$

$$\Rightarrow \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \frac{\partial(u, v)}{\partial(x, y)}$$

$\therefore |f'(z)|^2$, is the Jacobian u and v w.r.t. to x & y

—x—

Analytic functions are characterized by the condition

$$\boxed{\frac{\partial f}{\partial \bar{z}} = 0}$$

Let $z = x + iy$.

then $\bar{z} = x - iy$

$$\boxed{x = \frac{z + \bar{z}}{2}} \quad \boxed{y = \frac{z - \bar{z}}{2i}}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x} \cdot \left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{-1}{2i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Thus, $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$

which is the complex form of the CR-equations.

Thus, the CR-equations can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$.
Hence, the analytic functions are characterized by the conditions $\boxed{\frac{\partial f}{\partial \bar{z}} = 0}$.

Harmonic functions:

Let $u(x,y)$ be a function of two real variables x & y, defined in a region D. Then, $u(x,y)$ is said to be Harmonic function if $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$ and this equation is called as laplace equation.

Conjugate Harmonic function:

Let $f = u+iv$ be an analytic function in a region

$\leq f = u + iv$ $v(x,y)$

region D, then v is said to be a conjugate harmonic function of u.

Remark:

The real & Imaginary parts of an analytic functions are harmonic functions.

PI $u = x^2 - y^2$ is harmonic. Find it's harmonic conjugate. Also find the corresponding analytic function.

Proof: Given $u(x,y) = x^2 - y^2$

$u_x = 2x$ | $u_y = -2y$

$u_{xx} = 2$ | $u_{yy} = -2$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$2 - 2 = 0$

$\therefore u$ is a harmonic function.

Let $v(x,y)$ be a harmonic Conjugate of u. using CR equations, we have

$$\boxed{u_x = v_y} \Rightarrow 2x.$$

$$\Rightarrow v_y = 2x$$

Integrating w. r. to y

$$v = 2xy + \phi(x) \quad (1)$$

where $\phi(x)$ is a function of x.

Diff. (1). w. r. to x.

$$v_x = 2y + \phi'(x)$$

But $\boxed{v_x = -u_y}$. by CR-equation

$$\therefore \quad 2y + \phi'(x) = v_x$$

$$2y + \phi'(x) = 2y.$$

$$\phi'(x) = 0.$$

$$\phi(x) = \text{constant} = c \text{ (say)}$$

$$\boxed{\therefore \ v = 2xy + c}$$

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$f = u + iv.$

$= (x^2 - y^2) + i(2xy + c)$

$= x^2 - y^2 + 2ixy + ic$

$= (x + iy)^2 + ic$

$\boxed{f = z^2 + ic}$

Miline - Thompson method.

Let $u(x,y)$ be a given harmonic function. Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

then, $f'(z) = u_x(x,y) + iv_x(x,y)$

$= u_x(x,y) - iu_y(x,y)$

Let $\phi_1(x,y) = u_x(x,y)$ &

$\phi_2(x,y) = u_y(x,y)$

Hence, $f'(z) = \phi_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

$- i\phi_2\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$

Put $z = \bar{z}$, we obtain,

$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

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COMPLEX ANALYSIS

Edited by Anandh Balasubramanian

$f(z) = |z|^2$

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{|z|^2 - |z_0|^2}{z - z_0}$$

$$\Rightarrow \frac{z\bar{z} - z_0\bar{z_0}}{z - z_0}$$

$$\Rightarrow \bar{z} + \frac{z_0\bar{z} - z_0\bar{z_0}}{z - z_0}$$

$$\Rightarrow \bar{z} + \frac{z_0(\bar{z} - \bar{z_0})}{z - z_0}$$

Let, $z - z_0 = re^{i\theta}$ then

$\bar{z} - \bar{z_0} = re^{-i\theta}$

Rough

$$\frac{\bar{z}(z - z_0) + z_0\bar{z} - z_0\bar{z_0}}{z - z_0}$$

$$\frac{z\bar{z} - \cancel{z_0\bar{z}} + \cancel{z_0\bar{z}} - z_0\bar{z_0}}{z - z_0}$$

$$\frac{f(z)-f(z_0)}{z-z_0} = \bar{z} + \frac{z_0(\bar{z}-\bar{z}_0)}{z-z_0}$$

$$= \bar{z} + z_0 \frac{re^{-i\alpha}}{re^{i\alpha}}$$

$$= \bar{z} + z_0 \frac{e^{-i\alpha}}{e^{i\alpha}}$$

$$= \bar{z} + z_0 e^{-2i\alpha}$$

$$\Rightarrow \bar{z} + z_0 \left(\frac{1}{\cos 2\alpha + i \sin 2\alpha} \right)$$

$$= \bar{z} + z_0 \left(\frac{\cos 2\alpha - i \sin 2\alpha}{(\cos 2\alpha + i \sin 2\alpha)(\cos 2\alpha - i \sin}\right.$$

$$= \bar{z} + z_0 \left(\frac{\cos 2\alpha - i \sin 2\alpha}{\cos^2 2\alpha + \sin^2 2\alpha} \right)$$

$$= \bar{z} + z_0 ((\cos 2\alpha - i \sin 2\alpha)$$

At $z \to z_0$, the above expression doesn't tend to a unique limit

$\therefore f(z)$ is not differentiable at z_0.

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But, when z_0 is zero, this expression ~~then~~ tends to $\bar{z}$ which tends to zero with z.

—x—

Defn: **Analytic**

A single valued function $f(z)$ is said to be analytic at a point z_0, if it has a unique ~~differe~~ derivative at z_0.

$f(z)$ is said to ~~not~~ be analytic throughout a region R, if it has a derivative at every point of R.

—x—

Entire function

If a function is analytic in the entire complex plane, it is called an entire function.

—x—

Complex form of C-R equation:

General CR - equation

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Let $f(z) = u(x,y) + i v(x,y)$ be differentiable. Then the C-R equations can be put in the complex form as

$$f_x = -i f_y$$

$\rightarrow$ Let $f(z) = u(x,y) + i v(x,y)$

then $f_x = u_x + i v_x$

& $f_y = u_y + i v_y$

$$f_x = -i f_y \Leftrightarrow u_x + i v_x = -i u_y + v_y$$

$$\Leftrightarrow u_x = v_y \ \& \ u_y = -v_x$$

Thus, the two C-R equations equivalent to the equation

$$\boxed{f_x = -i f_y}$$

$\therefore$ $f_x = -i f_y$ is the complex form of C-R. equations.

— x —

C-R Min equations.

C-R Min equations provide a condition for analytic. So that this condition is sufficient for $f(z)$ be analytic provided u_x, u_y, v_x & v_y are continuous.

Prove that rigorously that the function $f(z)$ & $\overline{f(\bar{z})}$ are simultaneously analytic.

Soln.: Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in a region D.

Then the first order partial derivatives of u & v are continuous & satisfy the CR equations

i.e.

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \& \quad \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right] \text{---}(1)$$

$V(x,y) = x^4 - 6x^2y^2 + y^4$. Find $f(z) = u(x,y) + i\,v(x,y)$ such that $f(z)$ is analytic.

Soln :

Given $V(x,y) = x^4 - 6x^2y^2 + y^4$

$$\psi_1(x,y) = V_y = -12x^2y + 4y^3$$

$$\psi_2(x,y) = V_x = 4x^3 - 12xy^2$$

$$\therefore f(z) = \int \left[\psi_1(z,0) + i\psi_2(z,0)\right] dz$$

$$= \int \left\{\left[-12\,(z)^2(0) + 4(0)\right] + i\,\psi_2\left[4z^3 - 12.z(0)\right]\right\} dz$$

$$= \int i\,4z^3\,dz$$

$$= \frac{i\cancel{4}z^4}{\cancel{4}}$$

$$= z^4 i + c$$

$$\boxed{\therefore f(z) = z^4 i + c}$$

— x —

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S.T an analytic function $f(z)$ cannot have a constant absolute value without reducing a constant.

Proof.

Let $f(z) = u(x,y) + i v(x,y)$ in a region D.

Suppose $f(z)$ has a constant absolute value.

(i.e) $|f(z)|^2 = u^2 + v^2$ = Constant.

$|f(z)|^2 = u^2 + v^2 = c$ —— (1).

Diff. (1) w.r.to x. (Partially)

$2uu_x + 2vv_x = 0.$

Diff (1) w.r.to y.

$2uu_y + 2vv_y = 0.$

$\therefore f(z)$ is analytic.

$uu_x + vv_x = 0$

$uu_y + vv_y = 0$ —— (2).

Using CR-equations in (1) & (2)

$uu_x + vv_x = 0$

$uu_y + vv_y = 0$

$uu_x - vu_y = 0$ — (3)

$uu_y + vu_x = 0$ — (4)

Consider equation (3) $\times u$

$\Rightarrow u^2 u_x - v u u_y = 0$

Eqn. (4) $\times v \Rightarrow v u u_y + v^2 u_x = 0$

$\therefore u^2 u_x + v^2 u_x = 0$

$(u^2 + v^2) u_x = 0$

$\Rightarrow u^2 + v^2 = c$

$u_x = 0$

$|||^y$, we can prove that

$v_x = 0$

$\therefore f'(z) = u_x + i v_x = 0$

If $|f(z)|$ is constant then $f(z)$ is constant.

$\therefore$ An analytic function $f(z)$ cannot have a constant absolute value without reducing a constant.

Result:

We remark that $\overline{f(z)}$.

The conjugation analytic function $f(z)$ has derivative $\bar{z}$ w.r.t. z and so $\overline{f(z)}$ may be considered as a function of $\bar{z}$.

$$\boxed{\begin{aligned} f(x,y) &= u+iv \\ &= x+iy \\ \overline{f(z)} &= u-iv \\ &= (\bar{z}) \end{aligned}}$$

We denote $\overline{f(z)} = \bar{f}(\bar{z})$.

i.e. $\overline{f(x+iy)} = \bar{f}(x-iy)$

$\therefore u(x,y) = \frac{1}{2}\left[f(x+iy) + \bar{f}(x-iy)\right]$

$$\boxed{\begin{aligned} \text{Put } x &= z/2 \ \& \\ y &= z/2i \end{aligned}}$$

$u\left(z/2, \frac{z}{2i}\right) = \frac{1}{2}\left[f\left(z/2 + i\frac{z}{2i}\right)\right.$

$\left. + \bar{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right]$

$= \frac{1}{2}\left[f(z) + \bar{f}(0)\right]$

$\frac{1}{2} f(z) + \frac{1}{2}\bar{f}(0) = u\left(\frac{z}{2}, \frac{z}{2i}\right)$

$\frac{1}{2} f(z) + \frac{1}{2} \bar{f}(0) = u\left(\frac{z}{2}, \frac{z}{2i}\right)$

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0)} \quad \text{(1)}$$

Put $u(0,0)$

$= \frac{1}{2}\left[f(0) + \bar{f}(0)\right]$

$= \frac{1}{2}\left[f(0) + \bar{f}(0)\right]$

$u(0,0) = \operatorname{Re} f(0)$

$$\boxed{\bar{f}(0) = \operatorname{Re} f(0) + i \operatorname{Im} f(0)}$$

$= \operatorname{Re} f(0) + ic$

$\therefore f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ic$

Ly, it follows that in a neighbourhood of z_0, then the analytic func. $f = u + iv$, associated with the harmonic function $u(x,y)$ is given by

$$\boxed{f(z) = 2u\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) - u(x_0, y_0) + ic}$$

c is a real no.

—x—

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Find the analytic function $f(z) = u + iv$ given that $u - v = e^x(\cos y - \sin y)$.

—x—

Soln. Diff. w. r. to x.

$$u_x - v_x = e^x(\cos y - \sin y) \quad (1)$$

Diff. w. r. to y,

$$u_y - v_y = e^x(-\sin y - \cos y)$$
$$= -e^x(\sin y + \cos y) \quad (2)$$

Using CR. equation in (2) we get,

CR equn $\boxed{u_x = v_y \ \& \ u_y = -v_x}$

$$-v_x - u_x = -e^x(\sin y + \cos y) \quad (3)$$

Solve (1) & (3) we get,

$$u_x - \cancel{v_x} = e^x(\cos y - \sin y)$$
$$-u_x - \cancel{v_x} = -e^x(\sin y + \cos y)$$

$$2u_x = 2e^x \cos y.$$
$$u_x = e^x \cos y \quad (4).$$

for ex.

$f(z) = u + iv$

analytic $u - v = (\cos y - \sin y)$

$u_x - v_x = ?$

$u_{xx} - v_{xx} =$

$u_y - v_y = ?$

$u_{yy} - v_{yy} =$

$$\boxed{\begin{array}{l} u_{xx} + v_{xx} = 0 \\ u_{yy} + v_{yy} = 0 \end{array}}$$

$x(2a + c) = 0$

$x \neq 0 \Rightarrow$

$2a + c = 0$

$2a = -c$

$\boxed{a = -\frac{c}{2}}$

Problem: Show that a harmonic function satisfies the formal d.e. $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

Soln:

Let $z = x + iy$.

then $\bar{z} = x - iy$.

$$x = \frac{z + \bar{z}}{2}$$

$$y = \frac{z - \bar{z}}{2i}$$

$$u(x, y) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Thus, formally treating z & $\bar{z}$ as independent variables.

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial z}$$

$$= \frac{\partial u}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y}\left(\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\left(\frac{1}{i}\right)\right]$$

$$= \frac{1}{2}\left[u_x - i u_y\right]$$

$$\therefore \boxed{\frac{\partial u}{\partial z} = \frac{1}{2}\left[u_x - i u_y\right]}$$

$-2v_x = -2e^x \sin y.$

$$\boxed{v_x = e^x \sin y} \text{ --- } (5)$$

$$\boxed{u_x = e^x \cos y} \text{ --- } (4')$$

Integrating equations (4) & (5) w. r. to x.

$u_x = e^x \cos y.$

$\int u_x = \int e^x \cos y$ $\qquad \boxed{uv - \boxed{\int v\,du}}$

$u = e^x \cos y + \overset{c_1}{\phi(y)}$. $\quad u = e^x \quad dv = \cos y$

$du = e^x \quad v = ?$

$v_x = e^x \sin y.$

$\int v_x = \int e^x \sin y$

$V = e^x \sin y + c_2$

$u = e^x \cos y + c_1$

$f(z) = u + iv = e^x [\cos y + i \sin y] + c_1 + ic_2$

$= e^x e^{iy} + (c_1 + ic_2)$

$\boxed{e^z + \alpha} \Leftarrow = e^{x+iy} + \alpha \qquad \boxed{\alpha = c_1 + ic_2}$

Illy,

$$\overline{f(\bar{z})} = u(x,-y) - iv(x,-y).$$

is a analytic in D.

but $\overline{f(\bar{z})} = u_1(x,y) + v_1(x,y)$.

where $u_1(x,y) = u(x,-y)$.

& $v_1(x,y) = -v(x,-y)$.

$\therefore \overline{f(\bar{z})}$ is analytic.

we've, $\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y}$ &

$$\frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x}$$

$\therefore$ The first order partial derivatives u & v are continuous & satisfies the CR equation.

$\therefore f(z)$ is analytic.

$\therefore f(z)$ & $\overline{f(\bar{z})}$ are simultaneously analytic.

$f(z) = u(x,y) + iv(x,y)$

$f(\bar{z}) = u(x,-y) + iv(x,-y)$

$\overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$

$= u_1(x,y) + iv_1(x,y)$

$\boxed{\text{where } u(x,-y) = u_1(x,y) \text{ \& } -v(x,-y) = v_1(x,y)}$

$$\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v_1}{\partial y}$$

$$\frac{\partial u_1}{\partial y} = \frac{-\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$= -\frac{\partial u_1}{\partial x}$$

∴ The first order partial derivatives of u_1 & v_1 are continuous and satisfies CR-equations.

∴ $f(\bar{z})$ is analytic in a region D.

2)

$$\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial \bar{z}}\left(u_x - i u_y\right)$$

$$= \frac{1}{2}\left[\frac{\partial (u_x - i u_y)}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial (u_x - i u_y)}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right]$$

$$= \frac{1}{2}\left[(u_{xx} - i u_{xy})\left(\frac{1}{2}\right) + (u_{yx} - i u_{yy})\left(\frac{-1}{2i}\right)\right]$$

$$= \frac{1}{4}\left[u_{xx} - i u_{xy} + (u_{yx} - i u_{yy})\left(\frac{-1}{i}\right)\right]$$

$$= \frac{1}{4}\left[\quad \cdot \qquad \frac{i^2}{i}\right]$$

$$= \frac{1}{4}\left[u_{xx} - i u_{xy} + i u_{yx} - i^2 u_{yy}\right]$$

$$= \frac{1}{4}\left[u_{xx} - i u_{xy} + i u_{yx} + u_{yy}\right]$$

$$= \frac{1}{4}\left[(u_{xx} + u_{yy}) + i(u_{yx} - u_{xy})\right]$$

$$= \frac{1}{4}\left[0 + i\,0\right]$$

$$\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = 0.$$

from ① & ②.

$$-|z_1 - z_2| \le |z_1| - |z_2| \le |z_1 - z_2|.$$

(ie) $||z_1| - |z_2|| \le |z_1 - z_2|$

v) - Parallelogram law.

$$\boxed{|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]}.$$

This is for your exercise.

Result

i) zero is the only no. which is at once real & Imaginary parts.

ii) the reciprocal of a c.n. $\neq 0$ is given by $\frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}$

iii) i^n → They corresponds to value of n, which divided by 4, leaves the remainder 0,1,2,3.

$i^n \Rightarrow n = 0 \Rightarrow i^0 = 1$

$\Rightarrow n = 1 \Rightarrow i^1 = i$

$\Rightarrow n = 2 \Rightarrow i^2 = -1$

$\Rightarrow n = 3 \Rightarrow i^3 = i^2 \times i = -1 \times i = -i$

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Find the values of the following CN's:

i) $(1+2i)^3$

Soln: $(1+2i)^3 = (1+2i)^2 (1+2i)$

$\Rightarrow [1 + (2i)^2 + (2 \times 2i \times 1)] \ (1+2i)$

$\Rightarrow [(4i - 3) \ (1+2i)]$

$\Rightarrow 4i + 8i^2 - 3 - 6i$

$= 4i + 8(-1) - 3 - 6i$

$= 4i - 8 - 3 - 6i$

$$\boxed{\therefore (1+2i)^3 = -11 - 2i}$$

—— x ——

ii) $5/-3+4i$.

Soln: $\dfrac{5}{-3+4i} = \dfrac{5}{-3+4i} \times \dfrac{-3-4i}{-3-4i}$

$\Rightarrow \dfrac{5 \times (-3-4i)}{3^2 + 4^2} = \dfrac{5(-3-4i)}{9+16}$

$\Rightarrow \dfrac{5(-3-4i)}{25}$

$\Rightarrow \dfrac{-15-20i}{25}$

$\Rightarrow \dfrac{-15}{25} - \dfrac{20i}{25} \Rightarrow \dfrac{-3}{5} - \dfrac{4}{5}i$

(a) $\boxed{\dfrac{5}{-3+4i} = \dfrac{-3}{5} - \dfrac{4}{5}i}$

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POWER SERIES

A power series is of the form

$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n + \dots$

where the co-efficients a_n and the variable z are complex.

Consider the series of the function is of the form

$\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where z is a complex variable and z_0, a_n $n = 0, 1, \dots$ are fixed some complex no's.

This series is called power series with the centre z_0,

$\sum_{n=1}^{\infty} a_n z^{n-1}$ is called derived series of $\sum_{n=0}^{\infty} a_n z^n$.

Consider the geometric series

$1 + z + z^2 + z^3 + z^4 + \dots + z^n$.

Let (S_n) be the sequence of partial sum of the abover series.

$$\boxed{S_n = 1 + z + z^2 + z^3 + \dots + z^{n-1} = \frac{1 - z^n}{1 - z}, \text{ if } z < 1.}$$

$$\boxed{\text{Since, } z^n \to 0, \text{ for } |z| < 1, \; \& \; |z| \geq 1 \; \& \; |z| = 1.}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - z^n}{1 - z} = \frac{1}{1 - z}.$$

$\left[\because z^{\infty} = 0 \text{ if } z < 1\right]$

$\therefore$ The geometric series converges to $\frac{1}{1-z}$ for $|z|<1$, diverges for $|z| \geq 1$

—x—

Defn: The radius of convergence R of the given series $\sum_{n=0}^{\infty} a_n z^n$ is defined by,

$$R = \sup \left\{ \rho \,/\, \sum_{n=1}^{\infty} a_n z^n \text{ converges } \forall\, z \text{ satisfying } |z| \leq \rho \right\}.$$

Note: If $R=0$, then $\sum a_n z^n$ converges only for $z=0$.

—x—

Hadamard's formula:

For the radius of convergence is R is st $\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$

—x—

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COMPLEX ANALYSIS

A series $\sum a_n$ is said to be absolutely convergent if the series $\sum |a_n|$ is convergent.

$\sum |a_n| \to c$ if $\sum a_n \to c$
↳ Convergent

$\sum |a_n| \to d$ if $\sum a_n \to c$
↳ Conditionally con.

if $\sum_{n=1}^{\infty} a_n \to c \to |a_1| + |a_2| + |a_3| + \cdots + a_n \to c$
$|a_n| \to c \to |a_1| + |a_2| + |a_3| + \cdots + |a_n|$

Thm: Any absolutely convergent series is convergent.

Proof:
Let $\sum a_n$ be absolutely convergent.
Let $S_n = a_1 + a_2 + \cdots + a_n$
& $t_n = |a_1| + |a_2| + \cdots + |a_n|$
By hypothesis $\{t_n\}$ is convergent. Hence it is a cauchy sequence.

Hence, given $\epsilon > 0$, $\exists\ n_1 \ni : |t_n - t_m| < \epsilon$
$\forall\ n, m \geq n_1$

Now let $m > n$.

Then, $|S_n - S_m| = |a_{n+1} + \ldots + a_m|$
$\leq |a_{n+1}| + |a_{n+2}| + \ldots + |a_m|$
$= |t_n - t_m|$

$\therefore \{S_n\}$ is a cauchy sequence in R & hence it is converged.

$\therefore \Sigma a_n$ is a convergent series

Cauchy's first limit theorem

If $\lim_{n \to \infty} z_n = A$, P.T $\lim_{n \to \infty} \frac{1}{n}(z_1 + z_2 + \ldots + z_n) = A$.

Proof: Case (i): Let $A = 0$, in R.

Let $a_n = \frac{z_1 + z_2 + z_3 + \ldots + z_n}{n}$

Let $\epsilon > 0$, (given)

Since, $z_n \to 0$, then there exists $m \in N$. $\ni$: $|z_n| < \epsilon/2$
$\forall\, n \geq m$.

Then,

$$|a_n| = \left| \frac{z_1 + z_2 + \cdots + z_n}{n} \right|$$

$$= \left| \frac{z_1 + z_2 + z_3 + \cdots + z_m + z_{m+1} \cdots + z_n}{n} \right|$$

$$\leq \left| \frac{z_1 + z_2 + \cdots + z_m}{n} \right| + \left| \frac{z_{m+1} + z_{m+2} + \cdots + z_n}{n} \right|$$

$$\leq \frac{|z_1| + |z_2| + \cdots + |z_m|}{n} + \frac{|z_{m+1}| + |z_{m+2}| \cdots + |z_n|}{n}$$

$$\leq \frac{k}{n} + \frac{(n-m)}{n} \epsilon/2 \qquad \because |z_n| < \epsilon/2$$
$$|z_n - 0| < \epsilon/2$$

where $k = |a_1| + \cdots + |a_m|$

w.k. $\left(\frac{n-m}{n}\right) < 1$

$$\leq \frac{k}{n} + \epsilon/2$$

$$< \frac{k}{n} + \epsilon/2.$$

Since $\left(\frac{k}{n}\right) \to 0$ as $n \to \infty$. an $n_0 \in N$,
$\exists$: $|k/n| < \epsilon/2$, $\forall\, n \geq n_0$

Let $n_1 = \max\{m, n_0\}$

Then $|a_n| < \epsilon, \ \forall n \geq n_1$

$\therefore \|a_n\| \to 0$

— x —

<u>Case (ii)</u> $A \neq 0$.

Since $z_n \to A$, $z_n - A \to 0$.

$$\therefore \left(\frac{(z_1 - A) + (z_2 - A) + (z_3 - A) \cdots + (z_n - A)}{n}\right) \to 0$$

by case (i)

$$\therefore \left(\frac{z_1 + z_2 + z_3 + \cdots + z_n - nA}{n}\right) \to 0$$

$$\therefore \left(\frac{z_1 + z_2 + \cdots + z_n}{n} - \frac{nA}{n}\right) \to 0$$

$$\therefore \left(\frac{z_1 + z_2 + \cdots + z_n}{n} - A\right) \to 0.$$

$$\therefore \left(\frac{z_1 + z_2 + \cdots + z_n}{n}\right) \to A$$

$\therefore$ The above theorem is known as the Cauchy's first limit theorem.

— x —

Cauchy Criterion for Convergence

$\{z_n\}$

Let on $\mathbb{C}$, $\{z_n\}$ converges $\Leftrightarrow$ $\{z_n\}$ is a Cauchy sequence.

[$\Rightarrow$] Suppose $\{z_n\}$ converges to z_0.

i.e. $z_n \to z_0$ then $x_n = Re\, z_n \to x_0 = Re\, z_0$

$x_n \to x_0$

$Re\, z_n \to Re\, z_0$

$y_n = Im\, z_n \to y_0 = Im\, z_0$.

W.K.that, $\epsilon > 0$, be given

Every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.

Since $\{x_n\}$ is a Cauchy sequence, $\exists$ an $N_1(\epsilon) \ni$:

$|x_n - x_m| < \epsilon/2 \;\; \forall\, n, m > N_1(\epsilon)$

$\{y_n\}$ is a Cauchy sequence.

$\exists$ an $N_2(\epsilon)$ $\ni$: $|y_n - y_m| < \epsilon/2$,

$\forall$ $n, m > N_2(\epsilon)$.

Let $n = \max\{n_1, n_2\}$

$$\therefore |z_n - z_m| = |(x_n - x_m) + i(y_n - y_m)|$$

$$\leq |x_n - x_m| + |y_n - y_m|$$

$$= |Re(z_n - z_m)| + |Im(z_n - z_m)|$$

$$= \epsilon/2 + \epsilon/2$$

$$= \epsilon$$

$\therefore \{z_n\}$ is a Cauchy sequence.

51, 52, 63, 66, 67, ~~[illegible]~~, ~~[illegible]~~, 73

Conversely,

Suppose $\{z_n\}$ is a Cauchy seq.

To prove, $\{z_n\}$ is convergent in R.

$$|Re(z_n - z_m)| \leq |z_n - z_m|$$

$$\& \ |Im(z_n - z_m)| \leq |z_n - z_m|$$

$\because z_n$ is a cauchy sequence.

we've $|z_n - z_m| < \epsilon, \forall n, m > n$

where A is a +ive integer.

$\therefore |Re(z_n - z_m)| < \epsilon$

$\because |Re(z_n - z_m)| \leq |z_n - z_m| < \epsilon$

$\& \ |Im(z_n - z_m)| \leq \epsilon. \ \forall n, m > n$

$\therefore Re(z_n - z_m)$ is a cauchy seq in R.

$\&$ $Im(z_n - z_m)$ is a " " R.

[W.K.T. every Cauchy sequence in R is a bounded sequence, and also W.K.T. every bounded sequence of real no's has a limit. Therefore on Imaginary Parts of a Cauchy sequence they also converge..]

$\therefore \exists$ a subsequence $\{z_{n_k}\}$ of $\{z_n\}$

So that this subsequence $\{z_{n_k}\}$

Converges to $z_0 = x_0 + iy_0$

$\therefore$ The seq. $z_n \to z_0$

$\therefore$ Every Cauchy seq. is a convergent seq.

SERIES

$$\boxed{\begin{aligned} &\sum^{n} a_n = a_1 + a_2 + a_3 + \dots + a_n \\ &\prod a_n = a_1 . a_2 . a_3 . a_4 \dots a_n \end{aligned}}$$

An infinite series is a formal infinite sum.

$$\boxed{\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots}$$

Let $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$

$\therefore S_n = a_1 + \dots + a_n$

$\{s_n\}$ is called the sequence of Partial sums of the given series $\sum_{n=1}^{\infty} a_n$.

The series $\sum a_n$ is said to converge diverge or oscillate according as the Partial sums $\{s_n\}$ converge, diverge, oscillate.

S_n $S_1 + S_2 + S_3 + \dots \quad (S_{n-1})$

$a_1 \quad a_1 \quad a_1 + a_2 \quad a_1 + a_2 \dots$

If $S_n \to S$ then we say than Σa_n converges to the sums.

Cauchy's convergence test:

Thm: The series Σa_n is convergent iff, given $\epsilon > 0$, $\exists$ an $n_0 \ni$:
$|a_n + a_{n+1} + \dots + a_{n+p}| < \epsilon$, $\forall\, n \geq n_0$.
and for all $p \geq 0$

Proof: Let Σa_n be a convergent series.

Let $S_n = a_1 + a_2 + \dots + a_n$

$\therefore \{S_n\}$ is a convergent sequ.

$\therefore$ Every convergent sequence is a Cauchy sequence.

$\therefore \{S_n\}$ is a cauchy sequ.

$\therefore \exists\, n_0 : |S_{n+p} - S_{n-1}| < \epsilon \;\forall\, n \geq n_0$
& $\forall\, p \in \mathbb{N}$.

$\therefore |a_n + a_{n+1} + \dots + a_{n+p}| < \epsilon \;\forall\, n \geq n_0$
& $p \in \mathbb{N}$.

For, If $p = 0$, we find in particular that $|a_n| < \epsilon$.

Hence, the general term of a convergent series tends to zero.

This Condition is necessary, but not sufficient.

The sufficient condition is true for real no's.

If $\{a_n\}$ is a sequence of real no's.

& if $\{a_1 + a_2 + \dots + a_n\}$

Here, $|a_n + a_{n+1} + \dots + a_{n+p}| < \epsilon$, $\forall n \geq n_0$ & $p \in N$.

then $\{S_n\}$ is a cauchy sequence. $\because p > 0$.

—x—

Defn: A series $\sum a_n$ is said to be 'absolutely' convergent if the series $\sum |a_n|$ is convergent.

—x—

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UNIFORM CONVERGENCE.

The sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ on the set E, if to every $\epsilon > 0$, $\exists$ an n_0 $\ni: |f_n(x) - f(x)| < \epsilon$, $\forall n \geq n_0$, and for all $x \in E$.

— x —

Theorem:

The limit function of a uniformly convergent sequence of continuous function is itself continuous.

Proof:

Suppose that the function $f_n(x)$ are continuous and tends to uniformly to $f(x)$ on the set E.

For any $\epsilon > 0$, $\exists$ an $n \ni: |f_n(x) - f(x)| < \frac{\epsilon}{3}$ $\forall x \in E$.

Let x_0 be a point on E.

Since, $f_n(x)$ is continuous at x_0, we can find $\delta > 0$, $\ni: |f_n(x) - f(x)| < \epsilon/3$, $\forall x \in E$, with, $|x - x_0| < \delta$.

$$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$$

$$= |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

$\therefore f(x)$ is continuous at x_0.

— x —

Theorem:

The sequence $\{f_n(x)\}$ converges uniformly on $E \Leftrightarrow$ to every $\epsilon > 0$, $\exists$ an $n_0 \ni$:
$|f_m(x) - f_n(x)| < \epsilon$, $\forall\, n, m \geq n_0$ & $x \in E$.

Proof:
$\Rightarrow$ Suppose, the sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ on E.

For any $\epsilon > 0$, $\exists$ an $n_0 \ni$: $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ $\forall\, n \geq n_0$ for all $x \in E$.

For $n, m \geq n_0$ & all $x \in E$.

$$|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)|$$
$$\leq |f_m(x) - f(x)| + |f(x) - f_n(x)|$$
$$\leq \epsilon/2 + \epsilon/2$$
$$< \epsilon.$$

$\Leftarrow$ Suppose $|f_m(x) - f_n(x)| < \epsilon$ $\forall\, n, m \geq n_0$ & for all $x \in E$.

Fix n and allow m tends to ∞, we get,

$$|f(x) - f_n(x)| \leq \epsilon, \ \forall\, n \geq n_0.$$

$\therefore$ The given sequence $\{f_n(x)\}$ converges uniformly ~~on~~ to $f(x)$ on the set E.

—— x ——

$$f(z) = w = \frac{\alpha z + \beta}{\gamma z + \delta}$$

$$w(\gamma z + \delta) = \alpha z + \beta$$

$$w\gamma z + w\delta = \alpha z + \beta$$

$$w\gamma z - \alpha z = \beta - w\delta$$

$$z(w\gamma - \alpha) = \beta - w\delta$$

$$\therefore z = \frac{\beta - w\delta}{w\gamma - \alpha}$$

$$\boxed{z = \frac{\delta w - \beta}{-\gamma w + \alpha}}$$

$$\boxed{f^{-1}(w) = \frac{\delta w - \beta}{-\gamma w + \alpha}}$$

The transformations f & f^{-1} are inverse to each other.

By division, we've $R(z) = G(z) + H(z)$ where, $h(z)$ is a poly. without constant term and $H(z)$ is finite.

at α. if $R(z)$ has a pole at α. The degree of $G(z)$ is the order of the pole at α and the poly. $G(z)$ is called the singular part of $R(z)$ at α.

Let the distinct finite poles of $R(z)$ be denoted by

$\beta_1, \beta_2, \ldots \beta_q$.

The function $R(\beta_j + \frac{1}{\zeta})$ is a rational function of ζ with a pole at $\zeta = \alpha$.

By decomposition

$$R(\beta_j + \frac{1}{\zeta}) = G_j(\zeta) + H_j(\zeta)$$

(or) with a change of variable.

$$R(z) = G_j\left(\frac{1}{z-\beta_j}\right) + H_j\left(\frac{1}{z-\beta_j}\right)$$

$q_1, q_1, q_1+q_2, q_1+q_2 \ldots$

Here, $G_j\left(\frac{1}{z-\beta_j}\right)$ is a poly. in $\frac{1}{z-\beta_j}$ without constant term called the singular part of $R(z)$ at β_j.

The function, $H_j\left(\frac{1}{z-\beta_j}\right)$ is finite for $z=\beta_j$

— x —

Sequences:

The sequence $\{a_n\}_1^\infty$ has the limit A if to every $\epsilon > 0$, there exists a no. $\exists$: $|a_n - A| < \epsilon$ for $n \geq n_0$. A sequence with a finite limit is to be convergent, and any sequence which doesn't converge is divergent.

If $\lim_{n\to\infty} a_n = \infty$, the sequence may be said to be diverge of infinity.

—x—

Cauchy sequence:

A sequence will be called fundamental (or) Cauchy sequence if it satisfies the following condition, given any $\epsilon > 0$ $\exists$ an no. $\exists$: $|a_n - a_m| < \epsilon$, whenever $n \geq n_0$ & $m \geq n_0$

—x—

Theorem: Every convergent sequence is a Cauchy sequence.

Proof: Let $\{a_n\}$ be a convergent sequence.

Then take $\{a_n\} \to A$

Let $\epsilon > 0$, be given

Then $z_n \to z_0$.

Since $a_n \to A$, $\exists$ an n_0 $\ni$: $|z_n - z_0| < \frac{\epsilon}{2}$ $\forall n \geq n_0$.

(ii) $|a_n - a_m| = |a_n - A + A - a_m|$

$\leq |a_n - A| + |A - a_m|$

$< \epsilon/2 + \epsilon/2$

$< \epsilon$

$= \epsilon$. $\forall n, m \geq n_0$.

$\therefore \{a_n\}$ is a Cauchy seqn.

The sequence $z_n \to z_0$

$\Rightarrow Re\, z_n \to Re\, z_0$.

$\Rightarrow Im\, z_n \to Im\, z_0$.

Suppose $z_n = x_n + iy_n$ & $z_0 = x_0 + iy_0$

Then for ~~every~~ any given $\epsilon > 0$, $\exists$ n_0

$\ni$: $|z_n - z_0| = |(x_n + iy_n) - (x_0 + iy_0)|$

$\leq |x_n - x_0| + |i(y_n - y_0)|$

$\because z_n \to z_0$ we've $|z_n - z_0| < \epsilon$.

(i.e) $|x_n - x_0| + |y_n - y_0| < \epsilon$

(ii) $|x_n - x_0| < \epsilon/2$

& $|y_n - y_0| < \epsilon/2$

$\therefore$ $x_n \to x_0$

$y_n \to y_0$.

$\therefore$ $Re.z_n \to Re\, z_0$.

$Im\, z_n \to Im\, z_0$.

Since $x_n \to x_0$ & $y_n \to y_0$

$x_n \to x_0$ $\exists$ an n_1 $\exists$: $|x_n - x_0| < \epsilon/2$ $\forall n \geq n_1$

$y_n \to y_0$ $\exists$ an n_2 $\exists$: $|y_n - y_0| < \epsilon/2$ $\forall n \geq n_2$

Let $n_0 = \max\{n_1, n_2\}$

$$\therefore |z_n - z_0| = |(x_n + iy_n) - (x_0 + iy_0)|$$

$$\leq |x_n - x_0| + |i(y_n - y_0)|$$

$$= \epsilon/2 + \epsilon/2$$

$$= \epsilon \quad \forall n \geq n_0$$

$\therefore$ $z_n \to z_0$.

Analytic functions as mappings:

Sets and elements:

A set is a collection of identifiable objects. The objects are called elements.

A set also can be referred as a space. and an element as a point.

—— x ——

Distributive laws:

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$

—— x ——

De-Morgan's law:

$$\sim(X \cup Y) = \sim X \cap \sim Y$$

$$\sim(X \cap Y) = \sim X \cup \sim Y$$

Note: The complement of X is denoted by $\sim X$.

—— x ——

Metric space:

A metric space is a non-empty set 'X' together with a function $d : X \times X \to R$ satisfying the following conditions.

i) $d(x,y) \geq 0, \forall x,y \in X$

ii) $d(x,y) = 0$ iff $x = y$.

iii) $d(x,y) = d(y,x) \forall x,y \in X$

iv) $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in X$.

The (iv) holds triangle inequality. Here d is called a metric (or) metric function & $d(x,y)$ is called the distance between x & y.

Example:

In R, metric spaces with $d(x,y) = |x-y|$. This is called the usual metric on R.

III^{ly}, in $\mathbb{C}$, $d(z,w) = |z-w|$. This is called the usual metric in $\mathbb{C}$.

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②

The n-dimensional euclidean space $\mathbb{R}^n$,

$$(d(x-y))^2 = \sum_1^n (x_i - y_i)^2$$

where $x, y \in \mathbb{R}^n$

$x = (x_1, x_2, \dots x_n)$
$y = (y_1, y_2, \dots y_n)$

—x—

Open set:

A set $N \subset S$ is called a neighbourhood of $y \in S$ if it contains an open ball $B(x, \delta)$

—x—

A set is open if it is a neighbourhood of each of it's element.

—x—

Theorem:

Prove that Every open ball is an open set.

Proof:

Consider an open ball $B(y, \delta)$

Let $\in$ ~~x~~ $z \in B(y, \delta)$

$\therefore d(z, y) < \delta$.

(ie) $\delta - d(z, y) > 0$

Let $\delta_1 = \delta - d(z,y)$

To prove that $B(z,\delta_1) \subset B(y,\delta)$

Let $y \in B(z,\delta_1)$

$\therefore d(z,y) < \delta_1$

i.e) $< \delta - d(z,y)$

To prove $B(z,\delta_1) \subset B(y,\delta)$

$x \in B(z,\delta_1)$

$\therefore d(x,z) < \delta_1$

$d(x,z) + d(z,y) < \delta$

$d(x,y) \leq d(x,z) + d(z,y)$

$< \delta$

$\therefore x \in B(y,\delta)$

$\therefore B(z,\delta_1) \subset B(y,\delta)$

$\therefore B(y,\delta)$ is an open set.

—x—

Theorem: Prove that the intersection of a finite no. of open set is open.

Proof: Let (X,δ) be a metric space.

Let $A_1, A_2, \dots A_n$ be an open set in X.

This is called the usual metric in $\mathbb{R}$.

—x—

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(2)

Let $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$

Let $A = \phi$ then A is open.

If, Let $A \neq \phi$.

Let $x \in A$.

Since each A_i is open, $i = 1$ to n.

$\exists$ a real no. r_i such that

$B(x, r_i) \subset A_i$, $i = 1$ to $n \rightarrow$ (1).

Let $r_i = \min\{r_1, r_2, \dots r_n\}$.

clearly r is a positive real number and $B(x, r) \subseteq B(x, r_i)$ for $i = 1$ to n.

Hence $B(x, r_i) \subseteq A_i \; \forall \; i = 1$ to n.

$\therefore B(x, r) \subseteq \bigcap_{i=1}^{n} A_i$ by (1)

Hence A is an open.

—x—

Theorem.

P.T The union of any collection of open set is open.

Proof:

Let (X, d) be a metric space.

Let $\{A_i / i \in I\}$ be a family of opensets in X.

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Let $A = \bigcup_{i \in I} A_i$

If $A = \phi$ then A is open.

Let $A \neq \phi$.

Let $x \in A$.

Then $x \in A_i$ for some $i \in I$.

Since A_i is open.

$\exists$ an open ball $B(x, r) \subset A_i$

$\therefore B(x, r) \subset \bigcup_{i \in I} A_i$

Hence A_i is open.

— x —

Theorem:

Prove that the union of a finite no. of closed set is closed.

Proof:

Let (X, ρ) be a metric space. Let $A_1, A_2, \ldots A_n$ be a closed sets in X.

$\sim (A_1 \cup A_2 \ldots \cup A_n) = \sim A_1 \cap \sim A_2 \sim A_3 \sim \ldots \sim A_n$

(By Demorgan's law)

Since each A_i is closed, $\sim A_i$ is open.

Hence ...

By the theorem,

$A_1 \cup A_2 \cup \ldots \cup A_n$ is closed.

Theorem

Prove that the intersection of any collection of closed set is closed.

Proof:

Let (X, d) be a metric space. Let $\{A_i \mid i \in I\}$ be a collection of closed sets.

To prove, $\bigcap_{i \in I} A_i$ is closed.

W.K.T

$$\sim \bigcap_{i \in I} A_i = \bigcup_{i \in I} \sim A_i$$

(By demorgans law)

Since each A_i is closed. Complement of A_i is open.

(i.e) $\sim A_i$ is open.

Hence, $\bigcup_{i \in I} \sim A_i$ is open.

$\therefore \bigcap_{i \in I} A_i$ is closed.

[$\because$ The collection of any collection of open sets is open.]

Connectedness:

Let (X, d) be a metric space. X is said to be connected if X cannot be written as the two disjoint non-empty open sets.

— x —

Theorem:

Let non-empty connected subset of the real line of the intervals.

Proof:

Let A be a connected subset of R.

Suppose A is not an interval. Then $\exists\, a, b, c \in R \ni : a < b < c$, $a, c \in A$, but $b \notin A$.

Let $A_1 = (-\infty, b) \cap A$ &
$A_2 = (b, \infty) \cap A$

$\therefore (-\infty, b)$ & (b, ∞) are open in $\mathbb{R}$, A_1 & A_2 are open sets in A.

Also, $A_1 \cap A_2 = \phi$ & $A_1 \cup A_2 = A$.

Further, $a \in A_1$ & $c \in A_2$

Hence, $A_1 \neq \phi$ & $A_2 \neq \phi$.

Thus, A is the union of two disjoint non-empty open sets A_1 & A_2.

Hence A is not connected.

which is a $\Rightarrow\Leftarrow$.

Hence, A is an interval.

Conversely,

Let A be an interval.

we claim that,

~~Let~~ A is connected.

Let $A = A_1 \cup A_2$ where $A_1 \neq \phi$, $A_2 \neq \phi$, $A_1 \cap A_2 = \phi$. and A_1 & A_2 are closed sets in A.

chose, $x \in A_1$ & $z \in A_2$

Since, $A_1 \cap A_2 = \phi$ we've $x \neq z$

without loss of generality we assume that ~~$x < z$~~ $x < z$,

Now, since A is an interval $[x, z] \subseteq A$.

(ie) $[x, z] \subseteq A_1 \cap A_2$.

$\therefore$ Every element of $[x,z]$ is either in A_1 or A_2.

Now, let $y = \text{lub}\{[x,z] \cap A_2\}$.

clearly, $x \leq y \leq z$.

Hence $y \in A$.

Let $t > 0$, be given.

Then by the definition of l.u.b

$\exists$ $t \in [x,z] \cap A_1$ $\exists$! $y - t < t \leq y$

$\therefore$ $(y-t, y+t) \cap ([x,z] \cap A_1) \neq \phi$.

$\therefore$ $y \in \overline{[x,z] \cap A_1}$ by $x \in \bar{A} \Leftrightarrow B(x,r) \cap A \neq \phi$; $\forall r > 0$

$\therefore$ $y \in [x,z] \cap A_1$ $[\because [x,z] \cap A_1$ is closed in $A]$.

$\therefore$ $y \in A_1$

Again by the defn. of y,

$y + t \in A_2$, $\forall$ $t > 0$. $\exists$: $y + t \leq z$.

$\therefore$ $y \in \bar{A_2}$

$\therefore$ $y \in A_2$ ($\because A_2$ is closed) ②.

$y \in A_1 \cap A_2$ [by ① & ②]

which is a $\Rightarrow\Leftarrow$

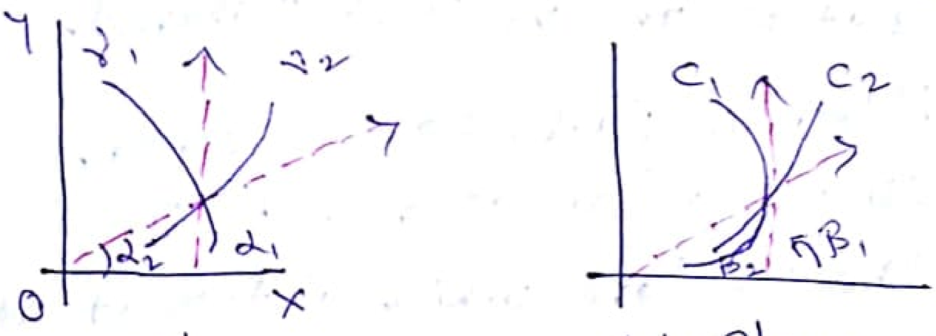
Since $A_1 \cap A_2 = \phi$. Hence A is connected.

Conformal mappings:

If the two curves in the z-plane intersect at the point z_0 at an angle θ, then if the two corresponding curves in the w-plane intersect at the point w_0 which corresponds to z_0 at the same angle θ, the transformation is said to be Isogonal.

Thus, if only the magnitude of the angle is preserved then the transformation is called Isogonal.

If the sense of the rotation as well as the magnitude of the angle is preserved, the transformation is said to be Conformal.



Z-Plane W-Plane.

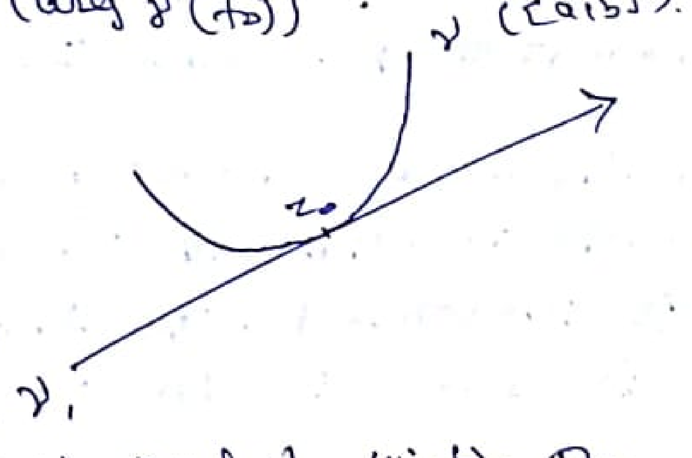
Description of angle preserving mapping:

Let $\gamma : [a,b] \to \mathbb{C}$ be a Jordan arc of class C.

Let $z_0 = \gamma(t_0)$ be a point on $\gamma([a,b])$ with $\gamma'(t_0) \neq 0$

Then γ has a tangent at z_0. The direction of such a line is defined to be the direction of $\gamma'(t_0)$.

(ie) The slope of this line is $\tan(\arg \gamma'(t_0))$.



Let γ_1 and γ_2 with parametric interval $[a,b]$ be two Jordan curves in an open set D. in the z-plane which intersect at $z_0 = \gamma_1(t_1) = \gamma_2(t_2)$

Then the angle of inclination θ between γ_1 and γ_2 at z_0, measured from γ_1 to γ_2 (anticlockwise) is the angle formed by their tangents at z_0.

$$\theta = \arg(\gamma_1'(t_1)) - \arg(\gamma_2'(t_2))$$

Suppose that f is analytic on D, such that $f'(z_0) \neq 0$ for $z_0 \in D$. Since each value of z in the z-plane gives an unique value of w in the w-plane, any curve in the z-plane must correspond to a curve in the w-plane.

Let T_1 & T_2 be the images of γ_1 & γ_2 under $w = f(z)$ respectively.

Let $z_1 = \gamma_1(t_1)$ $(z_1 \neq z_0)$ and $z_2 = \gamma_2(t_2)$, $(z_2 \neq z_0)$ be the variable points of γ_1 & γ_2 near z_0 respectively.

As $z_1 \to z_0$, $\arg(z_1 - z_0) \to \theta_1$, the inclination of the tangent to γ_1 at z_0.

In the ω-plane, as $z_1 \to z_0$

$w_1 = f(z_0) \to w_0\ f(z_0)$.

$\arg(w_1 - w_0) \to \phi_1$, the inclination of tangent at w_0 to T_1.

$$\theta_1 = \lim_{z_1 \to z_0} \arg(z_1 - z_0) \quad \&$$

$$\cancel{\phi(\omega) = \lim_{\omega \to \omega_0} \arg[f(z_1) - f(z_0)] - \arg(z_1 - z_0)]}$$

$$\phi_1 = \lim_{\omega \to \omega_0} \arg[f(z_1) - f(z_0)]$$

$$\therefore \phi_1 - \theta_1 = \lim_{z_1 \to z_0} \left[\arg(f(z_1) - f(z_0)) - \arg(z_1 - z_0)\right]$$

$$= \lim_{z_1 \to z_0} \arg\left(\frac{f(z_1) - f(z_0)}{z_1 - z_0}\right)$$

As f is analytic at z_0 and $f'(z_0)$ exist in neighbourhood at z_0.

$$f'(z_0) = \lim_{z_1 \to z_0} \frac{f(z_1) - f(z_0)}{z_1 - z_0}$$

$$= \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0}$$

If we write, $z_1 - z_0 = re^{i\theta}$

$w_1 - w_2 = Re^{i\phi}$ and $f'(z_0) = ke^{i\phi}$

⑨

$$K e^{i\psi} = f'(z_0) = \lim_{z_1 \to z_0} \left\{ \frac{R}{r} e^{i(\phi - \theta)} \right\}$$

$$\therefore K = |f'(z_0)| = \lim_{r \to 0} \frac{R}{r}.$$

$$\& \ \psi = \arg f'(z_0) = \lim_{r \to 0} (\phi - \theta)$$

$$= \phi_1 - \theta_1$$

M^{ly},

$$\arg f'(z_0) = \lim_{r \to 0} (\phi - \theta) = \phi_1 - \theta_1$$

11^{ly}, $\arg f'(z_0) = \phi_2 - \theta_2$, where ϕ_2 is the inclination of the tangent to Γ_2 at w_0 & θ_2, the inclination of the tangent to γ_2 at z_0.

$$\therefore \arg f'(z_0) = \phi_1 - \theta_1 = \phi_2 - \theta_2$$

$$\Rightarrow \phi_2 - \phi_1 = \theta_2 - \theta_1$$

(i) the angle between the two curves at the point of intersection z_0 is invariant under any analytic mapping f with $f'(z_0) \neq 0$.

If f is analytic on an open set D with $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping. —x—

Theorem :

The non-empty connected subset of the real line are the intervals.

Soln:

Suppose that the real line R represented as the union $R = A \cup B$, of two disjoint closed sets.

If $A \neq \phi$ & $B \neq \phi$. we can find $a_1 \in A$, and $b_1 \in B$, we may assume that $a_1 < b_1$ we biject the interval (a_1, b_1) as one of the two halves has its left ~~and~~ end point in A and its right and point in B.

we denote this interval by (a_2, b_2) and continue the Process indefinitely. In this way we obtain a sequence of nested intervals (a_n, b_n) with $a_n \in A$, $b_n \in B$. The sequences $\{a_n\}$ & $\{b_n\}$ have a common limit c.

Since A & B are closed c would have to be a common point of A & B..

(1

But, our assumption is $A \cap B = \phi$.
This contradiction shoulds that either A or B is empty.

Hence R is connected.

$\Leftarrow$ ly, suppose that R is connected. Let t_2 be an arbitrary subset of R.

We assume that E is a connected set. with the greatest lower bound 'a' and the least upper bound b.

All points of E lie b/w a & b. limits included. Suppose that a point ξ_c from the open interval (a, b) didn't belong to E. Then the open sets defined by $x < \xi$ and $x > \xi$ cover E, and because E is connected one of them must fail to meet E.

Suppose no point of E lies to the left of ξ. Then ξ would be a lower bound in contradiction with the fact that a is the greatest lower bound.

111[4],

we can prove ξ_1 is an upper bound. But b is the least upper bound.

Hence ξ_1 must belong to E.

It follows that E is an open, closed or semiclosed interval with the end points a & b and are included. $a = -\infty$, $b = +\infty$.

— x —

Theorem: Any closed and bounded non-empty set of real no's. has a minimum and a maximum.

Ans: Write the converse part of the above theorem.

— x —

Theorem: Any non-empty open set in the plane is connected. Iff any two of it's points can be joined by a polygan which lie in the set.

Proof: Necessary Part:

Let A be an open connected set choose a point $a \in A$.

Let A_1 be the subset of A whose points can be joined to a by

... common point of $\bar{A}$ & B.

Polygons in A & A_2 the subset of A whose point cannot be joined.

To prove,

A_1 is open.

If $a_1 \in A_1$, there ∃ a neighbourhood $|z-a| < \epsilon$ contained in A. All points in the neighbourhood can be joined to 'a_1' by a line segment and from there to a by a polygon.

Hence, the whole neighbourhood is contained in A_1 & A_2 is open.

Secondly if $a_2 \in A_2$, let $|z-a_2| < \epsilon$, be a neighbourhood in A. If a point in this neighbourhood could be joined to 'a' by a polygon then a_2 could be joined to this point by a line segment and from these to 'a'. This is a contrary to the definition of A_2 and we conclude that a_2 is open. Since A was connected either A_1 or A_2 must be empty.

But, A_1 contains a point 'a'.
Hence A_2 is empty and all points can be joined to a.

Finally, any two points in A can be joined by way of 'a'.

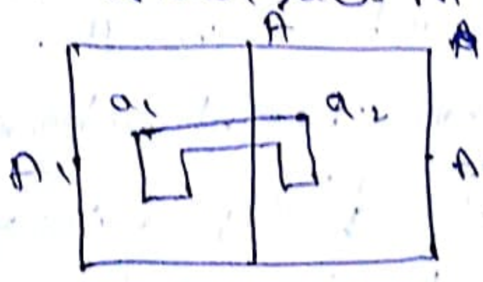
Hence the condition is necessary.

$\Leftarrow$ (sufficient condition)

Suppose A is not connected.

$\therefore A = A_1 \cup A_2$ where A_1 & A_2 are non-empty disconnected open sets.

We assume that any two points in A, can be joined by a polygon which lies in A.



Choose $a_1 \in A_1$, $a_2 \in A_2$.

Suppose a_1 & a_2 are joined by a polygon in A. one of the sides of the polygon must then join a point in A_1 to a point in A_2.

For this reason, it is sufficient to consider the case where a_1 & a_2 are joined by a line segment.

This segment has a parametric representation.

$$z = t\, a_2 + (1-t)\, a_1, \quad 0 \le t \le 1$$

(ie) $z = a_1 + t(a_2 - a_1), \quad 0 \le t \le 1.$

The subsets of the interval $0 < t < 1$ which corresponds to points in A_1 & A_2 respectively are open, disjoint & non void.

This contradicts, the connectedness of this condition of the theorem is sufficient.

—x—

Defn:

A non-empty connected open set is called a region.

—x—

Theorem:

Every set has a unique decomposition into components.

Defn:

A component of a set is a connected subset, which is not contained in any larger connected subset.

—x—

Proof:

Let E be a given set.
Consider, a point $a \in E$.
Let $C(a)$ denote the union of all connected subsets of E, that contain a, then $C(a)$ contain a, for the set consisting of the single point a is connected.

If we can show that $C(a)$ is connected, then it is a maximal, connected set, in otherword a component.

It would follows, then we prove that any two components are either disjoint or identical.

If $C(a) \cap C(b) \neq \phi$ &
$c \in C(a) \cap C(b)$. then
$C(a) \subset C(c)$ by the defn. of $C(c)$ and the connectedness of $C(a)$.

Hence $a \in C(c)$
$\therefore C(c) \subset C(a)$
$|||^{ly}$, $C(a) \subset C(c)$
$\therefore C(a) = C(c)$

$\therefore$ $C(a)$ is a connected subset, which is not contained in any larger connected subset.

$c \in C(a) \cap C(b)$

$\therefore$ $c \in C(a)$ & $C(b)$

If $c \in C(a)$,

$C(a) \subset C(c)$ — (1)

Now, $a \in C(c)$

$C(c) \subset C(a)$ — (2).

from (1) & (2) $\Rightarrow$ $C(a) = C(c)$

|||ly, $C(b) = C(c)$ and

$C(a) = C(b)$.

$\therefore$ $C(a)$ is the component of a.

To prove, $C(a)$ is connected.

Suppose, $C(a)$ were not connected.

Then $C(a) = A \cup B$ — (*) where A and B are non empty disjoint open sets.

Assume $a \in A$, $b \in B$.

from (*), $B \subset C(a)$.

$\therefore$ $B \in C(a)$, there is a connected set $E_0 \subset E$ which contains a and b.

The representation $E_0 = (E_0 \cap A) \cup (E_0 \cap B)$ decompose into relatively open subsets.

$\because a \in E_0 \cap A$, ~~$E_0 \cap B$~~

$b \in E_0 \cap B$.

neither part would be empty.

This is contrary the connected-ness of E_0.

— x —

<u>Theorem</u>:

In $\mathbb{R}^n$, the set of components is countable.

<u>Proof</u>:

In $\mathbb{R}^n$, every open set must contain a point with rational co-ordinates.

The set of points with rational co-ordinates is countable and it is expressed as a sequence $\{P_K\}$. For each component $e(a)$ determine the smallest $K \ni: P_K \in P(a)$ for giving different values for K, we get different components.

These components are in one to one correspondence with set of normal numbers.

$\therefore$ The set of components is countable.

— x —

Theorem: In $\mathbb{R}^n$, the components of any open set are open.

Proof:
Every δ-neighbourhoods in $\mathbb{R}^n$ are connected.

Consider $a \in C(a) \subset E$, where $C(a)$ is the union of all connected subsets of E that contain a.

claim: $C(a)$ is component.

If E is open then it contains $B(a, \delta)$.

(ie) $B(a,\delta) \in E$.

Since $B(a,\delta)$ is connected.

$B(a,\delta) \subset C(a)$.

Hence $C(a)$ is open

—x—

Defn: A metric space X is said to be completed if every Cauchy sequence in X converges to a point in X.

Cauchy sequence: The set $\{x_n\}$ is said to be a Cauchy sequence if $d(x_n, x_m) \to 0$

as n, m tends to ∞.

Defn:

A collection of open sets in an open covering of a set X if X is contained in the union of the open sets.

(ie) $\bigcup_{\alpha \in I} G_\alpha \supseteq X$ where each G_α is an open set.

A subcovering is a subcollection with the same property and a finite covering is one that consists of a finite no. of sets.

Defn:

A set X is compact iff every open covering of X contains a finite subcovering.

Defn:

A set X is totally bounded if for every $\epsilon > 0$, X can be covered by finitely many balls of radius ϵ.

(ie) $B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \dots \cup B(x_n, \epsilon) \supseteq X$

Defn:

Let f is continuous at a, if to every $\epsilon \geq 0$ $\exists$ $\delta > 0$, $\exists$: $d(x, a) < \delta$ implies $d(f(x), f(a)) < \epsilon$.

—x—

Theorem:

Under a continuous mapping the image of every compact set is compact.

Proof:

Suppose that f is defined and continuous on the compact set X.

Consider a covering of $f(X)$ by open set U.

The inverse image $f^{-1}(U)$ are open and form a covering of X.

Since X is a compact, we can select a finite subcovering.

$X \subset f^{-1}(U_1) \cup f^{-1}(U_2) \cup f^{-1}(U_3) \ldots \cup f^{-1}(U_n)$

(i) $f(X) \subset U_1 \cup U_2 \cup U_3 \ldots \cup U_n$.

Hence $f(X)$ is compact

—x—

Theorem:

Under a continuous mapping the image of any connected set is connected.

Proof:

We may assume that f is defined and continuous on the whole space S, and that $f(S)$ is all of S'.

Suppose that $S' = A \cup B$, where A & B are non-empty disjoint open sets.

Then $S = f^{-1}(A) \cup f^{-1}(B)$ is a ~~report~~ representation of S as a union of disjoint open sets.

If S is connected either $f^{-1}(A) = 0$ or $f^{-1}(B) = 0$ and hence $A = 0$ or $B = 0$.

$\therefore$ ~~S~~ $S' = f(S) = A$ or $f(S) = B$.

Hence $f(S)$ is connected

—x—