CHAPTER - I

Fuzzy Sets: Basic Types

Fuzzy set theory was introduced by Lotfi zadeh in the year 1965. He derived the fuzzy set as an extension of classical notation set.

⇒ The word fuzzy means vagueness (ambiguity)

⇒ Fuzziness occurs when the boundary of a piece of information is not clear-cut.

⇒ classical set theory allows the membership of the elements in the set binary terms

⇒ Fuzzy set theory permits membership function valued in the interval [0,1].

Membership function (or) Characteristic function:

A charasteristic function is defined as the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set. Large values denote higher degrees of set membership. Such function is called a membership function and the set defined by it a fuzzy set.

Definition: Fuzzy Set:

Let X be a non-empty set.
A function $\mu : X \rightarrow [0, 1]$ is called a fuzzy set on X for any $x \in X$, the number $\mu(x)$ is called the membership grade of x.

Example - 1

Words like young, tall, good or high are fuzzy

* There is no single quantitative value which defines the term young.
* For some people, age 25 is young, and for others, age 35 is young.
* The concept young has no clear boundary.
* Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

Example - 2

* Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the reference set of students.

* Let $\mu(x)$ be the fuzzy set of "Smart" students, where 'Smart' is fuzzy term.

$$\mu(x) = \{(x_1, 0.4)\,(x_2, 0.5)\,(x_3, 1)\,(x_4, 0.9)\,(x_5, 0.8)\}$$

Here $\mu(x)$ indicates that the smartness of x_1 is 0.4 and so on.

Example - 3

Let us consider four fuzzy sets whose membership function are shown in Figure 3.1. Each of these fuzzy sets expresses, in a particular form, the general conception of a class of real numbers that are close to 2.

The four fuzzy sets are similar in the sense that the following properties are possessed by each A_i $(i \in N_4)$:

(i) $A_i(2) = 1$ and $A_i(x) < 1$ for all $x \neq 2$;

(ii) A_i is symmetric with respect to $x = 2$, that is $A_i(2+x) = A_i(2-x)$ for all $x \in R$;

(iii) $A_i(x)$ decreases monotonically from 1 to 0 with the increasing difference $|2-x|$.

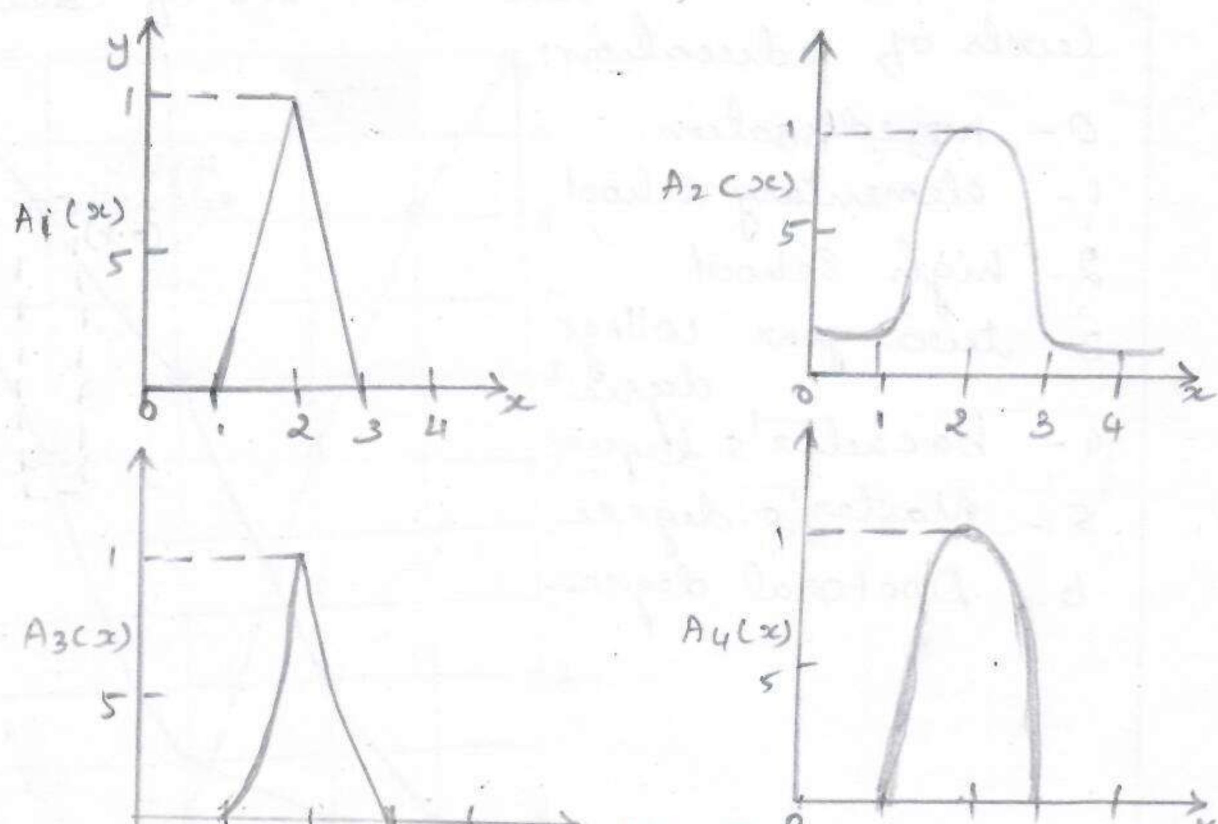


Fig: 3.1

Each function in Fig 3.1 is a member of parameterized family of functions.

$$A_1(x) = \begin{cases} P_1(x-r)+1 & \text{when } x \in [r - 1/P_1, r] \\ P_1(r-x)+1 & \text{when } x \in [r, r+1/P_1] \\ 0 & \text{otherwise} \end{cases}$$

$$A_2(x) = \frac{1}{1+P_2(x-r)^2}$$

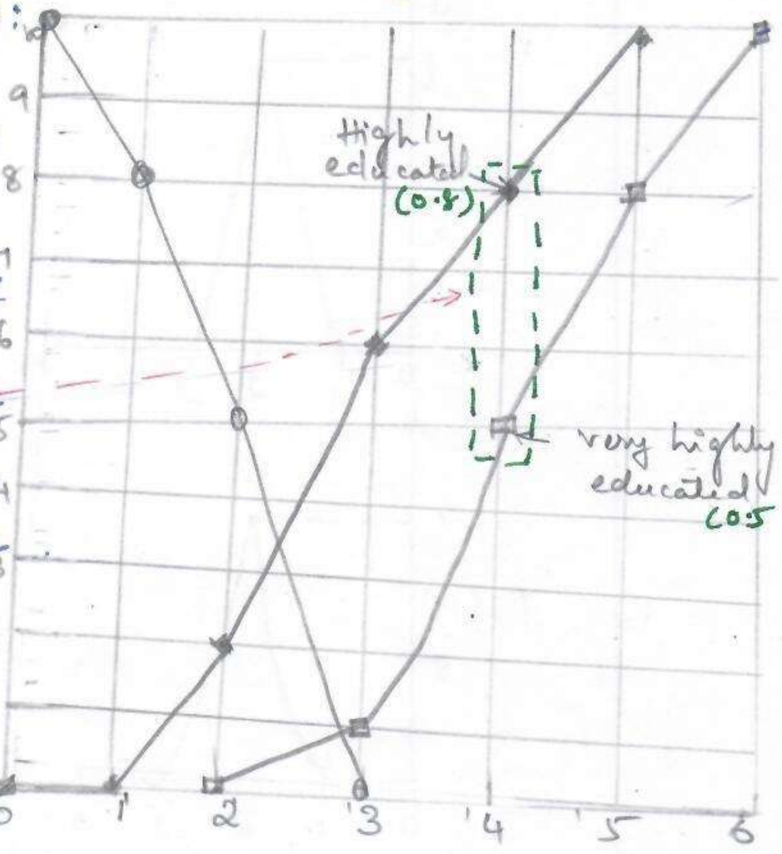
$$A_3(x) = e^{-|P_3(x-r)|}$$

$$A_4(x) = \begin{cases} (1+\cos(P_4\pi(x-r)))/2 & \text{when } x \in [r-1/P_4, r+1/P_4] \\ 0 & \text{otherwise} \end{cases}$$

Example - 4

Let us consider now, as a simple example, three fuzzy sets defined within a finite universial set that consists of seven levels of education:

0 - no education
1 - elementary school
2 - high school
3 - two-year college degree
4 - bachelor's degree
5 - Master's degree
6 - Doctoral degree



Fuzzy Variable :-

Several fuzzy sets representing linguistic concepts such as low, medium, high, and so on are often employed to define states of a variable. Such a variable is usually called a fuzzy variable.

Example -

The temperature within a range $[T_1, T_2]$ is characterized as a fuzzy variable, and it is contrasted in Figure (A & B) with comparable traditional variable (Non-fuzzy).

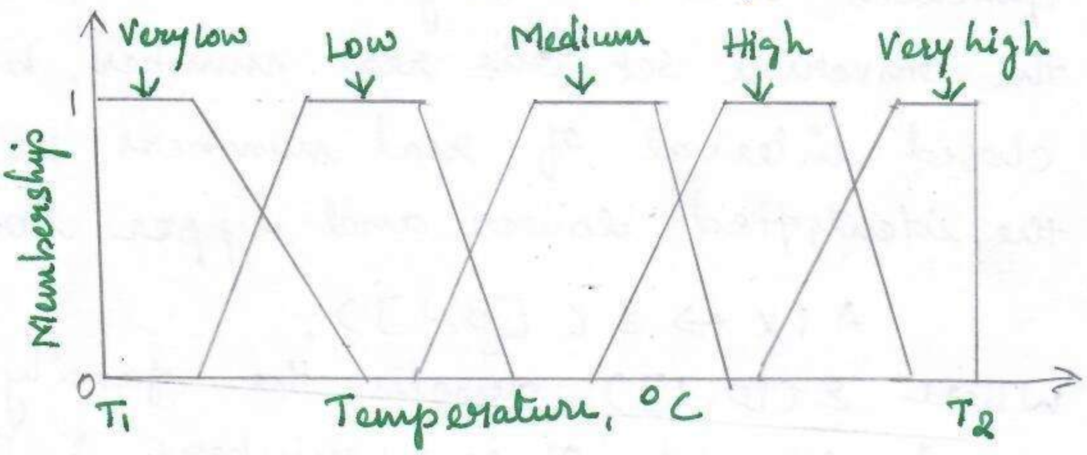


Figure (A)

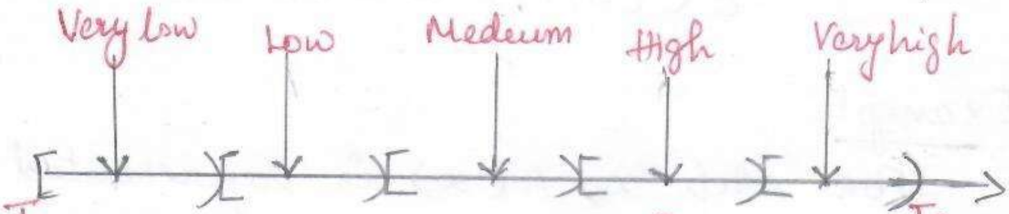


Figure (B)

States of fuzzy variable are fuzzy sets representing five linguistic concepts: very low low, medium, high, very high. They are all defined by membership functions of the form.

$$[T_1, T_2] \to [0, 1].$$

Graphs of these functions have trapezoida shapes, which together with triangular shapes

<u>Definition</u>: <u>Interval-valued fuzzy sets</u>:

A membership function based on the latt approach. "A fuzzy set whose membership functions does not assign to each element of the universal set one real number, but a closed interval of real numbers, between the identified lower and upper bounds.

$$A : X \to \mathcal{E}([0,1]).$$

Where $\mathcal{E}([0,1])$ denotes the family of all closed intervals of real numbers in $[0,1]$; Clearly,

$$\mathcal{E}([0,1]) \subset \mathcal{P}([0,1])."$$

<u>Example</u>

For each x, $A(x)$ is represented by a Segment between the two curves, which expres the identified lower and upper bounds. Thus, $A(a) = [\alpha_1, \alpha_2]$

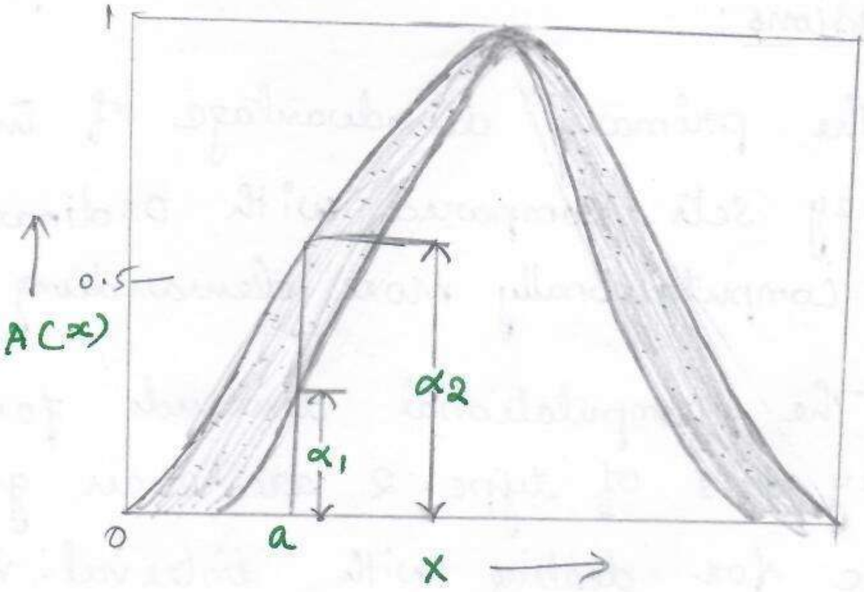


Figure : An example of an interval-valued fuzzy set $(A(a) = [\alpha_1, \alpha_2])$

Definition : Type - 2 fuzzy Set

Let X be a non-empty set. A function $A : X \to \mathcal{F}([0,1])$ where $F([0,1])$ denotes the set of all ordinary fuzzy sets that can be defined with the universal set $[0,1]$ is called a fuzzy set of type-2 and $\mathcal{F}([0,1])$ is also called a fuzzy power set of $[0,1]$.

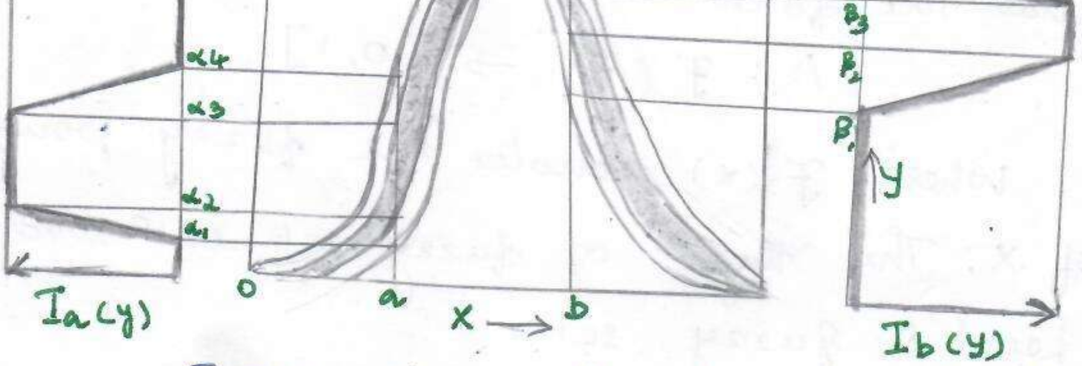


Figure shows the concept of fuzzy set of type-2

Discussions :

* The primary disadvantage of interval-value fuzzy sets compared with ordinary fuzzy sets, is computationally more demanding

* The computational demands for dealing with fuzzy sets of type-2 are even greater than those for dealing with interval-valued fuzzy sets.

* This is the primary reason why the fuzzy sets of type-2 have almost never been utilized in any applications.

Definition: L-fuzzy Set:

A Fuzzy set whose membership function has the form $A : X \rightarrow L$.

Where L is lattice or atleast a partially ordered set.

Definition: Level 2 fuzzy Set:

A Fuzzy set whose membership function has the form

$$A : \mathcal{F}(X) \rightarrow [0,1].$$

Where $\mathcal{F}(X)$ denotes the fuzzy power set of X. This ~~type~~ type of fuzzy set is called the Level 2 fuzzy set.

Example

Assuming that the proposition "x is close to r" is represented by an ordinary fuzzy set B, the membership grade of a value of x that is known to be close to r in the level 2 fuzzy sets A is given by A(B).

Definition: Fuzzy Sets of type 2 and Level 2:

A fuzzy set whose membership function has the form $A : \mathcal{F}(X) \to \mathcal{F}([0,1])$.

Where $\mathcal{F}(X)$ denotes the fuzzy power set of X. This type of fuzzy set is called the fuzzy set of type 2 and Level 2.

Discussions:

* These generalized types of fuzzy sets have not as yet played a significant role in applications of fuzzy set theory.

* Two reasons to introduce the generalized fuzzy sets in this section:

⇒ The reader can understand that fuzzy set theory does not stand or fall with ordinary fuzzy set.

⇒ The practical significance of some of the generalized types will increase.

2. FUZZY SETS : BASIC CONCEPTS

Definition: 2.1 – α-cut and Strong α-cut

Given a fuzzy set A defined on X and any number $\alpha \in [0, 1]$, the α-cut, ${}^{\alpha}A$ and the Strong α-cut, ${}^{\alpha+}A$, are the crisp sets:

$$ {}^{\alpha}A = \{x \mid A(x) \geq \alpha\} $$
$$ {}^{\alpha+}A = \{x \mid A(x) > \alpha\} $$

* The α-cut of a fuzzy set A is the crisp set that contains all the elements of the universal set X whose membership grades in A are greater than or equal to the specified value of α.

* The strong α-cut of a fuzzy set A is the crisp set that contains all the elements of the universal set X whose membership grades in A are only greater than the specified value of α.

Definition: 2.2 Level set of A

The set of all levels $\alpha \in [0, 1]$ that represent distinct α-cuts of a given fuzzy set A.

$$ \Lambda(A) = \{\alpha \mid A(x) = \alpha \text{ for some } x \in X\} $$

2. FUZZY SETS : BASIC CONCEPTS

In this Section, Some basic concepts and terminology of fuzzy sets are introduced. To illustrate the concepts, consider three fuzzy Sets that represent the concepts of young, middle-aged and old person.

The membership functions are defined on the interval [0, 80] as follows:

$$A_1(x) = \begin{cases} 1 & \text{when } x \le 20 \\ (35-x)/15 & \text{when } 20 < x < 35 \\ 0 & \text{when } x \ge 35 \end{cases}$$
(young)

$$A_2(x) = \begin{cases} 0 & \text{when either } x \le 20 \text{ or } \ge 60 \\ (x-20)/15 & \text{when } 20 < x < 35 \\ (60-x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \le x \le 45 \end{cases}$$
(middle-aged)

$$A_3(x) = \begin{cases} 0 & \text{when } x \le 45 \\ (x-45)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } x \ge 60 \end{cases}$$
(old)

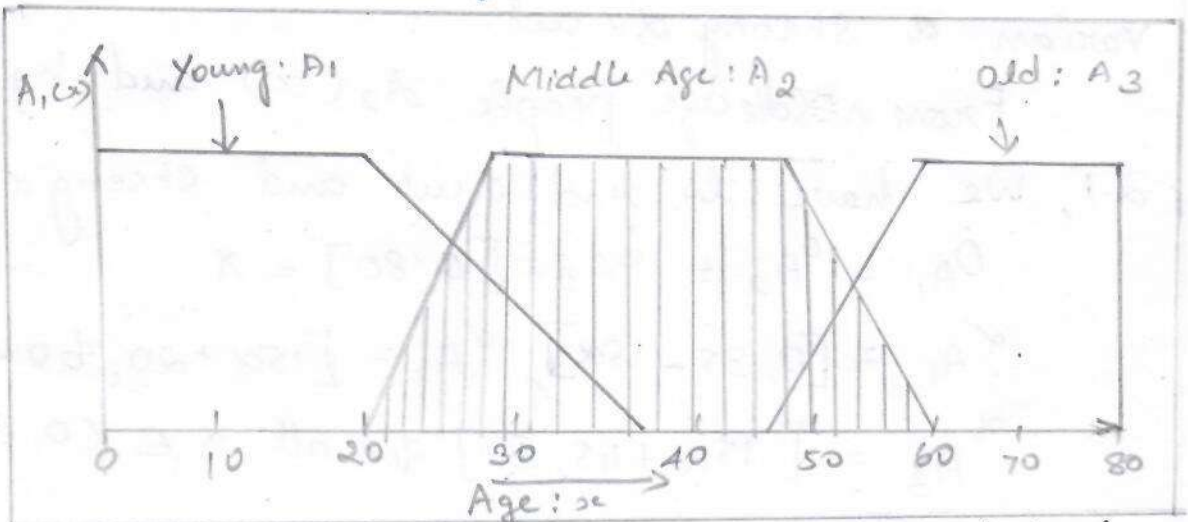


Fig: 2.1 (Shown discrete approximation D_2 of A_2 is defined numerically in Table 2.2).

Table : 2.2 : Discrete approximation of membership function A_2 (Fig. 2.1) by function D_2 of the form $D_2 : \{0, 2, 4 \dots 80\} \to [0, 1]$

x	$D_2(x)$
$x \notin \{22, 24, \cdots 58\}$	0.00
$x \in \{22, 58\}$	0.13
$x \in \{24, 56\}$	0.27
$x \in \{26, 54\}$	0.40
$x \in \{28, 52\}$	0.53
$x \in \{30, 50\}$	0.67
$x \in \{32, 48\}$	0.80
$x \in \{34, 46\}$	0.93
$x \in \{36, 38, \dots, 44\}$	1.00

α - cut and Strong α - cut

One of the most important concepts of fuzzy sets is the concept of an α - cut and its varian, a strong α - cut.

From middle age people $A_2(x)$ and by definition 2.1, we have to find α-cut and strong α-cut.

$${}^0A_1 = {}^0A_2 = {}^0A_3 = [0, 80] = X$$

$${}^\alpha A_1 = [0, 35 - 15\alpha], \quad {}^\alpha A_2 = [15\alpha + 20, 60 - 15\alpha],$$

$${}^\alpha A_3 = [15\alpha + 45, 80] \text{ for all } \alpha \in (0, 1];$$

$^{\alpha+}A_1 = (0, 35-15\alpha)$, $^{\alpha+}A_2 = (15\alpha+20, 60-15\alpha)$

$^{\alpha+}A_3 = (15\alpha+45, 80)$ for all $\alpha \in [0,1)$;

$^{1+}A_1 = {}^{1+}A_2 = {}^{1+}A_3 = \emptyset$

From middle age people $A_2(x)$ and by table 2.2, definition 2.2, we have to find the Level set.

$$\Lambda(A_1) = \Lambda(A_2) = \Lambda(A_3) = [0,1] \text{ and}$$

$$\Lambda(D_2) = \{0, 0.13, 0.27, 0.4, 0.53, 0.67, 0.8, 0.93, 1\}$$

The properties of α-cut and strong α-cut :-

* For any fuzzy set A and pair $\alpha_1, \alpha_2 \in [0,1]$ of distinct values such that $\alpha_1 < \alpha_2$, we have

$$^{\alpha_1}A \supseteq {}^{\alpha_2}A \text{ and } {}^{\alpha_1+}A \supseteq {}^{\alpha_2+}A$$

$$^{\alpha_1}A \cap {}^{\alpha_2}A = {}^{\alpha_2}A, \quad {}^{\alpha_1}A \cup {}^{\alpha_2}A = {}^{\alpha_1}A$$

$$^{\alpha_1+}A \cap {}^{\alpha_2+}A = {}^{\alpha_2+}A, \quad {}^{\alpha_1+}A \cup {}^{\alpha_2+}A = {}^{\alpha_1+}A$$

* All α-cuts and all strong α-cuts of any fuzzy set form two distinct families of nested crisp sets.

Definition: Support of fuzzy set A :-

The support of a fuzzy set A within a universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in A. The support of A is exactly the same as the strong α-cut of A for $\alpha = 0$

$$S(A) \text{ or } Supp(A) = {}^{0+}A.$$

The core of A:

The 1-cut of A (${}^{1}A$) is often called the core of A.

Definition: Height of fuzzy set A :-

Let A be an fuzzy set on X. Then the largest membership grade obtained by any element in that set. That set is called as the height of fuzzy set A. It is denoted by

$$h(A) = \sup_{x \in X} A(x)$$

Example-1

$A : N \to [0,1]$ defined by $A(x) = 1/x$

Example - 2

define $A : N \to [0,1]$ by $A(x) = \frac{x}{x+1}$

$h(A) = \sup A(x) \quad x \in X$

$= \sup \{ 1/2, 2/3, 3/4 \cdots \}$

$= 1$

Definition : Normal

A fuzzy set A is called normal when $h(A) = 1$

Example - 1

$A : N \to [0,1]$ defined by $A(x) = 1/x$

Definition : Subnormal.

A fuzzy set A is called subnormal when $h(A) < 1$

Example - 2

Define $A : N \to [0,1]$ by $A(x) = 1/2x$

$$h(A) = \sup_{x \in X} A(x)$$

$$= \sup_{x \in X} \{ 1/2, 1/4, 1/6, \cdots \}$$

$= 1/2$

Definition: Convexity :-

Fuzzy set whose α-cuts are convex crisp set for all $\alpha \in [0,1]$

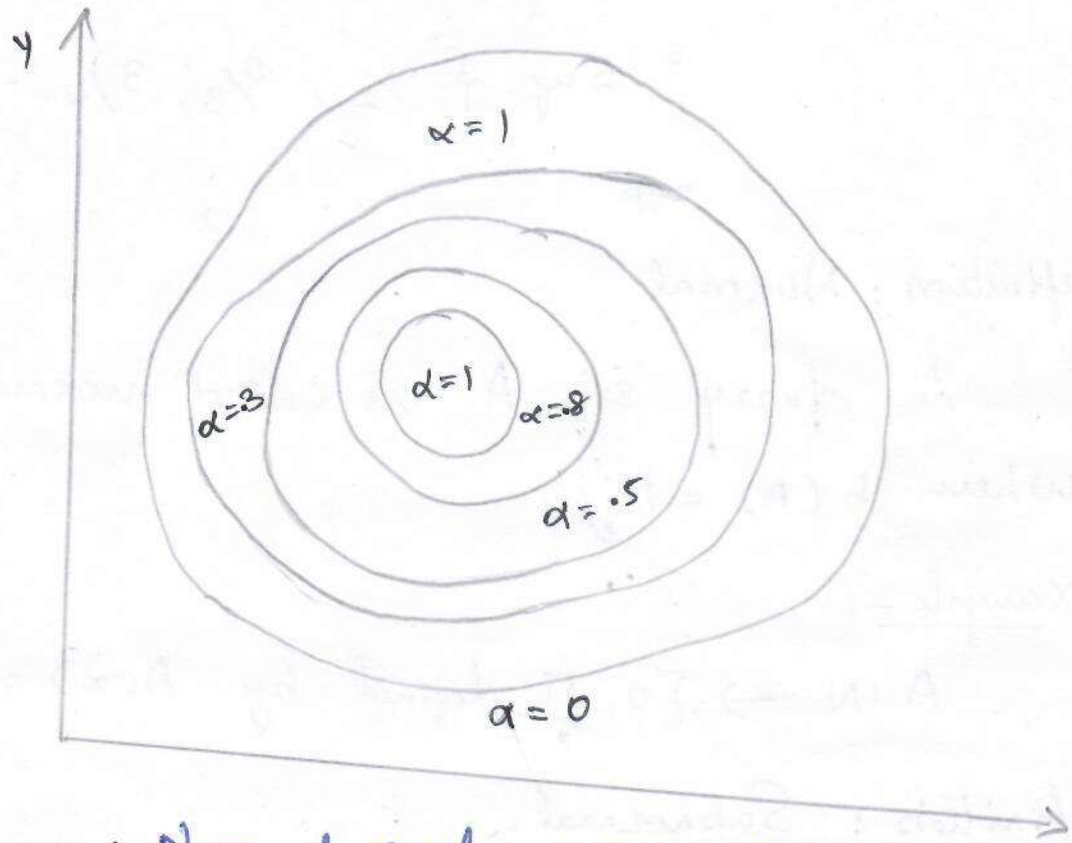


Figure: Normal and convex fuzzy set A^* defined by its α-cuts ${}^{1}A$, ${}^{.3}A$, ${}^{.5}A$, ${}^{.8}A$, ${}^{1}A$.

Definition: Convex crisp set in R^n:

For every pair of points $r = \{r_i / i \in N_n\}$ and $s = \{s_i / i \in N_n\}$ in A and every real number $\lambda \in [0,1]$ the point $t = \{\lambda r_i + (1-\lambda) s_i / i \in N_n\}$ is also in A.

Theorem :-

A fuzzy set A on $\mathbb{R}$ is convex iff

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)] \longrightarrow (1)$$

For all $x_1, x_2 \in R$ and all $\lambda \in [0,1]$ when minimum denotes the minimum operator.

Proof :-

(i) Assume that A is convex and let $\alpha = A_1(x) \leq A_2(x)$

Then, $x_2 \in {}^{\alpha}A$ and moreover $\lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}A$ for any $\lambda \in [0,1]$ by the convexity of A.

Consequently,

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = A(x_1) = \min[A(x_1), A(x_2)]$$

(ii) Assume that A satisfies (i)

we need to prove that for any $\alpha \in [0,1]$.

$\therefore$ ${}^{\alpha}A$ is convex.

Now, for any $x_1, x_2 \in {}^{\alpha}A$ (ie) $A(x_1) \geq \alpha$, $A(x_2) \geq \alpha$

and for any $\lambda \in [0,1]$ by (i)

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)]$$
$$= \min(\alpha, \alpha)$$
$$= \alpha$$

(ie) $\lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}A$

$\therefore$ ${}^{\alpha}A$ is convex for any $\alpha \in [0,1]$. Hence, A is convex.

DEFINITION: Cut Worthy :-

Any property generalized from classical set theory into the domain of fuzzy set theory.

(ie) preserved in all α-cuts for $\alpha \in [0,1]$ in the classical sense is called a cut worthy.

Property::

1. If it is preserved in all strong α-cuts for $\alpha \in [0,1]$.

2. Convexity of fuzzy set as defined above is an example of cut worthy property and as can be proven also a strong cut-worthy property.

Equilibirium Points:-

A standard complements, $\bar{A}$ of fuzzy set A with respect to the universal set X is defined for all $x \in X$ by the equation.

$$\bar{A}(x) = 1 - A(x)$$

Elements of X for which

$$A(x) = \bar{A}(x)$$

are called equilibrium points of A.

Definition: Union:-

Let A and B be two fuzzy sets on a non-empty set X. The union of A and B denoted as $A \cup B$ defined as $A \cup B(x) = \max \{A(x), B(x)\}$

$A \cup B : X \to [0,1]$ Hence $A \cup B$ is a fuzzy set on X. This is called an standard union of two fuzzy set.

Definition: Intersection:-

Let A and B be two fuzzy sets on a non-empty set X. The intersection of A and B denoted as $A \cap B$ defined as $A \cap B(x) = \min \{A(x), B(x)\}$. $A \cap B : X \to [0,1]$. Hence $A \cap B$ is a fuzzy set on X. This is called standard intersection of two fuzzy set.

Definition: Complement:-

Let A be a fuzzy set on a non-empty set X. The complement of A denoted as A^c is defined as $A^c(x) = 1 - A(x)$. Clearly $A^c : X \to [0,1]$

Hence A^c is a fuzzy set. This is called the standard complement of fuzzy set.

Scalar Cordinality:-

For any fuzzy set A defined on a finite universal set X. We define its scalar cordinality, by the formula

$$|A| = \sum_{x \in X} A(x)$$

Theorem:-

Let $A, B \in \mathcal{F}(x)$. Then the following Properties hold for all $\alpha, \beta \in [0,1]$.

(i) ${}^{\alpha+}A \subseteq {}^{\alpha}A$

(ii) $\alpha \leq \beta$ implies ${}^{\alpha}A \supseteq {}^{\beta}A$ and ${}^{\alpha+}A \supseteq {}^{\beta+}A$

(iii) ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$ and ${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$

(iv) ${}^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B$ and ${}^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B$

(v) ${}^{\alpha}(\bar{A}) = {}^{(1-\alpha)+}\bar{A}$

Proof:-

For (i): ${}^{\alpha+}A \subseteq {}^{\alpha}A$

Let $x \in {}^{\alpha+}A$

Then $A(x) > \alpha$

$\Rightarrow A(x) \geq \alpha$

$\Rightarrow x \in {}^{\alpha}A$

For (ii)

Let $\alpha, \beta \in [0,1]$ [from given $\alpha \leq \beta$]

$A(x) \geq \alpha$

$A(x) \geq \beta$

Hence, ${}^{\alpha}A \supseteq {}^{\beta}A$

${}^{\alpha+}A \supseteq {}^{\beta+}A$ because from given $\alpha \leq \beta$

For (iii) ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$

Let $x \in {}^{\alpha}(A \cap B)$

Then $(A \cap B)(x) \geq \alpha$

$\Leftrightarrow \min\{A(x), B(x)\} \geq \alpha$

$\Leftrightarrow A(x) \geq \alpha$ and $B(x) \geq \alpha$

$\Leftrightarrow x \in {}^{\alpha}A$ and $x \in {}^{\alpha}B$

$\Leftrightarrow x \in {}^{\alpha}A \cap {}^{\alpha}B$

$\therefore {}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B \longrightarrow (1)$

${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$

Let $x \in {}^{\alpha}(A \cup B)$

Then $(A \cup B)(x) \geq \alpha$

$\Leftrightarrow \max\{A(x), B(x)\} \geq \alpha$

$\Leftrightarrow A(x) \geq \alpha$ or $B(x) \geq \alpha$

$\Leftrightarrow x \in {}^{\alpha}A$ or $x \in {}^{\alpha}B$

$\Leftrightarrow x \in {}^{\alpha}A \cup {}^{\alpha}B$

$\therefore {}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B \longrightarrow (2)$

|

For (iv)

$$^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B$$

Let $x \in {}^{\alpha+}(A \cap B)$

Then $(A \cap B)(x) > \alpha$

$\Leftrightarrow \min \{A(x), B(x)\} > \alpha$

$\Leftrightarrow A(x) > \alpha$ and $B(x) > \alpha$

$\Leftrightarrow x \in {}^{\alpha+}A$ and $x \in {}^{\alpha+}B$

$\Leftrightarrow x \in {}^{\alpha+}A \cap {}^{\alpha+}B$

$\therefore \; {}^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B \longrightarrow ①$

$$^{\alpha+}(A \cup B) = {}^{\alpha+}A \cap {}^{\alpha+}B$$

$$^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B$$

Let $x \in {}^{\alpha+}(A \cup B)$

Then $(A \cup B)(x) > \alpha$

$\Leftrightarrow \max \{A(x), B(x)\} > \alpha$

$\Leftrightarrow A(x) > \alpha$ (or) $B(x) > \alpha$

$\Leftrightarrow x \in {}^{\alpha+}A$ (or) $x \in {}^{\alpha+}B$

$\Leftrightarrow x \in {}^{\alpha+}A \cup {}^{\alpha+}B$

$\therefore \; {}^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B \longrightarrow ②$

For (v) $\quad {}^{\alpha}(\bar{A}) = {}^{(1-\alpha)+}(\bar{A})$

Let $x \in {}^{\alpha}\bar{A}$

Then $\bar{A}(x) \geq \alpha$

$\Rightarrow 1 - A(x) \geq \alpha$

$\Rightarrow 1 - \alpha \geq A(x)$

$\Rightarrow x \in {}^{(1-\alpha)+}A$

$\Rightarrow x \in {}^{(1-\alpha)+}A$

$\therefore {}^{\alpha}\bar{A} \subseteq {}^{(1-\alpha)+}\bar{A} \longrightarrow (1)$

Conversly

Let $x \in {}^{(1-\alpha)+}\bar{A}$

Then $x \notin {}^{(1-\alpha)+}A$

$\Rightarrow A(x) \leq 1-\alpha$

$\Rightarrow 1 - A(x) \geq \alpha$

$\Rightarrow \bar{A}(x) \geq \alpha$

$\Rightarrow x \in {}^{\alpha}\bar{A}$

$\therefore {}^{(1-\alpha)+}\bar{A} \subseteq {}^{\alpha}\bar{A} \longrightarrow (2)$

From (1) and (2), we get

$${}^{(1-\alpha)+}\bar{A} = {}^{\alpha}\bar{A}$$

OPERATIONS ON FUZZY SETS :-

TYPES OF OPERATIONS:

The following special operations of fuzzy complement, intersection and union are introduced. $\overline{A}(x) = 1 - A(x)$

$$(A \cap B)x = \min(A(x), B(x))$$

$$(A \cup B)x = \max(A(x), B(x))$$

$$\forall x \in X$$

These operations are called the standard fuzzy operations.

FUZZY COMPLEMENT:

Let A be a fuzzy set on X. Then by definition A(x) is interpreted as the degree to which x belongs to A ($x \in A$). Let CA denote the fuzzy complement of A of type 'C'. Then CA(x) may be interpreted not only as the degree to which $x \in CA$. But also as the degree to which x does not belongs to A. Similarly A(x) may also be interpreted as the degree to which x does not belongs to CA. Let a complement CA be defined by a function.

$C : [0,1] \rightarrow [0,1]$ which assign a value $c(A(x))$ to each membership grade $A(x)$ of any given fuzzy set A. The value $c(A(x))$ is interpreted as the value of $c(A(x))$.

(ie) $c(A(x)) = cA(x) \; \forall \; x \in X$

The fuzzy complements function c must satisfy atleast the following two axiomatic requirements.

<u>Axiom</u> c_1 :

$c(0) = 1$ and $c(1) = 0$ (boundary condition)

<u>Axiom</u> c_2 :

For all $a, b \in [0,1]$, if $a \leq b$ then

$c(a) \geq c(b)$ (monotonically)

<u>Axiom</u> c_3 :

c is continuous function

<u>Axiom</u> c_4 :

c is involutive, which means that

$c(c(a)) = a$ for each $a \in [0,1]$.

<u>Theorem</u> 3.1

Let a function $c : [0,1] \to [0,1]$ satisfy Axioms c_2 and c_4. Then, c also satisfies axioms c_1 and c_3. Moreover, c must be a bijective function.

<u>Proof</u>

(i) Since the range of c is $[0,1]$.

$c(0) \leq 1$ and $c(1) \geq 0$.

By Axiom C_2, $c(c(0)) \geq c(1)$; and by Axiom C_4, $0 = c(c(0)) \geq c(1)$.

Hence $c(1) = 0$

Now again by axiom C_4, we have,

$$c(0) = c(c(1)) = 1$$

(ie) function c satisfies axiom C_1.

(ii) To prove that c is a bijective function. we observe that for all $a \in [0,1]$ there exists $b = c(a) \in [0,1]$ such that

$$c(b) = c(c(a)) = a$$

Hence c is an onto function.

Assume that

$$c(a_1) = c(a_2)$$

Then by axiom C_4,

$$a_1 = c(c(a_1)) = c(c(a_2)) = a_2$$

(ie) c is also a one-to-one function.

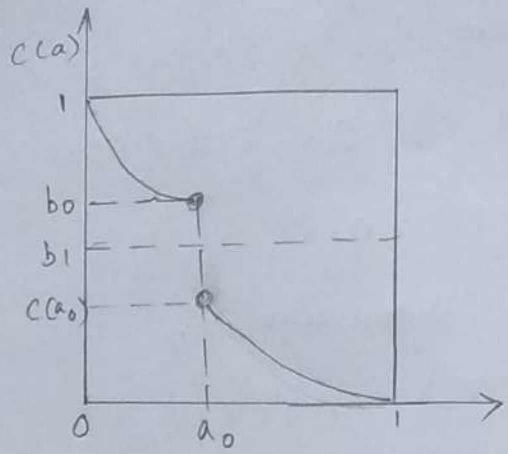
Consequently, it is a bijective function.

(iii) Since c is a bijective and satisfies axiom C_2, it cannot have any discontinuous points.

To show this, assume that c has a discontinuity at a_0.

Then we have

$$b_0 = \lim_{a \to a_0} c(a) > c(a_0)$$



and, clearly, there must exist $b_1 \in [0,1]$ such that $b_0 > b_1 > c(a_0)$ for which no $a_1 \in [0,1]$ exists such that $c(a_1) = b_1$.

This contradicts the fact that c is a bijective function

Theorem 3.2

Every fuzzy complement has atmost one equilibrium.

Proof

Let c be an arbitrary fuzzy complement. An equilibrium of c is a solution of the equation.

$$c(a) - a = 0$$

where $a \in [0,1]$, we can demonstrate that any equation

$$c(a) - a = b$$

where b is a real constant, must have atmost one solution.

Thus providing the theorem.

In order to do so, we assume that a_1 and a_2 are two different solutions of the equation

$$c(a) - a = b$$

such that $a_1 < a_2$

Then, since $c(a_1) - a_1 = b$

$$c(a_2) - a_2 = b$$

we get

$$c(a_1) - a_1 = c(a_2) - a_2 \longrightarrow (1)$$

However, because c is monotonic nonincreasing (by Axiom c_2)

$$c(a_1) \geq c(a_2)$$

and, since $a_1 < a_2$

$$c(a_1) - a_1 > c(a_2) - a_2$$

This inequality contradicts equation (1). Thus demonstrating that the equation must have at most one solution.

Theorem 3.3

Assume that a given fuzzy complement c has an equilibirium e_c, which by theorem 3.2 is unique. Then

$$a \leq c(a) \text{ iff } a \leq e_c \quad \&$$
$$a \geq c(a) \text{ iff } a \geq e_c$$

Proof

Let us assume that $a < e_c$ $a = e_c$ and $a > e_c$ in turn

Then, since c is monotonic nonincreasing (by Axiom c_2).

$$c(a) \geq c(e_c) \text{ for } a < e_c$$
$$c(a) = c(e_c) \text{ for } a = e_c$$
$$c(a) \leq c(e_c \text{ for } a > e_c$$

Because $c(e_c) = e_c$, we can rewrite this expression as

$$c(a) \geq e_c$$
$$c(a) = e_c$$
$$c(a) \leq e_c$$

respectively

In fact due to our initial assumption we can further rewrite these as

$$c(a) > a$$
$$c(a) = a$$
$$c(a) < a$$

respectively

Thus, $a \leq e_c \Rightarrow c(a) \geq a$ and
$a > e_c$

$\Rightarrow c(a) \leq a$

The inverse implications can be shown in a similar manner.

Theorem 3.4:

If c is a continuous fuzzy complement, then c has a unique equilibrium.

Proof.

The equilibrium e_c of a fuzzy complement c is the solution of the equation

$$c(a) - a = 0$$

This is a special case of the more general equation

$$c(a) - a = b$$

where $b \in [-1, 1]$ is a constant

By Axiom c_1, $c(0) - 0 = 1$ &

$$c(1) - 1 = -1$$

Since c is continuous complement. It follows from the intermediate value theorem for continuous functions that for each $b \in [-1, 1)$ there exists atleast one a such that $c(a) - a = b$.

This demonstrates the necessary existence of an equilibrium value for a continuous function and Theorem 3.2 gurantees its uniqueness.

Theorem 3.5

If a complement c has an equilibrium e_c, then

$$de_c = e_c$$

Proof

If $a = e_c$, then by definition of equilibrium $c(a) = a$ and thus

$$a - c(a) = 0$$

Additionally if $d_a = e_c$,

then $c(d_a) = d_a$ and

$$c(d_a) - d_a = 0$$

Therefore

$$c(d_a) - d_a = a - c(a)$$

This satisfies $c(d_a) - d_a = a - c(a)$

when $a = d_a = e_c$

Hence, the equilibrium of any complement is its own dual point.

Dual Point :

If we are given a fuzzy complement c and a membership grade whose value is represented by a real number $a \in [0,1]$, then any membership grade represented by the real number ${}^d a \in [0,1]$ such that $c({}^d a) - {}^d a = a - c(a)$ is called a dual point of a with respect to c.

Theorem 3.6

For each $a \in [0,1]$, ${}^d a = c(a)$ iff $c(c(a)) = a$, that is, when the complement is involutive.

Proof

Let ${}^d a = c(a)$

Then, substitution of c(a) for ${}^d a$ in

$$c({}^d a) - {}^d a = a - c(a) \longrightarrow 3.8$$

(3.8) produces

$$c(c(a)) - c(a) = a - c(a)$$

Therefore, $c(c(a)) = a$

For the reverse implication,

let $c(c(a)) = a$

Then substitution of $c(c(a))$ for a in (3.8) yields the functional equation

$$c({}^d a) - {}^d a = c(c(a)) - c(a)$$

for ${}^d a$ whose solution is ${}^d a = c(a)$

Theorem 3.7

(FIRST CHARACTERIZATION THEOREM OF FUZZY COMPLEMENTS)

Let c be a function from $[0,1]$ to $[0,1]$.
Then, c is a fuzzy complement (involutive) iff there exists a continuous function g from $[0,1]$ to $\mathbb{R}$ such that $g(0) = 0$, g is strictly increasing, and

$$c(a) = g^{-1}(g(1) - g(a)) \longrightarrow (1)$$

for all $a \in [0,1]$

Proof

(i) First we prove the inverse implication ($\Leftarrow$)
Let g be a continuous function from $[0,1]$ to $\mathbb{R}$ such that $g(0) = 0$ and g is strictly increasing
Then the pseudo inverse of g denoted by $g^{(-1)}$ is a function from $\mathbb{R}$ to $[0,1]$.
defined by

$$g^{(-1)}(a) = \begin{cases} a & \text{for } a \in (-\infty, 0] \\ g^{-1}(a) & \text{for } a \in [g(1), \infty] \\ 1 & \text{for } a \in [g(1), \infty] \end{cases}$$

Where g^{-1} is the ordinary inverse of g.
Let c be a function on $[0,1]$ define by eqn (1), we now prove that c is fuzzy complement.

First we show that 'c' satisfies axioms C_2 for any $a, b \in [0,1]$, if $a < b$, then $g(a) < b$ since g is strictly increasing.

Hence $g(1) - g(a) > g(1) - g(b)$ and consequently

$$c(a) = g^{-1}[g(1) - g(a)] > g^{-1}[g(1) - g(b)] > c(b)$$

There fore, c statisfies Axiom C_2.

Second, we show that 'c' is involutive for any $a \in [0,1]$,

$$c(c(a)) = g^{-1}[g(1) - g(c(a))]$$
$$= g^{-1}[g(1) - g(g^{-1}(g(1) - g(a)))]$$
$$= g^{-1}[g(1) - g(1) + g(a)]$$
$$= g^{-1}[g(a)]$$
$$= a$$

Thus, c is involutive

Theorem 3.8

[SECOND CHERACTERIZATION THEOREM OF FUZZY COMPLEMENTS]

Let c be a function from $[0,1]$ to $[0,1]$. Then c is a fuzzy complement iff there exists a continuous function f from $[0,1]$ to $\mathbb{R}$ such that $f(1) = 0$, f is strictly decreasing, and

$$c(a) = f^{-1}(f(0) - f(a)) \longrightarrow (1)$$

for all $a \in [0,1]$

Proof:

According to first characterization theorem of fuzzy complements (theorem 3.7) iff there exists an increasing generator g such that

$$c(a) = g^{-1}(g(1) - g(a))$$

Now, let $f(a) = g(1) - g(a)$

if we take $a = 1$

Then, $f(1) = a$ and, since g is strictly increasing, f is strictly decreasing.

Moreover,

$$f^{-1}(a) = g^{-1}(g(1) - (a))$$
$$= g^{-1}(f(0) - a)$$

since

$$f(0) = g(1) - g(0) = g(1)$$

$$f(f^{-1}(a)) = g(1) - g(f^{-1}(a))$$
$$= g(1) - g(g^{-1}(g(1) - a))$$
$$= a$$

and

$$g^{-1}(f(a)) = g^{-1}(g(1) - f(a))$$
$$= g^{-1}(g(1) - (g(1) - g(a)))$$
$$= g^{-1}(g(a))$$
$$= a$$

Now

$$c(a) = g^{-1}(g(1) - g(a))$$
$$= f^{-1}(g(a)$$
$$= f^{-1}(g(1) - (g(1) - g(a)))$$
$$= f^{-1}(f(0) - f(a))$$

If a decreasing generator f is given, we can define an increasing generator g as

$$g(a) = f(0) - f(a)$$

Then eqn (1) can be rewritten as

$$c(a) = f^{-1}(f(0) - f(a))$$
$$= g^{-1}(g(1) - g(a))$$

Hence, c defined by eqn (1) is a fuzzy complement.

Decreasing generators:-

Functions f defined in previous theorem are usually called decreasing generator. Each function f that qualifies as a decreasing generator also determines a fuzzy complement.

Increasing generators:-

Functions g defined in theorem 3.7 are usually called increasing generators. Each function g that qualifies as an increasing generator also determines a fuzzy complement.

Definition: Decreasing generator:-

The decreasing generator is a contineous and strictly decreasing function $f : [0,1] \to R$ such that $f(1) = 0$

Definition: Pseudo-inverse of decreasing generator:-

A pseudo-inverse of decreasing generator f denoted by $f^{(-1)}$ is a function from $R \to [0,1]$ given by

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in (0, f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$ (where f^{-1} is a ordinary inverse of f)

Example: 1

Define $f : [0,1] \to R$ defined by $f(a) = 1 - a^P$ for any $a \in [0,1]$ and $(P > 0)$

The pseudo inverse of $f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ (1-a)^{1/P} & \text{if } a \in [0,1] \\ 0 & \text{if } a \in (1, \infty) \end{cases}$

$\because f(a) = 1 - 0 = 1$

Example: 2

$f : [0,1] \to R$ defined by $f(a) = -\log a$ for any $a \in [0,1]$.
The pseudo inverse of f is defined by $f^{-1}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, \infty) \\ e^{-a} & \text{if } a \in (0, \infty) \end{cases}$

Definition: Increasing generator:

An increasing generator is a contineous function and Strictly increasing function $g : [0,1] \to R$ such that $g(0) = 0$ and the pseudo inverse of an increasing generator g denoted by $g^{(-1)}$ and is denoted by $R \to [0,1]$

$$= \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

where g^{-1} is the ordinary inverse of g.

Example:

1. Define $g : [0,1] \to R$ by $g(a) = a^P$, $a \in [0,1]$ and $P > 0$. The pseudo inverse of g is defined by

$$g^{(-1)} a = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ a^{1/P} & \text{if } a \in [0,1] \\ 1 & \text{if } a \in (1, \infty) \end{cases}$$

$\because g(1) = 1^P = 1$

2. Define $g : [0,1] \to R$ by $g(a) = -\log^{(1-a)}$ $a \in [0,1]$

$$g^{-1}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ 1 - e^{-a} & \text{if } a \in (0, \infty) \end{cases}$$

Fuzzy Intersection: t-norms :.

The intersection of two fuzzy sets A and B is specified in general by a binary operation on the unit interval. This the function of the form $i: [0,1] \times [0,1] \longrightarrow [0,1]$ for each x of the universal set, This function takes as it arguments the back consisting of the element membership grade in set A and in set B and yield the membership grade of the element in the set consisting of intersection of A and B.

Thus $(A \cap B)(x) = i(A(x), B(x)) \; \forall \; x \in X$
$(0x)$

An fuzzy intersection t-norm i is a binary operation on the unit interval that satisfies atleast the following axioms for all $a, b, d \in [0,1]$

(i)st Axiom : $i(a,1) = a$ (boundary)

(ii)nd Axiom : $b \leq d \Rightarrow i(a,b) \leq i(a,d)$
(monotonicity condition

(iii)rd Axiom : $i(a,b) = i(b,a)$
(commutativity)

(iv)th Axiom : $i(a, i(b,d)) = i(i(a,b), d)$
(Associativity)

Note :

Three of the most important requirements are expressed by following axioms

(v)th Axiom : i is a contineous function (vi)th Axiom : $i(a,a) < a$ (subidempotency)

(vii)th Axiom : $a_1 < a_2$ and $b_1 < b_2$

$\Rightarrow i(a_1, b_1) < i(a_2, b_2)$

(Strictly monotonicity)

Definition:

A contineous t-norms that satiesfies sub-idemptency is called an archimedian t-norm; if it also satiesfies strict monotonicity, it is called a strict Archimedean t-norm

Theorem:

Prove that the standard fuzzy intersection is the only idempotent t-norm.

Proof:

Clearly $\min(a,a) = a \quad \forall a \in [0,1]$

Assume that there exists a t-norm such that

$i(a,a) = a \quad \forall a \in [0,1]$

Then for any $a, b \in [0,1]$, if $a \leq b$

then $\min(a,b) = a$ and $a = i(a,a) \leq i(a,b)$

$\leq i(a,1) = a$

(ie) $i(a,b) \leq a$ and $a \leq i(a,b)$

Hence $i(a,b) = a$

Hence $\min(a,b) = i(a,b)$

whenever $a \leq b$

If $a \geq b$ then $\min(a,b) = b$ and

$b = i(b,b) \leq i(a,b) \leq i(1, b)$

(ie) $i(a,b) \leq b$ and $i(a,b) \geq b$

Hence $i(a,b) = b$

$\min(a,b) = i(a,b)$

whenever $a \geq b$

Hence standard fuzzy intersection is idempotent t-norm.

Note

Standard intersection : $i(a,b) = \min(a,b)$

Algebraic product : $i(a,b) = ab$

Bounded difference : $i(a,b) = \max(0, a+b-1)$

Drastic intersection : $i(a,b) = \begin{cases} a & \text{when } b=1 \\ b & \text{when } a=1 \\ 0 & \text{otherwise} \end{cases}$

Theorem:

For all $a, b \in [0,1]$

$i_{\min}(a,b) \leq i(a,b) \leq \min(a,b)$ —— (1)

Proof

(i) Upper bound

by boundary condition and monotonicity

$i(a,b) \leq i(a,1) = a$ and $i(a,b) = i(b,a)$
$\leq i(b,1) = b$

(commutative condition)

Hence $i(a,b) \leq a$ and $i(a,b) \leq b$

ie) $i(a,b) \leq \min(a,b) \longrightarrow$ (1)

Lower bound:

From the boundary condition $i(a,b) = a$ when $b = 1$ and $i(a,b) = b$ when $a = 1$

Since $i(a,b) \leq \min(a,b)$ and

$i(a,b) \in [0,1]$

Clearly $i(a,0) = i(0,b) = 0$

by monotonicity $i(a,b) \geq i(a,0) = i(0,b) = 0$

Hence the drastic intersection $i_{min}(a,b)$ is the lower bound of $i(a,b)$ for any $(a,b) \in [0,1]$

ie) $i_{min}(a,b) \leq i(a,b)$ — ②

From ① and ②

$$i_{min}(a,b) \leq i(a,b) \leq \min(a,b)$$

Note-1

A decreasing generator f and its pseudo inverse $f^{(-1)}$ satisfy $f^{-1}(f(a)) = a$ for any $a \in [0,1]$ and

$$f(f^{(-1)}(a)) = \begin{cases} 0 & a \in (-\infty,0) \\ a & a \in [0, f(0)] \\ f(0) & a \in (f(0), \infty) \end{cases}$$

Note-2

An increasing generator g and its pseudo inverse g inverse satisfy $g^{(-1)}(g(a)) = a$ for any $a, b \in [0,1]$ and

$$g(g^{-1}(a)) = \begin{cases} g(0) & \text{if } a \in (-\infty, 0) \\ g(g^{-1}(a)) & \text{if } a \in [0, g(1)] \\ g(1) & \text{if } a \in (g(1), \infty) \end{cases}$$

$$= \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ a & \text{if } a \in [0, g(1)] \\ g(1) & \text{if } a \in [g(1), \infty] \end{cases}$$

Theorem:

Let f be a decreasing generator then a function g defined by $g(a) = f(0) - f(a)$ for any $a \in [0,1]$ is an increasing generator with $g(1) = f(0)$ and its pseudo inverse is given by $g^{-1}(a) = f^{(-1)}(f(0)-a)$ for any $a \in R$

Proof:

Since f is a decreasing generator f is continuous, strictly increasing and such that $f(1) = 0$. Then $g(a) = f(0) - f(a)$, g is continuous for any $a, b \in [0,1]$ such that $a < b \Rightarrow f(a) > f(b)$ ($\because f$ is decreasing)

$$\Rightarrow -f(a) < -f(b)$$

$$\therefore g(a) = f(0) - f(a) < f(0) - f(b) = g(b)$$

ie $g(a) < g(b)$

Hence g is strictly increasing and

$$g(0) = f(0) - f(0) = 0$$

$\therefore g$ is continuous, strictly increasing and

$$g(0) = 0$$

Hence g is an increasing generator

Also
$$g(1) = f(0) - f(1)$$
$$= f(0) - 0$$
$$\therefore g(1) = f(0)$$

The pseudo inverse of g is given by

$$g^{(-1)} = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

Let $b = f(0) - f(a) = g(a)$

$\therefore f(a) = f(0) - b$

$a = f^{-1}(f(0) - b)$

$a = f^{-1}(f(0) - b)$

Where $a \in [0, g(1)] = [0, f(0)]$

Thus for any $a \in [0, f(0)]$, $g^{-1}(a) = f^{-1}(f(0) - a)$

Now, $f^{(-1)}(f(0), a) = \begin{cases} 1 & \text{if } f(0) - a \in (-\infty, 0) \\ f^{-1}(f(0) - a) & \text{if } f(0) - a \in [0, f(0)] \\ 0 & \text{if } f(0) - a \in (f(0), \infty) \end{cases}$

$$= \begin{cases} 1 & \text{if } a \in (f(0), \infty) \\ f^{-1}(f(0) - a) & \text{if } a \in [0, f(0)] \\ 0 & \text{if } a \in (-\infty, 0) \end{cases}$$

$$= \begin{cases} 1 & \text{if } a \in (g(1), \infty) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 0 & \text{if } a \in (-\infty, 0) \end{cases}$$

$f^{-1}(f(0) - a) = g^{-1}(a)$

Hence proved.

Theorem:

Let g be increasing generator. Then the function f defined by $f(a) = g(1) - g(a)$ for any $a \in [0,1]$ is a decreasing generator with $f(0) = g(1)$ and its pseudo inverse f^{-1} is given by $f^{(-1)}(a) = g^{-1}(g(1) - a)$ for any $a \in R$

Proof.

Since g be an increasing generator g is contineous and strictly decreasing and such that $g(0') = \infty$.

Then $f(a) = g(1) - g(a)$ g is contineous

for any $a, b \in [0,1]$ such that $a < b$

$$\Rightarrow (g(a) < g(b)$$

$$\Rightarrow -g(a) > -g(b)$$

$$f(a) = g(1) - g(a) > g(1) - g(b) = f(b)$$

$$f(a) > f(b)$$

Hence f is strictly decreasing and

$$f(a) = g(1) - g(a) = 1$$

f is contineous and strictly decreasing of $f(0) = 1$,

Hence f is decreasing generator

Also, $f(a) = g(1) - g(a)$

$$= 1 - g(a)$$

$$f(1) = g(a)$$

The pseudo inverse of f is given by

$$f^{-1}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [0, f(1)] \\ 1 & \text{if } a \in (0, \infty) \end{cases}$$

Let $b = g(1) - g(a) = g(a)$

$$g(a) = g(1) - b$$

$a = g^{-1}(g(1) - b)$ where $a \in [0, f(0)] = [0, g(1)]$

Thus for any $a \in [0, g(1)]$

$$f^{-1}(a) = g^{-1}(g(1) - a)$$

Now

$$g^{-1}(g(1) - a) = \begin{cases} 0 & \text{if } -a \in (-\infty, 0) \\ g^{-1}(g(1) - a) & \text{if } g(1) - a \in [0, g(1)] \\ 1 & \text{if } g(1) - a \in [g(1), \infty) \end{cases}$$

$$= \begin{cases} 0 & \text{if } a \in (g(1), \infty) \\ g^{-1}(g(1) - a) & \text{if } a \in (0, g(1)) \\ 1 & \text{if } a \in [-\infty, 0] \end{cases}$$

$$= \begin{cases} 1 & \text{if } a \in [-\infty, 0] \\ f^{-1}(a) & \text{if } a \in [0, f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

$$= f^{-1}(a)$$

Fuzzy Unions: t- Conorms

A fuzzy union/t- conorm u is a binary operation on the unit interval that satisfies atleast the following axioms for all $a, b, d \in [0,1]$:

Axiom u_1: $u(a, 0) = a$ (boundary condition)

Axiom u_2: $b \leq d$ implies $u(a,b) \leq u(a,d)$ (monotonicity)

Axiom u_3: $u(a,b) = u(b,a)$ (commutativity)

Axiom u_4: $u(a, u(b,d)) = u(u(a,b), d)$ (associativity)

Since this set of axioms is essential for fuzzy unions, we call it the axiomatic skeleton for fuzzy unions / t-conorms.

Axioms u_5: u is a continuous function (continuity)

Axiom u_6: $u(a,a) > a$ (superidempotency)

Axiom u_7: $a_1 < a_2$ and $b_1 < b_2$ implies $u(a_1, b_1) < u(a_2, b_2)$ (Strict monotonicity)

<u>Definition</u>:

Any continuous and super idempotent t-conorm is called Archimedean; if it is also strictly monotonic, it is called strictly Archimedean.

Theorem:

The standard fuzzy union is only idempotent t-conorm.

Proof:

Clearly, $\max(a,a) = a \ \forall a \in [0,1]$

Assume that there exist a t-conorm such that $u(a,a) = a \ \forall a \in [0,1]$

Then for any $a, b \in [0,1]$ if $b \le a$,

Then $\max(a,b) = a$ and

$$a = u(a,a) \ge u(a,b) \ge u(a,0) = a$$

(i.e) $u(a,b) \ge a$ and $a \ge u(a,b)$

Hence $u(a,b) = a$

Hence $\max(a,b) = u(a,b)$ whenever $b \le a$

if $a \le b$. Then $\max(a,b) = b$ and $b = u(b,b) \ge u(a,b) \ge u(0,b) = 0$

(i.e) $u(a,b) = b$

$\max(a,b) = u(a,b)$ whenever $a \le b$

Hence, standard fuzzy union is the idempotent t-conorm.

Examples of some t-conorms that all are frequently used as fuzzy unions (each defined for all $a, b \in [0,1]$

Standard union : $u(a,b) = \max(a,b)$

Algebraic sum : $u(a,b) = a+b-ab$

Bounded sum : $u(a,b) = \min(1, a+b)$

Drastic union : $u(a,b) = \begin{cases} a & \text{when } b=0 \\ b & \text{when } a=0 \\ 1 & \text{otherwise} \end{cases}$

Theorem:

For all $a, b \in [0,1]$ $\max(a,b) \le u(a,b) \le u_{max}(a,b)$

where U_{max} denoted the drastic union.

Proof

upper bound:

By boundary condition and monotonicity

$u(a,b) \ge u(a,0) = a$ and $u(a,b) = u(b,a) \ge u(b,0) = b$

Hence $u(a,b) \ge a$ and $u(a,b) \ge b$

(ie) $u(a,b) \ge \max(a,b)$ ——— (1)

Lower bound:

From the boundary condition $u(a,b) = a$ when $b=0$ and $u(a,b) = b$ when $a=0$

$u(a,b) \ge \max(a,b)$ and $u(a,b) \in [0,1]$

clearly $u(a,1) = u(1,b) = 1$

By monotonicity $u(a,b) \le u(a,1) = u(1,b) = 1$

Hence the drastic intersection

$\mu_{max}(a,b)$ is lower bound of $\mu(a,b)$ for any $a, b \in [0,1]$

(ie) $\mu_{max}(a,b) \geq \mu(a,b)$ —— (2)

From (1) and (2), we have

$$\mu_{max}(a,b) \geq \mu(a,b) \geq Max(a,b).$$

—x—

Theorem 3:11.

(CHARATERIZATION THEOREM OF t-NORMS)

Let i be a binary operation on the unit interval. Then, i is an Archimedean t-norm iff there exists a decreasing generator f such that

$$i(a, b) = f^{(-1)}(f(a) + f(b)) \quad \text{—} (1)$$

for all $a, b \in [0, 1]$

Proof:

[Schweizer and Sklar, 1963; Ling 1965]

Given a decreasing generator f, we can construct a t-norm i by (1). The following are examples of three parametrized classes of decreasing generators and the corresponding classes of t-norms. In each case,

the parameter is used as a subscript of f and i to distinguish different generators and t-norms are described in the literature, we identify them by their authors and relevant references.

1. [Schweizer & Sklar, 1963]: The class of decreasing generators distinguished by parameter P is defined by $f_P(a) = 1 - a^P$ ($P \neq 0$

Then

$$f_P^{(-1)}(z) = \begin{cases} 1 & \text{when } z \in (-\infty, 0) \\ (1-z)^{1/P} & \text{when } z \in [0, 1] \\ 0 & \text{when } z \in (1, \infty) \end{cases}$$

and we obtain the corresponding class of t-norms by applying (1)

$$i_p(a,b) = f_p^{(-1)}(f_p(a) + f_p(b))$$

$$= f_p^{(-1)}(2 - a^p - b^p)$$

$$= \begin{cases} (a^p + b^p - 1)^{1/p} & \text{when } 2 - a^p - b^p \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$= (\max(0, a^p + b^p - 1))^{1/p}$$

2. [Yager, 1980f]: Given a class of decreasing generator

$$f_w(a) = (1-a)^w \quad (w > 0)$$

we obtain

$$f_w^{(-1)} z = \begin{cases} 1 - z^{1/w} & \text{when } z \in [0,1] \\ 0 & \text{when } z \in (1, \infty) \end{cases}$$

and

$$i_w(a,b) = f_w^{(-1)}(f_w(a) + f_w(b))$$

$$= f_w^{(-1)}((1-a)^w + (1-b)^w)$$

$$= \begin{cases} 1 - ((1-a)^w + (1-b)^w)^{1/w} & \text{when } (1-a)^w + (1-b)^w \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

$$= 1 - \min(1, [(1-a)^w + (1-b)^w]^{1/w})$$

3. [Frank, 1979]: This class of t-norms is based on the class of decreasing generators.

$$f_s(a) = -\ln \frac{s^a - 1}{s - 1} \quad (s > 0, s \neq 1)$$

whose pseudo-inverse are given by

$$f_s^{(-1)} z = \log_s(1 + (s-1)e^{-z})$$

Employing (1), we obtain

$$i_s(a,b) = f_s^{(-1)}\left(f_s(a) + f_s(b)\right)$$

$$= f_s^{(-1)}\left[-\ln \frac{(s^a-1)(s^b-1)}{(s-1)^2}\right]$$

$$= \log_s\left[1 + (s-1)\frac{(s^a-1)(s^b-1)}{(s-1)^2}\right]$$

$$= \log_s\left[1 + \frac{(s^a-1)(s^b-1)}{s-1}\right]$$

Let us examine one of the three introduced classes of t-norms, the Yager class

$$i_w(a,b) = 1 - \min\left(1\left[(1-a)^w + (1-b)^w\right]^{1/w}\right) (w>0) \quad (2)$$

It is significant that this class covers the whole range of t-norms expressed by

$i_{min}(a,b) \leq i(a,b) \leq \min(a,b)$. This property of i_w is stated in the following theorem.

Theorem 3.12:

Let i_w denote the class of Yager t-norms defined by $[i_{min}(a,b) \leq i_w(a,b) \leq \min(a,b)]$.

Then

$$i_{min}(a,b) \leq i_w(a,b) \leq \min[a,b]$$

for all $a, b \in [0,1]$.

Proof:

Lower bound:

It is trivial that $i_w(1,b) = b$ and $i_w(a,1) = a$ independent of w. It is also easy to show that

$$\lim_{w \to 0} \left[(1-a)^{w} + (1-b)^{w}\right]^{1/w} = \infty$$

hence

$$\lim_{w \to 0} i_w(a,b) = 0$$

for all $a, b \in [0,1]$

Upper bound:

From the proof of theorem (10), W.K.T

$$\lim_{w \to \infty} \min\left[1, \left[(1-a)^{w} + (1-b)^{w}\right]^{1/w}\right] = \max\left[1-a, 1-b\right]$$

Thus, $i_\infty(a,b) = 1 - \max\left[1-a, 1-b\right]$

$= \min(a,b)$

which concludes the proof.

Theorem 3.16:

[Characterization theorem of t-conorms]

Let u be a binary operation on the unit interval. Then, u is an Archimedean t-conorm iff there exits an increasing generator such that

$$u(a,b) = g^{(-1)}\left[g(a) + g(b)\right] \longrightarrow (1)$$

for all $a, b \in [0,1]$.

Proof.

[Schweizer and Sklar, 1963; Ling. 1965].

Given an increasing generator g, we can construct a t-conorm u by (1). The following are examples of three parameterized classes of increasing generators and the corresponding classes of t-conorms, which are counterparts of the three classes of t-norms introduced in

FUZZY INTERSECTIONS : t-NORMS (3.3)

1. [Schweizer and Sklar, 1963]: The class of increasing generators is defined by

$$g_p(a) = 1-(1-a)^p \quad (p \neq 0)$$

Then

$$g_p^{(-1)}(z) = \begin{cases} 1-(1-z)^{1/p} & \text{when } z \in [0,1] \\ 1 & \text{when } z \in (1,\infty) \end{cases}$$

and we obtain the corresponding class of t-conorms by applying (1):

$$U_p(a,b) = g_p^{(-1)}(1-(1-a)^p + 1-(1-b)^p)$$

$$= \begin{cases} 1-[(1-a)^p + (1-b)^p - 1]^{1/p} & \text{when } 2-(1-a)^p-(1-b)^p \in [0,1] \\ 1 & \text{otherwise} \end{cases}$$

$$= 1 - \max[(0, (1-a)^p + (1-b)^p - 1]^{1/p}$$

2. [Yager, 1980f]: Given a class of increasing generators

$$g_w(a) = a^w \quad (w>0)$$

we obtain

$$g_w^{(-1)}(z) = \begin{cases} z^{1/w} & \text{when } z \in [0,1] \\ 1 & \text{when } z \in [1,\infty) \end{cases}$$

and

$$U_w(a,b) = g_w^{(-1)}(a^w + b^w)$$

$$= \min(1, (a^w + b^w)^{1/w})$$

3. [Frank, 1979]: Using the class of increasing generators

$$g_s(a) = -\ln \frac{s^{1-a} - 1}{s-1} \quad (s>0,\ s \neq 1)$$

whose pseudo-inverses are

$$g_s^{(-1)}(z) = 1 - \log_s (1 + (s-1) e^{-z})$$

we obtain

$$U_s(a,b) = 1 - \log_s \left\{ 1 + \frac{(s^{1-a} - 1)(s^{1-b} - 1)}{s-1} \right\}$$

Let us further examine only the Yager class of t-conorms.

$$U_w(a,b) = \min \left(1, (a^w + b^w)^{1/w}\right) (w > 0) \to ②$$

As stated by the following theorem, this class covers the whole range of t-conorms.

Theorem 3.17:

Let u_w denote the class of Yager t-conorms defined by before theorem (eqn ②)
Then

$$\max(a,b) \leq u_w(a,b) \leq u_{max}(a,b) \text{ — ①}$$

for all $a, b \in [0,1]$

Proof

Lower bound:

We have to prove that

$$\lim_{w \to \infty} \min \left[1, (a^w + b^w)^{1/w}\right] = \max(a,b) \to ②$$

This is obvious whenever (1) a or b equal 0, or (2) a=b, because the limit of $2^{1/w}$ was $w \to \infty$ equals 1.

If $a \neq b$ and the min equals $(a^{\omega}+b^{\omega})^{1/\omega}$, the proof reduces to the demonstration that

$$\lim_{\omega \to \infty} (a^{\omega}+b^{\omega})^{1/\omega} = \max(a,b).$$

Let us assume, with no loss of generality, that $a < b$, and let $Q = (a^{\omega}+b^{\omega})^{1/\omega}$

Then $$\lim_{\omega \to \infty} \ln Q = \lim_{\omega \to \infty} \frac{\ln (a^{\omega}+b^{\omega})}{\omega}$$

Using I' Hospitals rule, we obtain

$$\lim_{\omega \to \infty} \ln Q = \lim_{\omega \to \infty} \frac{a^{\omega} \ln a + b^{\omega} \ln b}{a^{\omega}+b^{\omega}}$$

$$= \lim_{\omega \to \infty} \frac{(a/b)^{\omega} \ln a + \ln b}{(a/b)^{\omega}+1}$$

$$= \ln b$$

Hence

$$\lim_{\omega \to \infty} Q = \lim_{\omega \to \infty} (a^{\omega}+b^{\omega})^{1/\omega} = b \ (= \max(a,b))$$

It remains to show that (2) is still valid when the min equals 1. In this case

$$(a^{\omega}+b^{\omega})^{1/\omega} \geq 1$$

(or)

$$a^{\omega}+b^{\omega} \geq 1$$

For all $\omega \in [0,\infty)$, when $\omega \to \infty$, the last inequality holds if $a=1$ (or) $b=1$ (since $a,b \in [0,1]$)

Hence (2) is again satisfied.

Upper bound:

It is trivial that $M(a,b) = b$ and $M(a,0) = a$ independent of ω. It is also easy to show that

$$\lim_{\omega \to 0} \left(a^{\omega} + b^{\omega}\right)^{1/\omega} = \infty$$

hence $\lim_{\omega \to 0} M_{\omega}(a,b) = 1$

for all $a, b \in [0,1]$

UNIT - III

COMBINATIONS OF OPERATIONS

Definition:

A t-norm i and t-conorm u are dual with respect to the fuzzy complement c iff

$$c\,(u,(a,b)) = i\,(c(a), c(b)) \text{ and}$$

$$c\,(i\,(a,b) = u\,(c(a), c(b))$$

Let the triple $\langle i, u, c \rangle$ denotes i and u are dual with respect to c and any such triple be called dual triple.

Note:

The following dual- t-norms and t- conorms with respect to the standard complement c_s (dual triples)

(i) $\langle \min(a,b), \max(a,b), c_s \rangle$

(ii) $\langle ab, a+b-ab, c_s \rangle$

(iii) $\langle \max(0, a+b-1), \min(1, a+b), c_s \rangle$

(iv) $\langle i_{min}(a,b), u_{max}(a,b), c_s \rangle$

Theorem:-

The triples $\langle \min, \max, c \rangle$ and $\langle i_{min}, u_{max}, c \rangle$ are dual with respect to any fuzzy complement c.

Proof:

Without loss of generality, we assume that $a \leq b$. Then $c(a) \geq c(b)$ for any fuzzy complement

$$\max\{c(a), c(b)\} = c(a) = c(\min(a,b))$$

$$\max\{c(a), c(b)\} = c(b) = c(\max(a,b))$$

$\therefore \langle \min, \max, c \rangle$ is dual, w.r.t any fuzzy complement.

Next we show that $\langle i_{min}, u_{max}, c \rangle$ is dual w.r.t any fuzzy complement c, where i_{min}, u_{max} respectively

$$c[i(a,b)] = u[c(a), c(b)]$$

If $b=1$, Then by drastic intersection $i(a,b) = 0$

$$\therefore c[i(a,b)] = c[i(a,1)] = c[a]$$

$$u[c(a), c(b)] = u[c(a), c(1)] = u[c(a), 0] = c(a)$$

$$\therefore c[i(a,b)] = u[c(a), c(b)]$$

If $a=1$, Then by drastic intersection $i(a,b) = b$

$$\therefore c[i(a,b)] = c[i(1,b)] = c(b)$$

$$u[c(a), c(b)] = u(c(1), c(b)] = u[0, c(b)] = c[b$$

$$\therefore c(i(a,b)) = u(c(a), c(b)]$$

if $a \neq$ (or) $b \neq 1$. Then by drastic intersection $i(a,b) = 0$ (ie) a (or) b must be 0

Suppose $a = 0$ then $c[i(a,b)] = c[(i(0,b))] = c(0) = 1$

$u(c(a), c(b))] = u[c(0), c(b)] = u(1, c(b)) = 1$

Suppose $b = 0$ then $c[i(a,b)] = c[i(a,0)] = c(0) = 1$

$u[c(a), c(b)] = u[c(a), c(0)] = u[c(a), 1] = 1$

Hence $c[i(a,b)] = u[c(a), c(b)]$

Next we prove that $c[u(a,b)] = i[c(a), c(b)]$

If $a = 0$, then by drastic intersection $u(a,b) = b$

$\therefore c[u(a,b)] = c(b)$

$i[c(a), c(b)] = i[c(0), c(b)] = i[1, c(b)]$

$\therefore i[c(a), c(b)] = c[u(a,b)]$

if $a \neq 0$, $b \neq 0$ then by drastic intersection

$u(a,b) = 1$

ie) a (or) b must be 1

if $a = 1$, then $c[u(0,b)] = c[u(1,b)] = c(1) = 0$

$i[c(a), c(b)] = i(c(1), c(b)) = i(0, c(b)) = 0$

if $b = 0$ then $c[u,(a,b)] = c[u(a,1)] = c(1) = 0$

$i[c(a), c(b)] = i[c(a), c(b)] = i(c(0), 0) = 0$

$\therefore i[c(a), c(b)] = u[c(a,b)]$

Umax is dual with respect to any fuzzy complement.

Theorem:

Given a t-norm and an involutive fuzzy complement c the binary operation u on [0,1] defined by

$$u(a,b) = c(i(c(a), c(b))) \quad \text{—— (1)}$$

for all $a,b \in [0,1]$ is a t-conorm such that $\langle i, u, c\rangle$ is a dual triple.

Proof:

To prove that u given by (1) is a t-conorm we have to show that it satisfies Axioms $u_1 - u_4$.

u_1 - Boundary Condition:

For any $a \in [0,1]$

$$u(a,0) = c(i(c(a), c(0))) \quad (\text{by } (1))$$
$$= c(i(c(a), 1)) \quad (\text{by axiom } c_1)$$
$$= c(c(a)) \quad (\text{by axiom } i_1)$$
$$= a \quad (\text{by axiom } c_4)$$

Hence, u satisfies Axiom u_1.

u_2 - Monotonicity condition:-

For any $a, b, d \in [0,1]$, if $b \leq d$, then

$c(b) > c(d)$.

Moreover,

$$i(c(a), c(b)) \geq i(c(a), c(d))$$

by axiom i_2, Hence by ①

$$u(a,b) = c(i(c(a), c(b))) \leq c(i(c(a), c(d)))$$
$$= u(a,b)$$

which shows that u satisfies axiom u_2.

u_3 - Commutative axiom:-

For any $a, b \in [0,1]$, we have

$$u(a,b) = c(i(c(a), c(b)) = c(i(c(b), c(a)))$$
$$= u(b,a)$$

By ① and Axiom i_3, that is, u satisfies Axiom u_3.

u_4 - Associative axiom:-

For any $a, b, d \in [0,1]$

$$u(a, u(b,d)) = c[i(c(a), c(u(b,d)))] \text{ (by ①)}$$
$$= c[i(c(a), c[c(i(c(b), c(d)))] \text{ (by ①)}$$
$$= c[i(c(a), i(c(b), c(d))] \text{ (by axiom } c_4)$$
$$= c[i(i(c(a), c(b)), c(d))] \text{ (by axiom } i_4)$$
$$= c[i(c c(i(c(a), c(b)))] \text{ (by axiom } c_4)$$
$$= u(u(a,b), d)$$

Hence, u satisfies Axiom u_4 and consequently, it is a t-conorm.

By employing ① and Axiom c_4, We can now show that u satisfies the Demorgon laws:

$$c(u(a,b)) = c(c(i(c(a), c(b)))) = i(c(a), c(b))$$

$$u(c(a), c(b)) = c(i(c(c(a)), c(c(b))))$$
$$= c(i(a,b))$$

Hence, i and u are dual with respect to c.

Theorem:

Given a t-conorm u and an involutive fuzzy complement the binary operation i on $[0,1]$ defined by $i(a,b) = c(u(c(a), c(b)))$ $\langle i, u, c\rangle$ is a dual triple.

Proof:

To prove that $i(a,b) = c[u(c(a), c(b))]$ is a t-norm.

ie) Show that it satisfies boundary, monotonicity, Commutative, Associative axioms.

1) Boundary conditions:-

For any $a \in [0,1]$, $i(a,1) = c[[u(c(a), c(1))]$
$$= c[u(c(a), 0)$$
$$= c[c(a)]$$
$$= a$$

2. Monotonicity condition:-

For any $a, b, d \in [0,1]$, if $b \leq d$ then $c(b) \geq c(d)$

$u(c(a), c(b)) \geq u(c(a), c(d))$

$\therefore\ i(a,b) = c[u(c(a)), c(b))] \leq c[u(c(a)), c(d))]$

$= i(a,d)$

Hence i satisfies monotonicity condition.

3. Commutative axiom:-

For any $a, b \in [0,1]$ we have

$i(a,b) = c[u(c(a)), c(b)] = c[u(c(b)), c(a))] = i(b,a)$

Hence i satisfies commutative axiom.

4. Associative axiom:-

For any $a, b, d \in [0,1]$

$i[a, i(b,d)] = c[u(c(a)), c(c(u[c(b), c(d)]))]$

$= c[u(c(a)), u(c(b), c(d))]$

$= c[u(u(c(a)), c(b)), c(d)]$

$= c[u(c(c(u(c(a)), c(b))))), c(d)]$

$= c[u(c(i(a,b))), c(d)]$

$= i(i(a,b), d)$

Hence i satisfies associative axiom.

Next we show that u satisfies Demorgan laws

$$c[i(a,b)] = c[c(u(c(a), c(b)))] = u[c(a), c(b)]$$

$$i[c(a), c(b)] = c[u(c(c(a)), c(c(b)))]$$

$$= c[u(a,b)]$$

Hence i and u are dual w.r.t. c.

Theorem:

Given an involutive fuzzy complement c and an increasing generator $g(c)$, the t-norm and t-conorm generated by g are dual with respect to c.

Proof:

For any $a, b \in [0,1]$, we have

$$c(a) = g^{-1}(g(1) - g(a)) \text{ and } i(a,b) = f^{-1}(f(a) + f(b))$$

$$u(a,b) = g^{-1}(g(a) + g(b))$$

by theorem

$$f^{-1}(a) = g^{-1}(g(1) - a) \text{ where } f(a) = g(1) - g(a)$$

$$\therefore f^{-1}(f(a) + f(b)) = g^{(-1)}(g(1) - (f(a) + f(b))$$

$$= g^{(-1)}(g(1) - (g(1) - g(a) + g(1) - g(b))$$

$$= g^{-1}(g(a) + g(b) - g(1))$$

$$c(n(a,b)) = i(c(a), c(b))$$

$$c(n(a,b)) = g^{-1}(g(1) - g(u(a,b)))$$

$$= g^{-1}(g(1) - g(g^{-1}(g(a) + g(b)))$$

$$= g^{-1}(g(1) - g(g^{-1}(g(a) + g(b)))$$

$$= g^{-1}(g(1) - g(a) - g(b)) \quad \text{--- } (1)$$

$$i(c(a), c(b)) = g^{-1}(g\,c(a)) + g(c(b)) - g(1))$$

$$= g^{-1}[g(g^{-1}(g(1) - g(a))] + g(g^{-1}(g(1) - g(b)) - g(1)]$$

$$= g^{-1}[g(1) - g(a) + \cancel{g(1)} - g(b) - \cancel{g(1)}]$$

$$= g^{-1}[g(1) - g(a) - g(b)] \quad \text{--- } (2)$$

From (1) and (2)

$$c(n(a,b)) = i(c(a), c(b))$$

To prove that

$$c(i(a,b)) = n(c(a), c(b))$$

$$c(i(a,b)) = g^{(-1)}(g(1) - g(i(a,b)))$$

$$= g^{(-1)}(g(1) - g(g^{-1}(g(a) + g(b)) - g(1))$$

$$= g^{(-1)}(g(1) - g(a) - g(b) + g(1))$$

$$= g^{(-1)}(2g(1) - (g(a) + g(b))) \quad \text{--- } (3)$$

$$u(c(a), c(b)) = g^{(-1)}(g(c(a)) + g(c(b)))$$

$$= g^{(-1)}[g(g^{-1}(g(1) - g(a))) + g(g^{-1}(g(1) - g(b)))]$$

$$= g^{-1}(g(1) - g(a) + g(1) - g(b))$$

$$= g^{-1}(2g(1) - g(a) + g(b))) \quad - (4)$$

From (3) & (4) we get

$$c(i(a,b)) = u(c(a), c(b))$$

Theorem :

Let $\langle i, u, c \rangle$ be a dual triple generated by an above theorem. Then, the fuzzy operations i, u, c satisfy the law of excluded middle and the law of contridiction.

Proof :

According to above theorem, we have

$$c(a) = g^{(-1)}(g(1) - g(a))$$

$$i(a,b) = g^{(-1)}(g(a) + g(b) - g(1))$$

$$u(a,b) = g^{(-1)}(g(a) + g(b))$$

Then

$$u(a, c(a)) = g^{(-1)}(g(a) + g(c(a)))$$

$$= g^{(-1)}(g(a) + g(g^{-1}(g(1) - g(a))))$$

$$= g^{(-1)}(g(a) + g(1) - g(a))$$

$$= g^{(-1)}(g(1))$$

$$= 1$$

for all $a \in [0,1]$. That is the law of excluded middles is satisfied. Moreover

$$i(a, c(a)) = g^{(-1)}(g(a) + g(c(a)) - g(1))$$

$$= g^{(-1)}(g(a) + g(g^{-1}(g(1) - g(a))) - g(1))$$

$$= g^{(-1)}(g(a) + g(1) - g(a) - g(1))$$

$$= g^{(-1)}(0)$$

$$= 0$$

for all $a \in [0,1]$. Hence, the law of contradiction is also satisfied.

Theorem

Let (i, u, u) be a dual triple that satisfies the law of excluded middle and the law of contradiction. Then (i, u, c) does not satisfy the distributive laws.

Proof:

Assume that the distributive law

$$i(a, u(b, d)) = u(i(a, b), i(a, d))$$

is satisfied for all $a, b, d \in [0,1]$.

Let e be the equilibrium of c

Clearly, $e \neq 0, 1$ since $c(0) = 1$ and $c(1) = 0$

By the law of excluded middle and the law of contradiction we obtain

$$u(e, e) = u(e, c(e)) = 1$$

$$i(e, e) = i(e, c(e)) = 0$$

Now, applying e to the above distributive law, we have

$$i(e, u(e, e)) = u(i(e, e), i(e, e)).$$

Substituting for $u(e, e)$ and $i(e, e)$

we obtain

$$i(e, 1) = u(0, 0)$$

which results (by axiom i, and u) in $e = 0$.

This contradicts the requirement that $e \neq 0$.

Hence, the distributive law and does not hold.

In an analogous way, we can prove that the dual distributive law does not hold either

Aggregation Operation :

Definition :

Any aggregation operation on ~~n~~ fuzzy Sets ($n \geq 2$) is defined by a function $h : [0,1]^n \to [0,1]$ when apply to fuzzy sets $A_1, A_2, \dots A_n$ defined on X. function h produce an aggregate fuzzy set A by operating on the membership grades of thus sets for each $x \in X$.

$A(x) = h(A_1(x), A_2(x), \dots A_n(x))$, for each $x \in X$. h must satisfies atleast ~~grades~~ of thus following the three axioms.

(i) $h(0, 0, 0, \dots 0) = 0$ and $h(1, 1, 1, \dots 1) = 1$
(bounded condition)

(ii) for any pair $(a_1, a_2, \dots a_n)$ and $(b_1, b_2, \dots b_n)$ an n dubles such that $(a_i, b_i) \in [0,1] \; \forall \; i \in N_n$
Then $h(a_1, a_2, \dots a_n) \leq h(b_1, b_2, \dots b_n)$
if $a_i \leq b_i \; \forall \; i$
(monotonicity increasing condition)

(iii) h is contineous function.

Note:

Aggregation operation on fuzzy sets are usually expected to satisfies the additional axiom

(iv) h is a symmetric function in all its arguments; that is,

$$h(a, a_2 \dots a_n) = h(a_{p(1)}, a_{p(2)} \dots a_{p(n)})$$

for any permutation p on N_n

(v) h is an idempotent function, that is

$$h(a, a, \dots a) = a$$

for all $a \in [0,1]$.

Note:

Any aggregation operation h that satisfies monotonicity increasing and idempotent condition the inequality minimum of $(a_1, a_2, \dots a_n) \leq h(a_1, a_2, \dots a_n) \leq$ maximum $(a_1, a_2, \dots a_n)$

Averaging operation: Definition:

Function h satisfy $\min(a_1, a_2, \dots a_n) \leq h(a_1, a_2, \dots a_n) \leq \max\{a_1, a_2, \dots a_n\}$ are the only aggregation operation there are idempotent. These aggregation operation are called averaging operation.

Definition:

One class of Averaging operation that covers the entire interval between min and max operation consists of generalized means these are defined on the formula

$$h_\alpha(a_1, a_2, \dots a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha}{n}\right)^{1/\alpha}$$

where $\alpha \in R$ $(\alpha \neq 0)$ and $a_i \neq 0 \ \forall \ i \in N_n$

Note: 1

For $\alpha < 0$ and $a_i \to 0$ for any $i \in N_n$ then $h_\alpha(a_1, a_2, \dots a_n)$ converges to zero.

Note: 2

For $\alpha \to 0$ the function h_α converges to the geomentrical mean.

Proof:

$$h_\alpha(a_1, a_2, \dots a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha}{n}\right)^{1/\alpha}$$

$$\log h_\alpha(a_1, a_2, \dots a_n) = \log\left(\frac{a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha}{n}\right)^{1/\alpha}$$

$$\lim_{\alpha \to 0} \log h_\alpha(a_1, a_2, \dots a_n) = \lim_{\alpha \to 0} \frac{\left(\log(a_1^\alpha + a_2^\alpha + \dots a_n^\alpha) - \log n\right]}{\alpha}$$

$$= \frac{0}{0}$$

using L-hospital rule.

$$\lim_{\alpha \to 0} \log h_\alpha(a_1, a_2 \ldots a_n) = \lim_{\alpha \to 0} \frac{(a_1^\alpha \log a_1 + a_2^\alpha \log a_2 + \ldots + a_n^\alpha \log a_n)}{a_1^\alpha + a_2^\alpha + \ldots a_n^\alpha}$$

$$= \frac{\log a_1 + \log a_2 + \ldots \log a_n}{n}$$

$$= \frac{\log (a_1, a_2, \ldots a_n)}{n}$$

$$\lim_{\alpha \to 0} \log h_\alpha (a_1, a_2, \ldots a_n) = \log (a_1, a_2, \ldots a_n)^{1/n}$$

$$\lim_{\alpha \to 0} h_\alpha (a_1, a_2, \ldots a_n) = (a_1, a_2 \ldots a_n)^{1/n}$$

$$h_\alpha (a_1 a_2, \ldots a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \ldots a_n^\alpha}{n}\right)^{1/n}$$

$$\alpha = 1 \Rightarrow h_1 (a_1 a_2, \ldots a_n) = \frac{a_1 + a_2 + \ldots a_n}{n}$$

Which is the arithematic mean

$$\alpha = -1 \Rightarrow h_{-1} (\alpha_1, a_2, \ldots a_n) = \frac{n}{1/a_1 + 1/a_2 + \ldots 1/a_n}$$

Which is the Harmonic mean

Definition: Ordered weighted averaging (OWA)

Let $w = (w_1, w_2, \ldots w_n)$ be a weighting vector such that $w_i \in [0,1]$ for $i \in N_n$ and $\sum w_i = 1$ then an OWA operation associated with w is the function $h_w (a_1, a_2, \ldots a_n) = w_1 b_1 + w_2 b_2 + \ldots + w_n b_n$ where for any $b_i \in N_n$ is the i^{th} largest element in $a_1, a_2, \ldots a_n$.

Theorem:

Let $h : [0,1]^n \to R^+$ be a function that satisfy $h(0,0,0 \ldots 0)$ and $h(1,1,\ldots 1) = 1$ monotonic increasing and the property

$$h(a_1+b_1, a_2+b_2, \ldots a_n+b_n) = h(a_1, a_2, \ldots a_n) + h$$

where $a_i, b_i, (a_i+b_i) \in [0,1]$ $\forall i \in N_n$. Then

$(b_1, b_2, \ldots b_n)$

$$h(a_1, a_2, \ldots a_n) = \sum_{i=1}^{n} w_i a_i \text{ where } w_i > 0 \ \forall i \in N_n$$

Proof:

Let $h_i(a_i) = h(0,0,\ldots a_i, 0, 0)$ $\forall i \in N_n$

Then for any $a, b, a+b \in [0,1]$

$$h_i(a+b) = h_i(a) + h_i(b)$$

This is a well investigated functional equation to as Cauchy's functional equation. Its solution is

$$h_i(a) = w_i a \text{ for any } a \in [0,1]$$

where $w_i > 0$

$$h(a_1, a_2, \ldots a_n) = h(a_1, 0, 0, \ldots 0) + h(0, a_2, \ldots 0) + h(0, 0, \ldots a_n)$$

$$= h_1(a_1) + h_2(a_2) + \ldots + h_n(a_n)$$

$$= w_1(a_1) + w_2(a_2) + \ldots + w_n(a_n)$$

$$h(a_1, a_2, \ldots a_n) = \sum_{i=1}^{n} w_i a_i$$

Note:

h is defined by above theorem becomes a weighted average. If h is also satisfies axiom h_5. In this case, when $a_1 = a_2 = \dots = a_n = a \neq 0$ we have $a = h(a, a, \dots a) = \sum_{i=1}^{n} w_i a = a \sum_{i=1}^{n} w_i$

$$\sum_{i=1}^{n} w_i = 1$$

Theorem:

Let $h : [0,1]^n \longrightarrow [0,1]$ be a function that satisfies axiom h_1, axiom h_3 and the properties
$h(\max(a_1)(b_1), \dots \max(a_n, b_n)) = \max(h(a_1, a_2, \dots a_n), h(b_1, b_2, \dots b_n))$

$h_i(h_i(a_i)) = h_i(a_i)$, where $h_i(a_i) = h(0, 0 \dots a_i, 0, 0)$ $\forall i \in N_n$ ②

Then $h(a_1, a_2, \dots a_n) = \max(\min(w_1, a_1) \dots \min(w_n, a_n))$
where $w_i \in [0,1] \ \forall i \in N_n$

Proof:

Clearly $h(a_1, a_2, \dots a_n) = h(\max(a_1, 0), \max(0, a_2) \dots \max(0, a_n))$

Given $h(\max(a_1, b_1), \dots \max(a_n, b_n)) =$
$\max[h(a_1, a_2, \dots a_n)\ h(b_1, b_2, \dots b_n)]$

$\therefore h(a_1, a_2, \dots a_n) = \max(h(a_1, 0, 0 \dots 0)\ h(0, a_2 \dots 0)\ h(0, a_3 \dots 0) \dots)$

$$= \max\left(h(0, a_2, 0 \dots 0) \max h(0, 0, a_3, 0) \max h(0,0,0,a_4,0), \max(0,0, \dots a_n)\right)$$

$$= \max\left(h(0, a_2 \dots 0)\ h(0,0,a_3, \dots)\ h(0,0,0 \dots a_n)\right)$$

$$\therefore h(a_1, a_2, \dots a_n) = \max\left(h(a_1, 0, 0 \dots), h(0, a_2, 0 \dots) \dots h(0,0, \dots a_n)\right)$$

$$= \max\left(h_1(a_1), h_2(a_2) \dots h_n(a_n)\right)$$

To prove that $h_i(a_i) = \min(w_i, a_i)\ \forall\ i \in N_n$

Clearly $h_i(a)$ is contineous and non-decreaing and such that $h_i(0) = 0$ and

$$h_i(h_i(a)) = h_i(a_i$$

Let $h_i(1) = w_i$; then, the range of h_i is $[0, w_i]$

For any $a_i \in [0, w_i]$, there exists b_i such that $a_i = h_i(b_i)$ and hence,

$$h_i(a_i) = h_i(h_i(b_i)) = h_i(b_i) = a_i$$
$$= \min(w_i, a_i); \text{ (by (2))}$$

for any $a_i \in [w_i, 1]$, $w_i = h_i(1) = h_i(h_i(1))$
$$= h_i(w_i)$$
$$\leq h_i(a_i)$$
$$\leq h(1)$$
$$= w_i$$

and, consequently, $h_i(a_i) = w_i = \min(w_i, a_i)$

Theorem:

Let a norm operation h be contineous and idempotent then there exists $\lambda \in [0,1]$ Show that

$$h(a,b) = \begin{cases} \max(a,b) & \text{when } a,b \in [0,\lambda] \\ \min(a,b) & \text{when } a,b \in [\lambda,1] \\ \lambda & \text{otherwise } a,b \in [0,1] \end{cases}$$

Proof.

Suppose that h is a contineous and idempontent norm operation. Then h satisfies the properties of continuity, monotonicity, Associativity, commutativity, & weak boundary conditions ($h(0,0)=0$, $h(1,1)=1$, $h(a,a)=a$) $\forall a \in [0,1]$

Let $\lambda = h(0,1) \in [0,1]$

To prove that h sateisfies the following two properties

(i) $h(0,a) = a, \ \forall a \in [0,\lambda]$

(ii) $h(1,a) = a, \ \forall a \in [\lambda,1]$

(i) Let f_1 be a function defined by $f_1(x) = h(0,x)$

Then $f_1(0) = h(0,0) = 0$ and $f_1(1) = h(0,1) = \lambda \ \forall x \in [0,1]$

Since f_1 is contineous and monotonically increasing for any $a \in [0,\lambda]$, $\exists \ x_0 \in [0,1]$ such that

$f_1(x_0) = a$

Then $h(0,a) = h(0, f_1(x_0)) = h(0, h(0,x_0))$
$= h(h(0,0), x_0)$
$= f_1(x_0) = a$

$\therefore h(0,a) = a, \ \forall a \in [0,\lambda]$

ii) Let f_2 be a function defined by

$$f_2(x) = h(x, 1), \forall x \in [0,1]$$

Then $f_2(0) = h(0,1) = \lambda$ and

$$f_2(1) = h(1,1) = 1$$

Since f_2 is contineous a monotonically increasing for any $a \in [\lambda, 1]$, there exist $x_0 \in [0,1]$ such that $f_2(x_0) = a$

Then $h(1,a) = h(1, f_2(x_0)) = h(1, h(x_0, 1))$

$= h(1, h(1, x_0))$ (commutativity)

$= h(h(1,1), x_0)$ (Associativity)

$= h(1, x_0) = h(x_0, 1) = f_2(x_0) = a$

$$h(1,a) = a \quad \forall a \in (\lambda, 1)$$

Part-I

If $a, b \in [0, \lambda]$, then $a = h(a,0) \leq h(a,b)$

and $b = h(0,b) \leq h(a,b)$ (monotonicity $0 \leq b, 0 \leq a$)

Thus $\max(a,b) \leq h(a,b) \rightarrow$ (1)

On the other hand $h(a,b) \leq h(\max(a,b), b)$

$\leq (\max(a,b), \max(a,b))$

$= \max(a,b) \rightarrow$ (2)

From (1) and (2) we get

$$h(a,b) = \max(a,b) \quad \forall a, b \in [0, \lambda]$$

Part ii

if $a, b \in [\lambda, 1]$ then $h(a,b) \leq h(a,1) = a$

$h(a,b) \leq h(1,b) = b$ (by monotonicity $a \leq 1, b \leq 1$)

$\therefore h(a,b) \leq \min(a,b) \ \forall \ a, b \in [\lambda, 1] \rightarrow (3)$

On the otherhand

$$h(a,b) \geq h(\min(a,b), b)$$
$$\geq h(\min(a,b), \min(a,b))$$
$$= \min(a,b) \longrightarrow (4)$$

from (3) and (4) we get

$$h(a,b) = \min(a,b) \quad \forall \ a, b \in [\lambda, 1]$$

Part iii

if $a \in [0, \lambda]$ & $b \in [\lambda, 1]$ Then

$\lambda = h(a, \lambda) \leq h(a,b) \leq h(0, \lambda, b) = \lambda$

ie) $h(a,b) \leq \lambda$ & $h(a,b) \geq \lambda$

$\therefore h(a,b) = \lambda$

($\because h(0,a) = a, \ \forall \ a \in [0, \lambda]$
$h(1,a) = a, \ a \in [\lambda, 1]$)

if $a \in [\lambda, 1]$ and $b \in [0, \lambda]$

Then $\lambda = h(\lambda, b) \leq h(a,b) \leq h(a, \lambda) = \lambda$

ie) $h(a,b) \leq \lambda$ & $h(a,b) \geq \lambda$

$\therefore h(a,b) = \lambda \ \forall \ a, b \in [0, 1]$

$$h(a,b) = \begin{cases} \max(a,b) & \text{when } a, b \in [0, 1] \\ \min(a,b) & \text{when } a, b \in [\lambda, 1] \\ \lambda & \text{otherewise} \end{cases}$$

Definition :

A Special kind of aggrication operations are binary operation h on [0,1] that satisfy the properties of monotonicity, commutative and associativity of t-norm and t-conorm but replace the boundary conditions of t-norms and t-conorms with weaker boundary condition $h(0,0)=1$ and $h(1,1)=1$ these aggrication operation is called norm operation.

Note:

A norm operation has the property $h(a,1)=a$ It becomes a t-norm and the norm operation satisfy $h(a,0)=a$, It becomes t-conorm otherwise It is an associative averaging operation.

Definition :

A paramelesised class of norm operation that are neither t-norm nor t-conorm is the class of binary operations on [0,1] defined by
$$h_\lambda(a,b) = \begin{cases} \min(\lambda, u(a,b)) & \text{if } a,b \in [0,\lambda] \\ \max(\lambda, i(a,b)) & \text{if } a,b \in [\lambda,1] \\ \lambda & \text{Otherwise} \end{cases}$$
$\forall\, a,b \in [0,1], \lambda \in (0,1)$

i is an t-norm and u is an t-conorm. These operations is called λ-averages.

Definition : Fuzzy Number

A fuzzy Number is a fuzzy set A on R must posses atleast the following three properties

(i) A must be normal fuzzy set

(ii) α_A must be a closed interval for $\alpha \in [0,1]$

(iii) The support of A, 0_{+A} must be bounded

Example

Special case fuzzy number include ordinary real numbers and intervals of real numbers

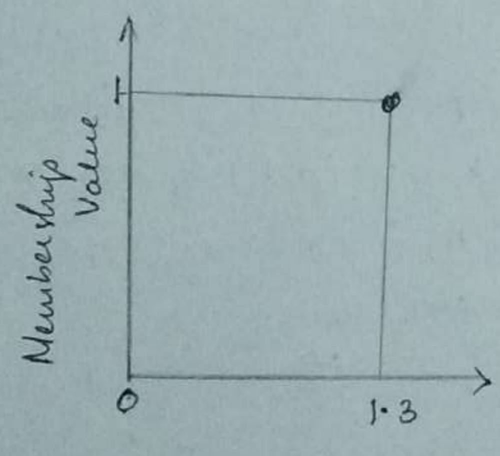
a) An ordinary real number 1.3

b) An ordinary closed interval [1.25, 1.35]

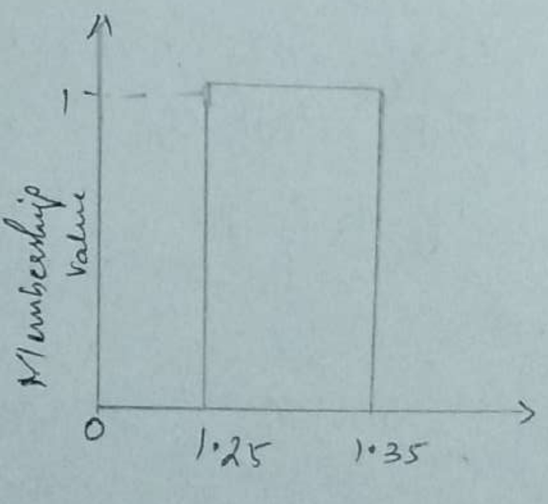
c) A fuzzy number expressing the proportion closed to 1.3 and

d) A fuzzy number with a flat region (fuzzy interval)

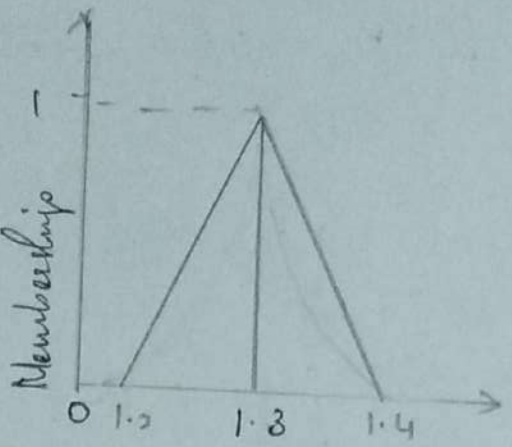
a)



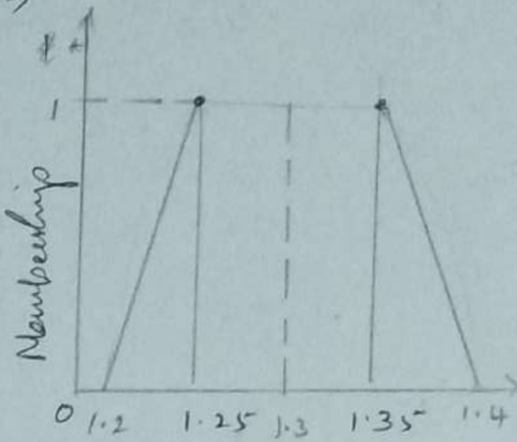
b)



c)



d)



Theorem:

Let $A \in \mathcal{F}(R)$ then A is a fuzzy number iff there exist a closed interval $[a,b] \neq \phi$ such that

$$A(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ l(x) & \text{if } x \in (-\infty, a) \\ r(x) & \text{if } x \in (b, \infty) \end{cases} \quad \text{——— (4.1)}$$

Where l is a function from $(-\infty, a)$ to $[0,1]$

ie) Monotonic increasing contineous from the right and such that $l(x) = 0$ for $x \in (-\infty, \omega_1)$ and r is a function from (b, ∞) to $[0,1]$ that is monotonic decreasing contineous from the left and such that $r(x) = 0 \ \forall \ x \in (\omega_2, \infty)$

Proof

Assume that A is a fuzzy number

ie) ${}^{\alpha}A$ is closed interval for ever $\alpha \in [0,1]$

For $\alpha = 1$, ${}^{1}A$ is non empty

($\because$ A is normal)

Hence there exist a pair $a, b \in R$ such that

$^1A = [a,b]$ where $a \leq b$

ie) $A(x) = 1$ for $x \in [a,b]$ and

$A(x) < 1$ for $x \notin [a,b]$

Now, let $l(x) = A(x)$ for any $x \in (-\infty, a)$

Then, $0 \leq l(x) < 1$ since $0 \leq A(x) < 1$

for every $x \in (-\infty, \alpha)$.

Let $x \leq y < a$; then

$$A(y) \geq \min[A(x), A(a)] = A(x)$$

by Theorem 1.1. Since A is convex and $A(a) = 1$.

Hence, $l(y) \geq l(x)$; that is, l is monotonic increasing.

Assume now that $l(x)$ is not continuous from the right. This means that for some $x_0 \in (-\infty, a)$ there exist a sequence of numbers $\{x_n\}$ such that $x_n \geq x_0$ for any n and $\lim_{n \to \infty} x_n = x_0$.

but

$$\lim_{n \to \infty} l(x_n) = \lim_{n \to \infty} A(x_n) = \alpha > l(x_0) = A(x_0)$$

Now, $x_n \in {}^\alpha A$ for any n since ${}^\alpha A$ is a closed interval and hence, also $x_0 \in {}^\alpha A$. Therefore, $l(x_0) = A(x_0) \geq \alpha$, which is a contradiction. That is, $l(x)$ is continuous from the right.

The proof that function r in (4.1) is monotonic decreasing and continuous from the left is similar.

Since A is a fuzzy number, ${}^{0+}A$ is bounded. Hence, there exists a pair $w_1, w_2 \in \mathbb{R}$ of finite numbers such that $A(x) = 0$ for $x \in (-\infty, w_1) \cup (w_2, \infty)$

$A(x) = 0$ for $x \in (-\infty, w_1) \cup (w_2, \infty)$.

Sufficiency. Every fuzzy set A defined by (4.1) is clearly normal, and its support, ${}^{0+}A$, is bounded since ${}^{0+}A \subseteq [w_1, w_2]$.

It remains to prove that ${}^{\alpha}A$ is a closed interval for any $\alpha \in (0, 1]$.

Let

$$x_\alpha = \inf\{x \mid l(x) \geq \alpha,\ x < a\},$$
$$y_\alpha = \sup\{x \mid r(x) \geq \alpha,\ x > b\}$$

for each $\alpha \in (0, 1]$.

We need to prove that ${}^{\alpha}A = [x_\alpha, y_\alpha]\ \forall \alpha \in (0, 1]$.

For any $x_0 \in {}^{\alpha}A$, if $x_0 < a$, then $l(x_0) = A(x_0) \geq \alpha$.

That is, $x_0 \in \{x \mid l(x) \geq \alpha,\ x < a\}$ and, consequently, $x_0 \geq \inf\{x \mid l(x) \geq \alpha,\ x < a\} = x_\alpha$

If $x_0 > b$, then $r(x_0) = A(x_0) \geq \alpha$

that is, $x_0 \in \{x \mid r(x) \geq \alpha,\ x > b\}$ and consequently, $x_0 \leq \sup\{x \mid r(x) \geq \alpha,\ x > b\} = y_\alpha$.

Obviously, $x_\alpha \leq a$ and $y_\alpha \geq b$;
that is, $[a, b] \subseteq [x_\alpha, y_\alpha]$
Therefore, $x_0 \in [x_\alpha, y_\alpha]$ and hence,
${}^\alpha A \subseteq [x_\alpha, y_\alpha]$. It remains to prove that
$x_\alpha, y_\alpha \in {}^\alpha A$.

By the definition of x_α, there must exist a sequence $\{x_n\}$ in $\{x \mid l(x) \geq \alpha, x < a\}$ such that $\lim_{n \to \infty} x_n = x_\alpha$, where $x_n \geq x_\alpha$ for any n.

Since l is continuous from the right, we have

$$l(x_\alpha) = l(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} l(x_n) \geq \alpha$$

Hence, $x_\alpha \in {}^\alpha A$. We can prove that $y_\alpha \in {}^\alpha A$ in a similar way.

UNIT - IV

ARITHMETIC OPERATIONS ON INTERVAL

Let $*$ denote any of the four arithmetic operations on closed intervals: addition $+$, subtraction $-$, multiplication $\cdot$, & division $/$.

Then $[a, b] * [d, e] = \{f * g \mid a \le f \le b,\ d \le g \le e\}$.

is a general property of all arithmetic operations on closed intervals.

$$[a, b] + [d, e] = [a+d,\ b+e]$$

$$[a, b] - [d, e] = [a-e,\ b-d]$$

$$[a, b] \cdot [d, e] = [\min(ad, ae, bd, be),\ \max(ad, ae, bd, be)],$$

and, provided that $0 \notin [d, e]$,

$$\frac{[a, b]}{[d, e]} = [a, b] \cdot [1/e, 1/d]$$

$$= [\min(a/d, a/e, b/d, b/e),\ \max(a/d, a/e, b/d, b/e)]$$

$[a, b]/[d, e] = F$

Arithmetic operations on closed intervals satisfy some useful properties.

Let $A = [a_1, a_2]$, $B = [b_1, b_2]$, $C = [c_1, c_2]$, $0 = [0, 0]$, $1 = [1, 1]$.

Using these symbols, the properties are formulated as follows:-

1. $A + B = B + A$

   $A \cdot B = B \cdot A$ (commutativity)

2. $(A+B)+C = A+(B+C)$

$(A\cdot B)\cdot C = A\cdot(B\cdot C)$ (associativity)

3. $A = 0+A = A+0$

$A = 1\cdot A = A\cdot 1$ (Identity)

4. $A\cdot(B+C) \subseteq A\cdot B + A\cdot C$ (sub distributivity)

5. If $b\cdot c \geq 0$ for every $b \in B$ & $c \in C$, then

$A\cdot(B+C) = A\cdot B + A\cdot C$ (distributivity).

Furthermore, If $A = [a, a]$, the $a\cdot(B+C) = a\cdot B + a\cdot C$.

6. $0 \in A-A$ & $1 \in A/A$

7. If $A \subseteq E$ & $B \subseteq F$, then:

$A+B \subseteq E+F$

$A-B \subseteq E-F$

$A\cdot B \subseteq E\cdot F$

$A/B \subseteq E/F$ (inclusion monotonicity)

Theorem 4.2

Let $* \in \{+, -, \cdot, /\}$, and let A, B denote continuous fuzzy numbers. Then, the fuzzy set $A*B$ defined by

$$(A*B)(z) = \sup_{z = x*y} \min[A(x), B(y)]$$

is a continuous fuzzy number.

Proof:-

First, we prove ${}^{\alpha}(A*B) = {}^{\alpha}A * {}^{\alpha}B$ by showing that ${}^{\alpha}(A*B)$ is a close interval for every $\alpha \in (0,1]$.

For any $z \in {}^{\alpha}A * {}^{\alpha}B$, there exist some $x_0 \in {}^{\alpha}A$ & $y_0 \in {}^{\alpha}B$ such that $z = x_0 * y_0$.

81) Thus,

$$(A * B)(z) = \sup_{z = x * y} \min [A(x), B(y)]$$

$$\geq \min [A(x_0), B(y_0)]$$

$$\geq \alpha.$$

Hence, $z \in {}^{\alpha}(A * B)$ and consequently,

$$^{\alpha}A * {}^{\alpha}B \subseteq {}^{\alpha}(A * B).$$

For any $z \in {}^{\alpha}(A * B)$, we have

$$(A * B)(z) = \sup_{z = x * y} \min [A(x), B(y)]$$

$$\geq \alpha.$$

Moreover, for any $n > [1/\alpha] + 1$, where $[1/\alpha]$ denotes the largest integer that is less than or equal to $1/\alpha$, there exist x_n & y_n such that $z = x_n * y_n$ &

$$\min [A(x_n), B(y_n)] > \alpha - 1/n$$

That is, $x_n \in {}^{\alpha - 1/n}A$, $y_n \in {}^{\alpha - 1/n}B$ & we may consider two sequences, $\{x_n\}$ & $\{y_n\}$.

Since,
$$\alpha - 1/n \leq \alpha - \frac{1}{n+1}$$

we have

$$^{\alpha - 1/(n+1)}A \subseteq {}^{\alpha - 1/n}A, \quad {}^{\alpha - 1/n+1}B \subseteq {}^{\alpha - 1/n}B.$$

Hence, $\{x_n\}$ & $\{y_n\}$ fall into some $^{\alpha - 1/n}A$ & $^{\alpha - 1/n}B$, respectively.

Since the latter are closed intervals, $\{x_n\}$ & $\{y_n\}$ are bounded sequences. Thus, there exist a convergent subsequence $\{x_{n,i}\}$ such that $x_{n,i} \to x_0$.

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To the corresponding subsequence $\{y_{n,i,j}\}$ here also exist a convergent subsequence $\{y_{n,i,j}\}$ such that $y_{n,i,j} \to y_0$.

If we take the corresponding subsequence, $\{x_{n,i,j}\}$ from $\{x_{n,i}\}$, then $x_{n,i,j} \to x_0$.

Thus, we have two sequence $\{y_{n,i,j}\}$ & $\{y_{n,i,j}\}$, such that $x_{n,i,j} \to x_0$, $y_{n,i,j} \to y_0$, & $x_{n,i,j} * y_{n,i,j} = z$.

Now, since $*$ is continuous

$$z = \lim_{j\to\infty} x_{n,i,j} * y_{n,i,j} = \left(\lim_{j\to\infty} x_{n,i,j}\right) * \left(\lim_{j\to\infty} y_{n,i,j}\right)$$

$$= x_0 * y_0.$$

Also, since $A(x_{n,i,j}) > \alpha - \frac{1}{n_{i,j}}$ &

$$B(y_{n,i,j}) > \alpha - \frac{1}{n_{i,j}}$$

$$A(x_0) = A\left(\lim_{j\to\infty} x_{n,i,j}\right) = \lim_{j\to\infty} A(x_{n,i,j}) \geq \lim_{j\to\infty}\left(\alpha - \frac{1}{n_{i,j}}\right)$$

$$= \alpha$$

and

$$B(y_0) = B\left(\lim_{j\to\infty} y_{n,i,j}\right) = \lim_{j\to\infty} B(y_{n,i,j}) \geq \lim_{j\to\infty}\left(\alpha - \frac{1}{n_{i,j}}\right)$$

$$= \alpha.$$

Therefore, there exist $x_0 \in {}^\alpha A$, $y_0 \in {}^\alpha B$ such that $z = x_0 * y_0$.

That is, $z \in {}^\alpha A * {}^\alpha B$.

Thus, ${}^\alpha(A*B) \subseteq {}^\alpha A * {}^\alpha B$,

& consequently,

$${}^\alpha(A*B) = {}^\alpha A * {}^\alpha B.$$

83) Now we prove that $A*B$ must be continuous

By theorem (4.1), the membership function of $A*B$ must be of the general form.

Assume $A*B$ is not continuous at z_0.

(ie) $\lim_{z \to z_0} (A*B)(z) < (A*B)(z_0) = \sup_{z_0 = x*y} \min[A(x), B(y)]$.

Then there must exists x_0 & y_0 such that $z_0 = x_0 * y_0$ &

$$\lim_{z \to z_0} (A*B)(z) < \min[A(x_0), B(y_0)] \longrightarrow ①$$

Since the operation $* \in \{+, -, \cdot, /\}$ is monotonic with respect to the first & the second argument respectively.

We can always find two sequences $\{x_n\}$ & $\{y_n\}$ such that $x_n \to x_0$, $y_n \to y$ as $n \to \infty$.

Thus, $x_n * y_n < z_0$ for any n.

Let $z_n = x_n * y_n$: then $z_n \to z_0$ as $n \to \infty$.

Thus,

$$\begin{aligned}
\lim_{z \to z_0} (A*B)(z) &= \lim_{n\to\infty} (A*B) z_n) \\
&= \lim_{n\to\infty} \sup_{z_n = x*y} \min[A(x), B(y)] \\
&> \lim_{n\to\infty} \min[A(x_n), B(y_n)] \\
&= \min[A(\lim_{n\to\infty} x_n), B(\lim_{n\to\infty} y_n)] \\
&= \min[A(x_0), B(y_0)].
\end{aligned}$$

This contradicts (4.1) &, therefore, $A*B$ must be continuous fuzzy number.

This completes the proof.

BINARY FUZZY RELATIONS:-

A fuzzy relation $R(X, X)$, its domain is a fuzzy set on X, dom R, whose membership function is defined by

$$\text{dom } R(x) = \max_{y \in Y} R(x, y) \text{ for each } x \in X$$

The range of $R(X, Y)$ is a fuzzy relation on Y, ran R, whose membership function is defined by

$$\text{ran } R(y) = \max_{x \in X} R(x, y) \text{ for each } y \in Y$$

The height of a fuzzy relation $R(X, Y)$ is a number, $h(R)$, defined by

$$h(R) = \max_{y \in Y} \max_{x \in X} R(x, y).$$

That is, $h(R)$ is the largest membership grade attained by any pair $\langle x, y \rangle$ in R.

A convenient representation of binary relation $R(X, Y)$ are membership matrices $R = [r_{xy}]$, where $r_{xy} = R(x, y)$.

The inverse of a fuzzy relation $R(X, Y)$ which is denoted by $R^{-1}(Y, X)$ is a relation on $Y \times X$ defined by

$$R^{-1}(y, x) = R(x, y)$$

for all $x \in X$ & all $y \in Y$.

The standard composition of these relations, which is denoted by $P(X, Y) \circ Q(Y, Z)$, Produces a binary relation $R(X, Z)$ on $X \times Z$ defined by

$$R(x, z) = [P \circ Q](x, z) = \max_{y \in Y} \min [P(x, y), Q(y, z)]$$

for all $x \in X$ and all $z \in Z$.

This composition, which is based on the standard t-norm & t-conorm, is often referred to as the max-min composition.

10/17 FUZZY EQUIVALENCE RELATIONS:

A crisp binary relation $R(X, X)$ that is reflexive, symmetric and transitive is called an equivalence relation. For each element x in X, we can define a crisp set A_x, which contains all the elements of X that are related to x by the equivalence relation. Formally,

$$A_x = \{ y / (x, y) \in R(X, X) \}.$$

Example 5.9

Let $X = \{1, 2, \ldots, 10\}$. The cartesian product $X \times Y$ contains 100 members: $\langle 1,1 \rangle$ $\langle 1,2 \rangle \cdots \langle 10,10 \rangle$. Let $R(X, X) = \{ \langle x, y \rangle \mid x$ & y have the same remainder when divided by $\{3\}$. The relation is easily shown to be reflexive, symmetric & transitive & is therefore an equivalence relation on X. The three equivalence classes defined by this relation are:

$A_1 = A_2 = A_7 = A_{10} = \{1, 4, 7, 10\}$.

$A_2 = A_5 = A_8 = \{2, 5, 8\}$

$A_3 = A_6 = A_9 = \{3, 6, 9\}$

Hence, in this example, $X/R = \{\{1,4,7,10\}\{2,5,8\}, \{3,6,9\}$

— x —

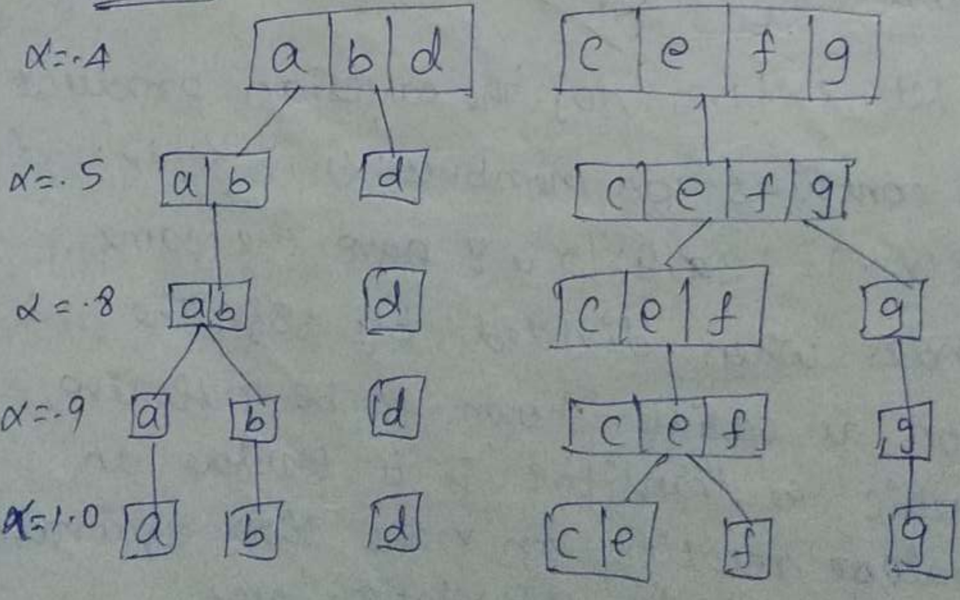
A fuzzy binary relation that is reflexive, symmetric and transitive is known as a <u>fuzzy equivalence relation (or) Similarity relation.</u>

<u>Example 5.10</u>

The fuzzy relation $R(X, X)$ represented by the membership matrix.

$$
\begin{array}{c|ccccccc}
 & a & b & c & d & e & f & g \\
\hline
a & 1 & .8 & 0 & .4 & 0 & 0 & 0 \\
b & .8 & 1 & 0 & .4 & 0 & 0 & 0 \\
c & 0 & 0 & 1 & 0 & 1 & .9 & .5 \\
d & .4 & .4 & 0 & 1 & 0 & 0 & 0 \\
e & 0 & 0 & 1 & 0 & 1 & .9 & .5 \\
f & 0 & 0 & .9 & 0 & .9 & 1 & .5 \\
g & 0 & 0 & .5 & 0 & .5 & .5 & 1
\end{array}
$$

<u>Partition tree</u>



Partition tree for the similarity relation.

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UNIT - V

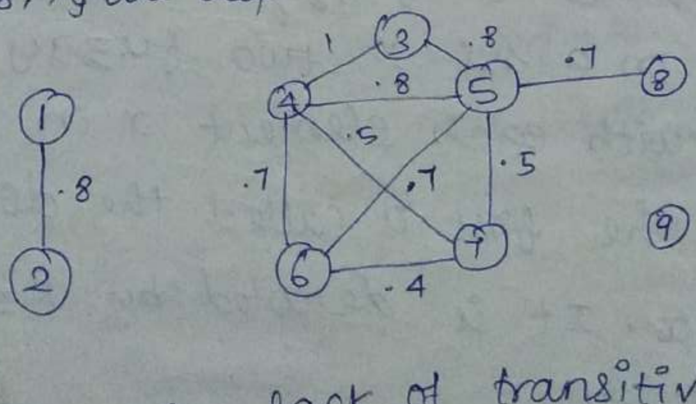
Example 5.11

Consider a fuzzy relation $R(X, X)$ defined on $X = N_9$ by the following membership matrix:

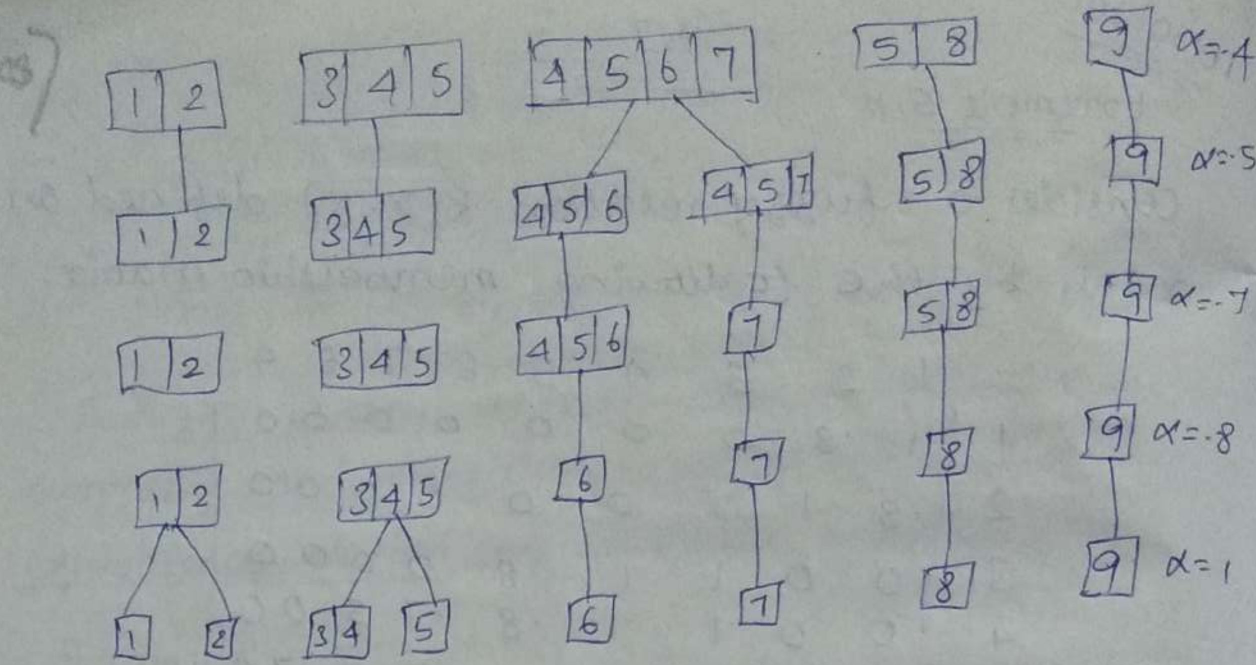
$$
\begin{array}{c|c}
 & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} &
\begin{bmatrix}
1 & .8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & .8 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & .8 & .7 & .5 & 0 & 0 \\
0 & 0 & .8 & .8 & 1 & .7 & .5 & .7 & 0 \\
0 & 0 & 0 & .7 & .7 & 1 & .4 & 0 & 0 \\
0 & 0 & 0 & .5 & .5 & .4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & .7 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Since the matrix is symmetric and all entries on the main diagonal are equal to 1, the relation represented is reflexive & symmetric; therefore, it is a compatibility relation.

Its comple α-covers for $\alpha > 0$ & $\alpha \in \Lambda_R = \{0, .4, .5, .7, .8, 1\}$ are depicted.



Since the lack of transitivity distinguishes compatability relations from similarity relations, the transitive closures of compatibility relations are similarity relations.



All complete α-covers for the compatibility relation R.

Fuzzy partial ordering

A fuzzy binary relation R on a set X is a fuzzy partial ordering iff it is reflexive, antisymmetric, & transitive under some form of fuzzy transitivity.

Dominating class:-

When a fuzzy partial ordering is defined on a set X, two fuzzy sets are associated with each element x in X.

The first is called the dominating class of x. It is denoted by $R_{\geq[x]}$ & is defined by

$$R_{\geq[x]}(y) = R(x, y)$$

where $y \in X$. In other words, the dominating class of x contains the members of X to the degree to which they dominate x.

39) The second fuzzy set of concern in the class dominated by x, which is denoted by $R_{\leq}[x]$ & defined by

$$R_{\leq[x]}(y) = R(y, x),$$

where $y \in X$. The class dominated by x contains the elements of X to the degree to which they are dominated by x.

An element $x \in X$ is undominated iff

$$R(x, y) = 0$$

for all $y \in X$ & $x \neq y$; an element x is undominating iff

$$R(y, x) = 0 -$$

for all $y \in X$ & $y \neq x$.

For a crisp subset A of a set X on which a fuzzy partial ordering R is defined, the fuzzy upper bound for A is the fuzzy set denoted by $U(R, A)$ & defined by

$$U(R, A) = \bigcap_{x \in A} R_{\geq[x]}$$

where $\cap$ denotes an appropriate fuzzy intersection.

If a least upper bound of the set A exists, it is the unique element x in $U(R, A)$ such that

$$U(R, A)(x) > 0 \quad \&$$

$$R(x, y) > 0,$$

for all elements y in the support of $U(R, A)$.

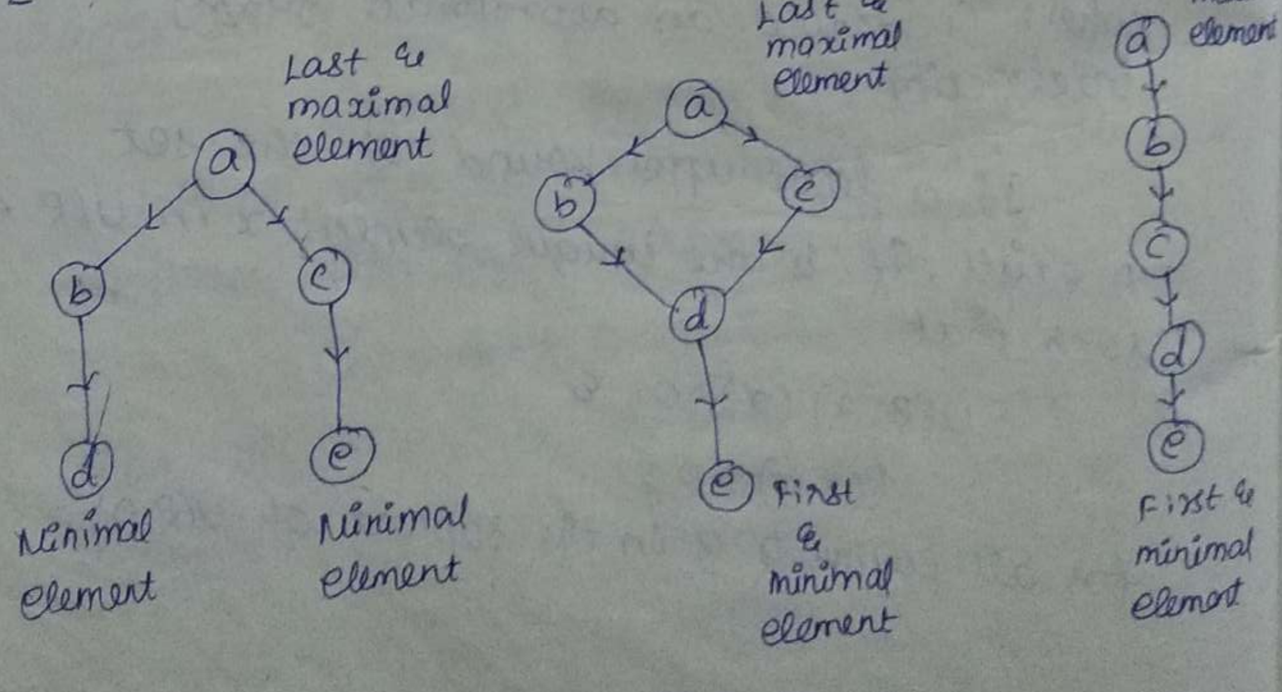
Example 5.12

Three crisp partial orderings P, Q & R on the set $X = \{a, b, c, d, e\}$ are defined by their membership matrices (crisp) and their Hasse diagrams. The underlined entries in each matrix indicate the relationship of the immediate predecessor & successor employed in the corresponding Hasse diagram. P has no special properties, Q is a lattice and R is an example of a lattice that represents a linear ordering.

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \underline{1} & 1 & 0 & 0 & 0 \\ \underline{1} & 0 & 1 & 0 & 0 \\ 1 & \underline{1} & 0 & 1 & 0 \\ 1 & 0 & \underline{1} & 0 & 1 \end{bmatrix} \end{array}$$

$$Q = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \underline{1} & 1 & 0 & 0 & 0 \\ \underline{1} & 0 & 1 & 0 & 0 \\ 1 & \underline{1} & \underline{1} & 1 & 0 \\ 1 & 1 & 1 & \underline{1} & 1 \end{bmatrix} \end{array}$$

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \underline{1} & 1 & 0 & 0 & 0 \\ 1 & \underline{1} & 1 & 0 & 0 \\ 1 & 1 & \underline{1} & 1 & 0 \\ 1 & 1 & 1 & \underline{1} & 1 \end{bmatrix} \end{array}$$



Examples of partial ordering

Example 5.13

The following membership matrix defines a fuzzy partial ordering R on the set $X = \{a, b, c, d, e\}$:

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & .7 & 0 & 1 & .7 \\ 0 & 1 & 0 & .9 & 0 \\ .5 & .7 & 1 & 1 & .8 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & .1 & 0 & .9 & 1 \end{bmatrix} \end{array}$$

The dominating class for each element is given by the row of the matrix corresponding to that element.

The columns of the matrix give the dominated class for each element.

Under this ordering, the element d is undominated and the element c is undominating.

For the subset $A = \{a, b\}$, the upper bound is the fuzzy set produced by the intersection of the dominating classes for a & b.

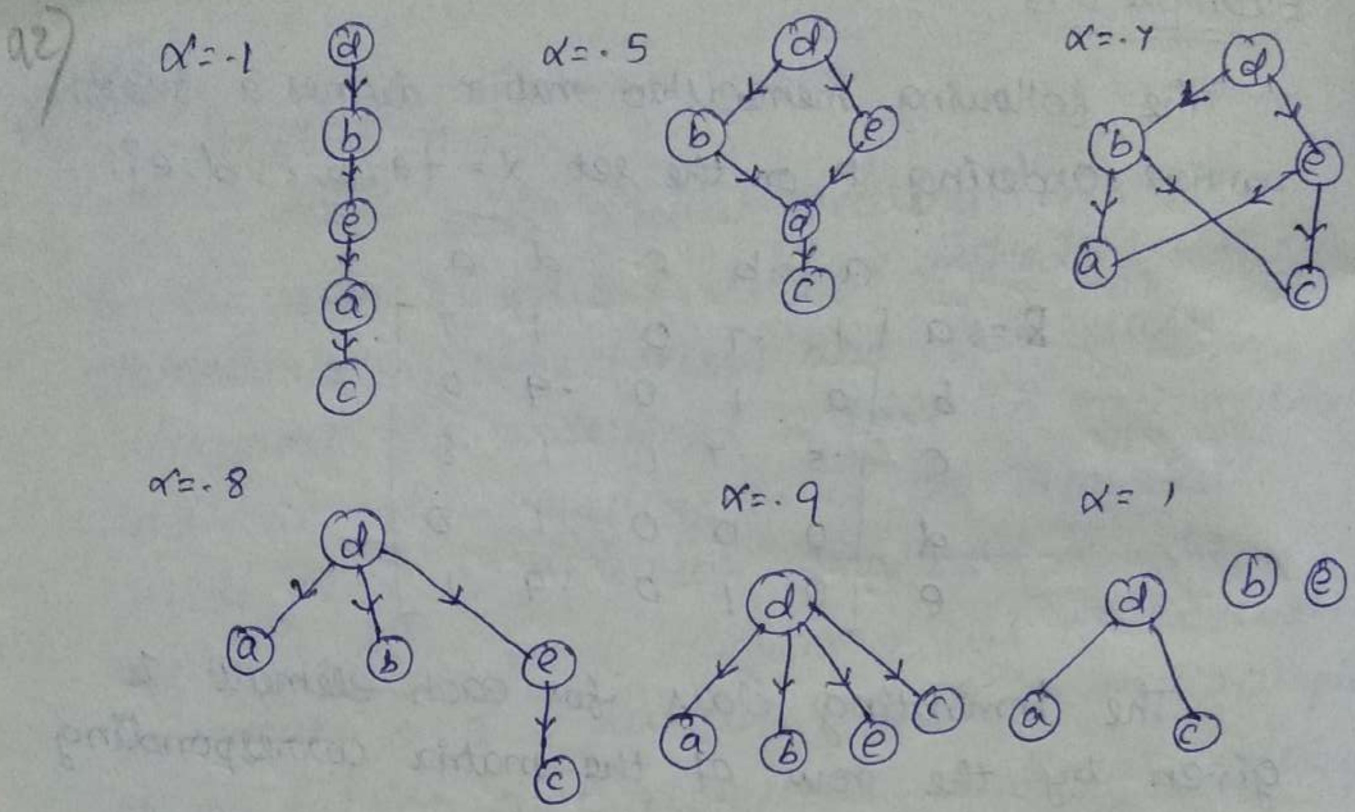
Employing the min operator for fuzzy intersection, we obtain

$$U(R, \{a, b\}) = .7/b + .9/d.$$

The unique least upper bound for the set A is the element b.

All distinct crisp orderings captured by the given fuzzy partial ordering R.

We can see that the orderings became weaker with the increasing value of α.



The set of all distinct crisp orderings captured by the fuzzy partial ordering R.

FUZZY MORPHISMS:-

If two crisp binary relations $R(X, X)$ & $Q(Y, Y)$ are defined on sets X & Y, respectively, then a function $h: X \to Y$ is said to be a homomorphism from $\langle X, R \rangle$ to $\langle Y, Q \rangle$ if

$$\langle x_1, x_2 \rangle \in R \text{ implies } \langle h(x_1), h(x_2) \rangle \in Q,$$

for all $x_1, x_2 \in X$.

In other words, a homomorphism implies that for every two elements of set X which are related under the relation R, their homomorphic images $h(x_1), h(x_2)$ in set Y are related under the relation Q.

$$R(x_1, x_2) \le Q(h(x_1), h(x_2)),$$

93) for all $x_1, x_2 \in X$ & their images $h(x_1), h(x_2) \in Y$.

The homomorphic function h is said to be strong homomorphism, It satisfies the two implications,

$$\langle x_1, x_2\rangle \in R \text{ implies } \langle h(x_1), h(x_2)\rangle \in Q$$

for all $x_1, x_2 \in X$ &

$$\langle y_1, y_2\rangle \in Q \text{ implies } \langle x_1, x_2\rangle \in R,$$

for all $y_1, y_2 \in Y$, where $x_1 \in h^{-1}(y_1)$ & $x_2 \in h^{-1}(y_2)$.

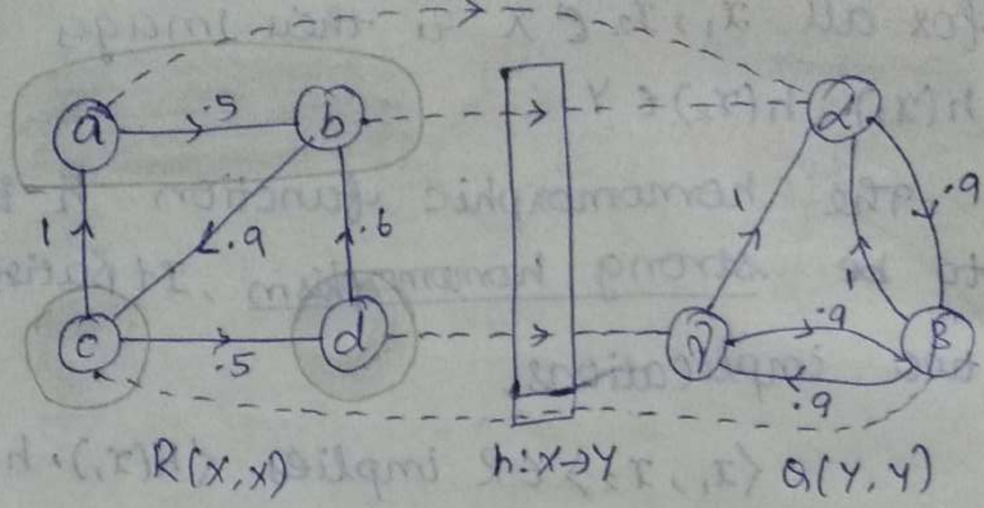
Example 5.14

The following membership matrices represent fuzzy relations $R(X, X)$ & $Q(Y, Y)$ defined on sets $X = \{a, b, c, d\}$ & $Y = \{\alpha, \beta, \gamma\}$, respectively.

$$R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & .5 & 0 & 0 \\ b & 0 & 0 & .9 & 0 \\ c & 1 & 0 & 0 & .5 \\ d & 0 & .6 & 0 & 0 \end{array} \qquad Q = \begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline \alpha & .5 & .9 & 0 \\ \beta & 1 & 0 & .9 \\ \gamma & 1 & .9 & 0 \end{array}$$

The second relation, Q, is the homomorphic image of R under the homomorphic function h, which maps elements a & b to element α, element c to element β, & element d to element γ. The pair $\langle\gamma, \beta\rangle$ in Q does not correspond to any pair in the relation R. Figure illustrates the fuzzy homomorphism.

94)



(a) ordinary fuzzy homomorphism

DEFINITION:-

If $h : X \to Y$ is a homomorphism from $\langle X, R \rangle$ to $\langle Y, Q \rangle$, & if h is completely specified, one-to-one, and onto, then it is called an isomorphism.

This is effectively a translation or direct relabeling of elements of the set X into elements of the set Y that preserves all the properties of R in Q.

If $Y \subseteq X$, then h is called an endomorphism.

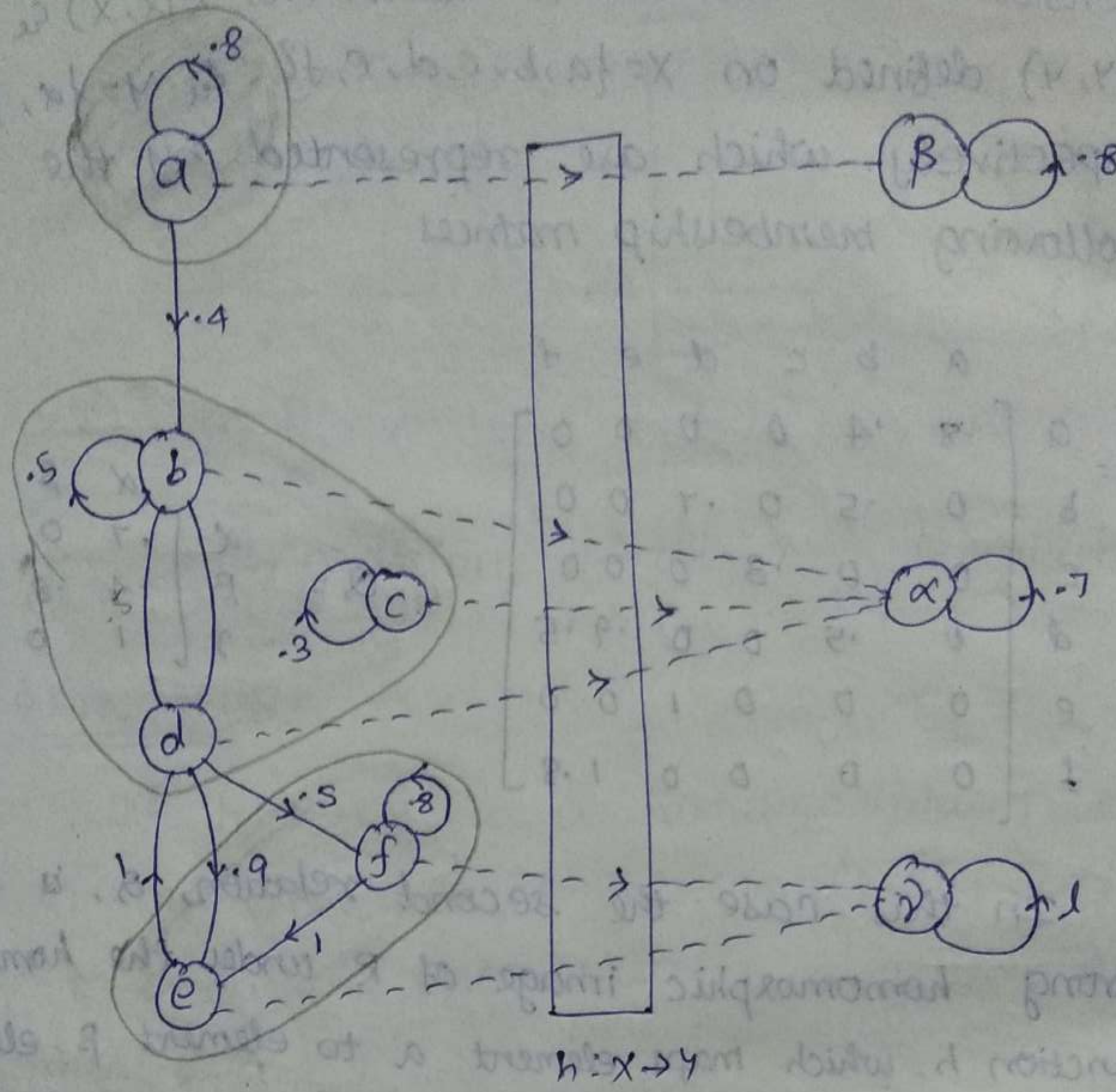
A function that is both an isomorphism & an endomorphism is called an automorphism

95/ consider now alternative relations $R(X, X)$ & $Q(Y, Y)$ defined on $X = \{a, b, c, d, e, f\}$ & $Y = \{\alpha, \beta, \gamma\}$, respectively, which are represented by the following membership matrices.

$$R = \begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} .8 & .4 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & .7 & 0 & 0 \\ 0 & 0 & .8 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & .9 & .5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & .8 \end{bmatrix} \end{array} \qquad Q = \begin{array}{cc} & \begin{array}{ccc} \alpha & \beta & \gamma \end{array} \\ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} & \begin{bmatrix} .7 & 0 & .9 \\ .4 & .8 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

In this case the second relation, Q, is a strong homomorphic image of R under the homomorphic function h which maps element a to element β, element b, c and d to element α, and elements e & f to elements γ. The set X is therefore partitioned into the three equivalence classes $\{a\}, \{b, c, d\}$ & $\{e, f\}$. figure depicts this strong homomorphism.

96)



R(X : X)

Q(Y, Y)

(b) Strong fuzzy homomorphism.

11/10/17

—— X ——