Calculus and Trigonometry

Unit I

Successive Differentiation

Formulas

	$y = f(x)$	$\frac{dy}{dx}$
	$y = c$ (constant)	$\frac{dy}{dx} = 0$
1,	$y = x^n$	$\frac{dy}{dx} = n x^{n-1}$
2	$y = e^x$	e^x.
3	$y = \log_e x$	$\frac{1}{x}$.
4	$y = \sin x$	$\cos x$.
5	$\cos x$	$-\sin x$
6	$\tan x$	$\sec^2 x$
7	$\cot x$	$-\operatorname{cosec}^2 x$
8	$\sec x$	$\sec x \tan x$
9,	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
10,	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$.
11,	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
12,	$\tan^{-1} x$	$\frac{1}{1+x^2}$
13,	$\cot^{-1} x$	$-\frac{1}{1+x^2}$
14,	$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
15,	$\operatorname{cosec}^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$
16,	$\sinh x$	$\cosh x$
17,	$\cosh x$	$\sinh x$

Rules: where u and v are functions of x.

1, If $y = u \pm v$, where u and v are functions of x.
then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

2, Product Rule

If $y = uv$, where u and v are functions of x

then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

III^ly If $y = uvw$

then $\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

3 Quotient Rule

If $y = \frac{u}{v}$, where u and v are functions of x

then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Successive Differentiation

If $y = 4x^5$

Diff w.r.t x

$$\frac{dy}{dx} = 4 \cdot 5\, x^4$$
$$= 20\, x^4$$
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 20 \cdot 4\, x^3$$
$$\frac{d^2y}{dx^2} = 80\, x^3.$$

The n^{th} derivative

For certain functions a general expression involving n may be found for the n^{th} derivative

Problem (1) If $y = e^{ax}$ find $\frac{d^n y}{dx^n}$

Sol $y = e^{ax}$

Diff w.r.t x

$$\frac{dy}{dx} = e^{ax} \cdot a$$
$$= a\, e^{ax}$$
$$\frac{d^2y}{dx^2} = a \cdot e^{ax} \cdot a$$
$$= a^2 e^{ax}$$
$$\frac{d^3y}{dx^3} = a^2 e^{ax} \cdot a$$
$$= a^3 e^{ax}$$
$$\therefore \frac{d^n y}{dx^n} = a^n e^{ax}$$

Standard Results

(1) If $y = (ax+b)^m$ then

$$y_1 = \frac{dy}{dx} = m\,(ax+b)^{m-1} \cdot a$$
$$= m \cdot a\,(ax+b)^{m-1}$$
$$y_2 = \frac{d^2y}{dx^2} = m \cdot (m-1)\,a\,(ax+b)^{m-2} \cdot a$$
$$= m\,(m-1)\,a^2\,(ax+b)^{m-2}$$

$|||^{ly}$
$$y_3 = \frac{d^3y}{dx^3} = m\,(m-1)(m-2)\,a^3\,(ax+b)^{m-3}$$
$$y_n = m\,(m-1)(m-2)\ldots(m-\overline{n-1})\,a^n\,(ax+b)^{m-n}$$
$$= m\,(m-1)\,(m-2)\ldots(m-n+1)\,a^n\,(ax+b)^{m-n}$$

(2) If $y = \frac{1}{(ax+b)}$ then

$$y = (ax+b)^{-1}$$

$$y = (ax+b)^{-1}$$

$$\therefore\ y_n = (-1)(-1-1)(-1-2)\cdots(-1-n+1)\, a^n (ax+b)^{-n-1}$$

$$= (-1)(-2)(-3)\cdots(-n)\, a^n (ax+b)^{-n-1}$$

$$= (-1)^n (1)(2)\cdots(n)\, a^n (ax+b)^{-n-1}$$

$$= (-1)^n n!\, a^n (ax+b)^{-n-1}$$

$$= \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$$

Problem 1 Find y_n, where $y = \dfrac{3}{(x+1)(2x-1)}$

Sol

$$y = \frac{3}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1} \rightarrow (1)$$

$$= \frac{A(2x-1) + B(x+1)}{(x+1)(2x-1)}$$

$$A(2x-1) + B(x+1) = 3$$

Put $x = -1$

$$A[2(-1)-1] = 3$$

$$A(-3) = 3$$

$$\boxed{A = -1}$$

Put $x = 1/2$

$$B(1/2 + 1) = 3$$

$$B\left(\tfrac{3}{2}\right) = 3$$

$$\boxed{B = 2}$$

Using in (1)

$$y = \frac{2}{2x-1} - \frac{1}{x+1}$$

$$y = 2(2x-1)^{-1} - (1)(x+1)^{-1} \rightarrow (2)$$

If $y = (ax+b)^{-1}$ then $y_n = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$\therefore$ using in (2)

$$y_n = \frac{(2)(-1)^n (2)^n n!}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

$$= \frac{2^{n+1} (-1)^n n!}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

$$= (-1)^n n! \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}} \right]$$

② Find y_n when $y = \dfrac{x^2}{(x-1)^2(x+2)}$

Sol. $y = \dfrac{x^2}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2} \rightarrow ①$

$$= \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)^2(x+2)}$$

$$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = x^2$$

Put $x = 1$,

$A(0) + B(1+2) = 1^2$

$B(3) = 1$

$B = \frac{1}{3}$

Put $x = -2$

$C(-2-1)^2 = (-2)^2$

$C(-3)^2 = (-2)^2$

$9C = 4$

$\boxed{C = 4/9}$

Put $x = 0$

$-2A + 2B + C = 0$

$-2A + 2/3 + \frac{4}{9} = 0$

$-2A = -\left(\frac{2}{3} + \frac{4}{9}\right)$

$= -\frac{6+4}{9} = \frac{-10}{9}$

$-\cancel{2}A = \frac{-\cancel{10}5}{9}$

$\boxed{A = 5/9}$

Using in ①

$$y = \frac{5}{9}\frac{1}{x-1} + \frac{1}{3}\frac{1}{(x-1)^2} + \frac{4}{9}\frac{1}{x+2} \rightarrow ②$$

If $y = \dfrac{1}{ax+b}$ then $y_n = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

Using in ②

$$y_n = \frac{5}{9}\frac{n!(-1)^n}{(x-1)^{n+1}} + \frac{1}{3}\frac{(n+1)!(-1)^n}{(x-1)^{n+2}} + \frac{4}{9}\frac{(-1)^n (n!)}{(x+2)^{n+1}}$$

$$y_n = (-1)^n n! \left[\frac{5}{9(x-1)^{n+1}} + \frac{n+1}{3(x-1)^{n+2}} + \frac{4}{9(x+2)^{n+1}} \right]$$

③ Find y_n when $y = \dfrac{1}{x^2+a^2}$

Sol $y = \dfrac{1}{x^2+a^2}$

$$\frac{1}{a^2-b^2} = \frac{1}{(a-b)(a+b)}$$

$$= \frac{1}{[x^2-(ai)^2]}$$

$$= \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right] \longrightarrow ①$$

If $y = \dfrac{1}{ax+b}$ then $y_n = \dfrac{(-1)^n n! (a)^n}{(ax+b)^{n+1}}$

Using in ①

$$y_n = \frac{1}{2ai}\left[\frac{(-1)^n n! (1)^n}{(x-ai)^{n+1}} - \frac{(-1)^n n! (1)^n}{(x+ai)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right]$$

Find the nth differential coefficient of $\frac{x^2}{(x+1)^2(x+2)}$

Sol

$$y = \frac{x^2}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} \rightarrow (1)$$

$$= \frac{A(x+1)(x+2) + B(x+2) + C(x+1)^2}{(x+1)^2(x+2)}$$

$$A(x+1)(x+2) + B(x+2) + C(x+1)^2 = x^2$$

Put $x=-1$

$B(-1+2) = 1$

$B(1) = 1$

$\boxed{B = 1}$

Put $x = -2$

$C(-2+1)^2 = 4$

$C(-1)^2 = 4$

$\boxed{C = 4}$

Put $x = 0$

$~~2A~~$ $2A + 2B + 2C = 0$

$2A + 2 + 8 = 0$

$2A + 10 = 0$

$2A = -10$

$\boxed{A = -5}$

Using in (1)

$$y = \frac{-5}{x+1} + \frac{1}{(x+1)^2} + \frac{4}{x+2} \rightarrow (2)$$

If $y = \frac{1}{ax+b}$ then $y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

Using in (2)

$$y_n = \frac{-5\, n!\,(-1)^n}{(x+1)^{n+1}} + \frac{(n+1)!\,(-1)^n}{(x+1)^{n+2}} + 4\,\frac{(-1)^n n!}{(x+2)^{n+1}}$$

If $y = \sin(ax+b)$ then find y_n

Sol Proof $y = \sin(ax+b)$

Diff w.r.t x.

$y_1 = \cancel{\sin}\cos(ax+b) \cdot a$

$= a\cos(ax+b)$ $\therefore \sin(\pi/2+\theta) = \cos\theta$

$= a\sin\left(\frac{\pi}{2} + ax+b\right)$

$y_2 = a^2\cos\left(\frac{\pi}{2}+ax+b\right)$

$= a^2\sin\left(\frac{\pi}{2} + \frac{\pi}{2} + ax+b\right)$

$= a^2\sin\left(2\cdot\frac{\pi}{2} + ax+b\right)$

$y_3 = a^3\sin(3\pi/2 + ax+b)$

$\therefore y_n = a^n\sin\left(\frac{n\pi}{2} + ax+b\right)$

$\|^{ly}$ If $y = \cos(ax+b)$ then $y_n = a^n\cos\left(\frac{n\pi}{2} + ax+b\right)$

Result Put $a=1$, and $b=0$

$D^n(\sin x) = \sin\left(\frac{n\pi}{2} + x\right)$

$D^n(\cos x) = \cos\left(\frac{n\pi}{2} + x\right)$

Problem Find the n^{th} differential coefficient of

$\cos x \cdot \cos 2x \cos 3x$

Sol

$\cos x \cos 2x \cos 3x$ $\quad (3x+x) \quad (3x-x)$

$= \frac{1}{2}\cos 2x[\cos 4x + \cos 2x]$

$= \frac{1}{2}\cos 2x\cos 4x + \frac{1}{2}\cos^2 2x$

$= \frac{1}{2}\left[\frac{1}{2}(\cos 6x + \cos 2x)\right] + \frac{1}{2}\cos^2 2x$

$= \frac{1}{4}[\cos 6x + \cos 2x] + \frac{1}{2}\left[\frac{1+\cos 2(2x)}{2}\right]$

$\cos(A+B)$
$= \cos A\cos B - \sin A\sin B$

$\cos(A-B)$
$= \cos A\cos B + \sin A\sin B$

$\cos(A+B) + \cos(A-B)$
$= 2\cos A\cos B$

$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$= \frac{1}{4}\left[\cos 6x + \cos 2x\right] + \frac{1}{4}\left[1 + \cos 4x\right]$$

$$= \frac{1}{4}\left[\cos 6x + \cos 2x\right] + \frac{1}{4} + \frac{1}{4}\cos 4x$$

$$= \frac{1}{4}\left[\cos 2x + \cos 4x + \cos 6x\right] + \frac{1}{4}$$

$D^n(\cos x \cos 2x \cos 3x)$ $\quad\left[\because D^n[\cos(ax+b)] = a^n \cos\left[\frac{n\pi}{2} + ax+b\right]\right]$

$$= \frac{1}{4}\left[2^n \cos\left(\frac{n\pi}{2} + 2x\right) + 4^n \cos\left(\frac{n\pi}{2} + 4x\right) + 6^n \cos\left(\frac{n\pi}{2} + 6x\right)\right]$$

P.P Find the nth differential coefficient of

$$\frac{x^4}{(x-1)(x-2)}$$ **Ans** $$\frac{16(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}}$$

Formation of Equations involving derivatives

Problem ① If $xy = ae^x + be^{-x}$ Prove that

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0.$$

Sol

$$xy = ae^x + be^{-x} \rightarrow ①$$

Diff w.r.t x

$$y + x\frac{dy}{dx} = ae^x - be^{-x}$$

Diff w.r.t x

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx}(1) = ae^x + be^{-x}$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy \quad \text{using ①}$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$$

(2) Prove that if $y = \sin(m \sin^{-1} x)$

$(1-x^2)y_2 - xy_1 + m^2 y = 0.$

Sol ~~If~~ Let $y = \sin(m \sin^{-1} x)$

$$\sin^{-1} y = m \sin^{-1} x$$

Diff w.r.t x on both Sides

$$\frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = m \frac{1}{\sqrt{1-x^2}}$$

Squaring on both Sides

$$\frac{1}{(1-y^2)}\left(\frac{dy}{dx}\right)^2 = \frac{m^2}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$$

Diff w.r.t x

$$(1-x^2)\, 2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2(-2x) = m^2\left(-2y\frac{dy}{dx}\right)$$

$\div 2\frac{dy}{dx}$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2 y$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$$

If $x = \sin\theta$, $y = \cos p\theta$ Prove that

$(1-x^2)y_2 - xy_1 + p^2 y = 0$

Sol $x = \sin\theta$ $\qquad y = \cos p\theta$

$$\frac{dx}{d\theta} = \cos\theta \qquad \frac{dy}{d\theta} = -p\sin p\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-p\sin p\theta}{\cos\theta}$$

$$= \frac{-p\sqrt{1-\cos^2 p\theta}}{\sqrt{1-\sin^2\theta}}$$

$$\frac{dy}{dx} = -p \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Squaring on both sides

$$\left(\frac{dy}{dx}\right)^2 = p^2 \frac{(1-y^2)}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2)$$

i.e, $(1-x^2)(y_1)^2 = p^2(1-y^2)$

Diff w.r.t x

$$(1-x^2)\,2y_1y_2 + (y_1)^2(-2x) = p^2(-2yy_1)$$

$\div 2y_1$ $\quad (1-x^2)y_2 - xy_1 = -p^2y$

$$\therefore (1-x^2)y_2 - xy_1 + p^2y = 0$$

$$\frac{dy}{dx} = -p\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Squaring on both sides

$$\left(\frac{dy}{dx}\right)^2 = p^2\frac{(1-y^2)}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2)$$

i.e., $(1-x^2)(y_1)^2 = p^2(1-y^2)$

Diff w.r.t x

$$(1-x^2)\,2y_1y_2 + (y_1)^2(-2x) = p^2(-2yy_1)$$

$\div 2y_1$ $\quad (1-x^2)y_2 - xy_1 = -p^2y$

$$\therefore (1-x^2)y_2 - xy_1 + p^2y = 0$$

If $y = e^{-x}\cos x$ Prove that $\frac{d^4y}{dx^4} + 4y = 0$.

Sol Let $y = e^{-x}\cos x$

Diff w.r.t x.

$$\frac{dy}{dx} = e^{-x}(-\sin x) + \cos x(e^{-x})(-1)$$

$$= -e^{-x}\sin x - e^{-x}\cos x$$

$$\frac{d^2y}{dx^2} = -e^{-x}\cos x + \sin x(-e^{-x})(-1) \cancel{+} e^{-x}(-\sin x) - \cos x(e^{-x})(-1)$$

$$= -\cancel{e^{-x}\cos x} + e^{-x}\sin x + e^{-x}\sin x + \cancel{e^{-x}\cos x}$$

$$= 2e^{-x}\sin x$$

$$\frac{d^3y}{dx^3} = 2e^{-x}(\cos x) + 2\sin x(e^{-x})(-1)$$

$$= 2e^{-x}\cos x - 2e^{-x}\sin x$$

$$\frac{d^4y}{dx^4} = 2e^{-x}(-\sin x) + 2\cos x(e^{-x})(-1) - 2e^{-x}(\cos x) - 2\sin x(e^{-x})(-1)$$

$$= \cancel{-2e^{-x}\sin x} - 2e^{-x}\cos x - 2e^{-x}\cos x + \cancel{2e^{-x}\sin x}$$

$$= -4e^{-x}\cos x.$$

$= -4y$

$$\frac{d^2y}{dx^2} + 4y = 0$$

If $y = (\sin^{-1}x)^2$, Show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$

Sol $y = (\sin^{-1}x)^2 \rightarrow ①$

Diff w.r.t x

$$\frac{dy}{dx} = 2\sin^{-1}x \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1}x$$

Squaring on both Sides

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1}x)^2$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y \quad \text{from ①}$$

Diff w.r.t x

$$(1-x^2)\,2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = 4\frac{dy}{dx}$$

$\div\ 2\frac{dy}{dx}$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

If $y = (\tan^{-1}x)^2$ Prove that $(x^2+1)y_2 + 2x(x^2+1)y_1 - 2 = 0$

Sol $y = (\tan^{-1}x)^2 \rightarrow ①$

Diff w.r.t x

$$y_1 = 2\tan^{-1}x\left(\frac{1}{1+x^2}\right)$$

$$(1+x^2)y_1 = 2\tan^{-1}x$$

Squaring on both Sides

$$(1+x^2)^2 y_1^2 = 4(\tan^{-1}x)^2$$

$$(1+x^2)^2 y_1^2 = 4y \qquad \text{from } ①$$

Diff w.r.t x

$$(1+x^2)^2 2y_1y_2 + y_1^2 (2)(1+x^2)(2x) = 4y_1$$

$\div 2y_1$

$$(1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$$

$$(x^2+1)^2 y_2 + 2(x^2+1)y_1 - 2 = 0$$

If $y = a\cos(\log x) + b\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Sol $\quad y = a\cos(\log x) + b\sin(\log x) \rightarrow ①$

Diff w.r.t x.

$$\frac{dy}{dx} = -a\sin(\log x)\left(\frac{1}{x}\right) + b\cos(\log x)\frac{1}{x}.$$

$$x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Again Diff w.r.t x.

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}(1) = -a\cos(\log x)\frac{1}{x} + b(-\sin(\log x))\frac{1}{x}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -a\cos(\log x) - b\sin(\log x)$$

$$= -[a\cos(\log x) + b\sin(\log x)]$$

$$= -y \qquad \text{from } ①$$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Leibnitz formula for the nth derivative of a Product

Statement: If u and v are functions of x

Then $D^n(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots$

$+ {}^nC_{r-1} u_{n-r+1} v_{r-1} + {}^nC_r u_{n-r} v_r + \dots + u v_n$

Proof: The theorem prove by induction method

If $n = 1$

$D^n(uv) = D(uv)$

$= vDu + uDv$

If $n = 2$

$D^n(uv) = D^2(uv)$

$= D[D(uv)]$

$= D[(vDu) + (uDv)]$

$= D(vDu) + D(uDv)$

$= vD^2u + DuDv + DuDv + uD^2v$

$= vD^2u + 2DuDv + uD^2v$

$|||^{ly}$ $D^3(uv) = vD^3u + 3D^2u \cdot Dv + 3DuD^2v + uD^3v$

Assume the theorem is true for n

ie, $D^n(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_r u_{n-r} v_r$
$+ u v_n$

Diff w.r.t x

$D^{n+1}(uv) = (u_{n+1} v + u_n v) + {}^nC_1 (u_n v_1 + u_{n-1} v_2)$

$+ {}^nC_2 (u_{n-1} v_2 + u_{n-2} v_3) + \dots +$

$+ {}^nC_{r-1} (u_{n-r+2} v_{r-1} + u_{n-r+1} v_r)$

$+ {}^nC_r (u_{n-r+1} v_r + u_{n-r} v_{r+1}) + \dots +$

$+ (u_1 v_n + u v_{n+1})$

$$= u_{n+1}v + (1+{}^nc_1)u_n v_1 + ({}^nc_1 + {}^nc_2)u_{n-1}v_2 + \dots + ({}^nc_{r-1} + {}^nc_r)u_{n-r+1}v_r + \dots + uv_{n+1} \rightarrow (1)$$

Now $({}^nc_{r-1} + {}^nc_r) = (n+1)c_r$

$$1 + {}^nc_1 = (n+1)c_1$$
$${}^nc_1 + {}^nc_2 = (n+1)c_2$$
$${}^nc_2 + {}^nc_3 = (n+1)c_3$$

Using in (1), we have

$$D^{n+1}(uv) = u_{n+1}v + (n+1)c_1 u_n v_1 + \dots + (n+1)c_r u_{n-r+1}v_r + \dots + uv_{n+1}$$

Problem (1) If $y = \sin(m\sin^{-1}x)$ Prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Sol If $y = \sin(m\sin^{-1}x)$

we proved $(1-x^2)y_2 - xy_1 + m^2y = 0 \rightarrow (1)$

Using Leibnitz's theorem for nth derivative

$$(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n - xy_{n+1} - {}^nc_1(1)y_n + m^2y_n = 0$$

$$(1-x^2)y_{n+2} - n2xy_{n+1} - 2\frac{n(n-1)}{2!}y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

(2) If $y = xe^x$ find y_n

Sol $y = xe^x$

using Leibnitz's theorem

$$y_n = xe^x + n_{c_1}(1)e^x$$

$$= xe^x + ne^x$$

$$= (n+x)e^x$$

(3) If $y = x^2e^{3x}$ find y_n

Sol $y = x^2e^{3x}$

using Leibnitz's theorem

$$y_n = x^2e^{3x}.3^n + n_{c_1}(2x)e^{3x}.3^{n-1} + n_{c_2}(2)e^{3x}.3^{n-2}$$

$$= x^2e^{3x}.3^n + n2x3^{n-1}e^{3x} + \frac{n(n-1)}{2!}2e^{3x}3^{n-2}$$

$$= x^2e^{3x}.3^n + 3^{n-1}2nxe^{3x} + n^2e^{3x}\cancel{e^{3x}}_{3^{n-2}} - ne^{3x}3^{n-2}$$

$$= e^{3x}[3^nx^2 + 3^{n-1}2nx + 3^{n-2}n^2 - 3^{n-2}n]$$

(4) If $y = x \sin x$ find y_n

Sol $y = x \sin x$

Using Leibnitz's theorem

$$y = x\sin\left(\frac{n\pi}{2}+x\right) + n_{c_1}(1)\sin\left(\frac{(n-1)\pi}{2}+x\right)$$

$$= x\sin\left(\frac{n\pi}{2}+n\right) + n\sin\left[\frac{(n-1)\pi}{2}+x\right]$$

If $y = \sin^{-1}x$ Prove $(1-x^2)y_2 - xy_1 = 0$

and $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$.

Sol If $y = \sin^{-1}x$

Diff w.r.t x

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = 1$$

~~Di~~ Squaring both Sides

$$(1-x^2)\, y_1^2 = 1$$

Diff w.r.t x

$$(1-x^2)\, 2y_1y_2 + y_1^2(-2x) = 0.$$

$\div 2y_1$ $$(1-x^2)y_2 - xy_1 = 0$$

Using Leibnitz's theorem.

$$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n$$

$$- xy_{n+1} - \cancel{(-1)}\, {}^nC_1(+1)y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - \cancel{2}\,\frac{n(n-1)}{2!}y_n$$

$$- xy_{n+1} - ny_n = 0$$

$$(1-x^2)y_{n+1} - 2nxy_{n+1} - n^2y_n + \cancel{ny_n} - \overset{xy_{n+1}}{\cancel{xy_{n+1}}}\, \cancel{-ny_n} = 0$$

$$(1-x^2)y_{n+1} - (2n+1)xy_{n+1} - n^2y_n = 0$$

If $y = e^{a\sin^{-1}x}$ Prove that $(1-x^2)y_2 - xy_1 - a^2y = 0$

Hence show that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Sol If $y = e^{a\sin^{-1}x} \longrightarrow (1)$

Diff w.r.t x.

$$y_1 = e^{a\sin^{-1}x}\frac{a}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = a e^{a\sin^{-1}x}$$

$$\sqrt{1-x^2}\,y_1 = ay \qquad \text{from (1)}$$

Squaring on both sides

$$(1-x^2)y_1^2 = a^2y^2$$

Diff w.r.t x

$$(1-x^2)2y_1y_2 + y_1^2(-2x) = a^2 2yy_1$$

$$\div y_1 \quad (1-x^2)y_2 - xy_1 = a^2y$$

$$(1-x^2)y_2 - xy_1 - a^2y = 0$$

Using Leibnitz's theorem

$$(1-x^2)y_{n+2} + nc_1(-2x)y_{n+1} + nc_2(-2)y_n - xy_{n+1} - nc_1(1)y_n - a^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{\cancel{2}!}\cancel{2}y_n - xy_{n+1} - ny_n - a^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + \cancel{ny_n} - xy_{n+1} - \cancel{ny_n} - a^2y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

P.P ① If $y = a\cos(\log x) + b\sin(\log x)$ show that

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$$

② If $y = (x + \sqrt{1+x^2})^m$ Prove that

$$(1+x^2) y_{n+2} + (2n+1)x y_{n+1} + (n^2 - m^2) y_n = 0$$

If $y^{1/m} + y^{-1/m} = 2x$ Prove that

$$(x^2-1) y_{n+2} + (2n+1)x y_{n+1} + (n^2 - m^2) y_n = 0$$

Sol

P.P ① If $y = a\cos(\log x) + b\sin(\log x)$ show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

② If $y = (x+\sqrt{1+x^2})^m$ Prove that

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

If $y^{1/m} + y^{-1/m} = 2x$ Prove that

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Sol

$$y^{1/m} + y^{-1/m} = 2x \rightarrow ①$$

Let $A = y^{1/m}$

Then $\frac{1}{A} = y^{-1/m}$

using in ①, we have

$$A + \frac{1}{A} = 2x$$

$$A^2 + 1 = 2xA$$

$A^2 - 2xA + 1 = 0$. which is quadratic in A

This is of the form $ax^2+bx+c=0$.

$\therefore x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\therefore A = \frac{2x \pm \sqrt{4x^2-4(1)(1)}}{2}$$

$$A = \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$A = x \pm \sqrt{x^2-1}$$

$$y^{1/m} = x \pm \sqrt{x^2-1}$$

$$y = (x \pm \sqrt{x^2-1})^m \rightarrow ②$$

Diff w.r.t x

$$y_1 = m(x \pm \sqrt{x^2-1})^{m-1} \cdot \left(1 \pm \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}\right)$$

$$= m(x \pm \sqrt{x^2-1})^{m-1} \frac{(\sqrt{x^2-1} \pm x)}{\sqrt{x^2-1}}$$

$$y_1 = \frac{m(x \pm \sqrt{x^2-1})^m}{\sqrt{x^2-1}}$$

$$\sqrt{x^2-1}\, y_1 = m(x \pm \sqrt{y^2-1})$$

$$\sqrt{x^2-1}\, y_1 = \cancel{x} \pm y \qquad \text{from } (2)$$

Squaring on both sides

$$(x^2-1)\, y_1^2 = m^2 y^2$$

Diff w.r.t x

$$(x^2-1)\, 2y_1 y_2 + y_1^2 (2x) = m^2 2 y y_1$$

$\div 2y_1$ $\cancel{(x^2-1)y_2 + xy_1 - m^2y = 0}$ $(x^2-1)y_2 + xy_1 - m^2 y = 0$

Using Leibnitz's theorem

$$(x^2-1)\, y_{n+2} + n_{C_1}(2x)y_{n+1} + n_{C_2}(2)\, y_n + x y_{n+1} + n_{C_1}(1)\, y_n - m^2 y_n = 0$$

$\cancel{ny_n = 0}$

$$(x^2-1)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2!}\, 2\, y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$

$$(x^2-1)\, y_{n+2} + 2nx\, y_{n+1} + n^2 y_n - \cancel{n y_n} + x y_{n+1} + \cancel{n y_n} - m^2 y_n = 0$$

$$(x^2-1)y_{n+2} + (2n+1)\, x y_{n+1} + (n^2 - m^2)\, y_n = 0$$

P.P If $y = \dfrac{\sinh^{-1} x}{\sqrt{1+x^2}}$ Prove that

$$(1+x^2)\, y_{n+2} + (2n+3)\, x y_{n+1} + (n+1)^2 y_n = 0.$$

ENVELOPES, CURVATURE OF PLANE CURVES

ENVELOPES:

The equation $f(x,y,t)=0$ determines a curve corresponding to each particular value of t. The totality of all such curves by gaining different values of t, the totality is said to be a family of curves and the variable t which is different for different curves is said to be the parameter for the family.

Consider the equation $x\cos\theta + y\sin\theta = a$, where a is constant. For different values of θ the equation represents a family of straight lines touching the circle $x^2 \; y^2 = a^2$. Here θ is the parameter of the family of straight lines

$x \cos\theta$. ''

Similarly ents a family of straight lines with the rameter m touching the parabola $y^2 = 4ax$.

Similarly $(x-a)^2 + y^2 = r^2$ where r is a constant, is a family with the parameter a touching the

We have seen that in the three illustrations the family of curves touches a curve, in the first case a circle, in the second case a parabola and in the third case a pair of lines. The curve E which is touched by a family of curves C is called the envelope of the family of curves C.

Ex - 1

Find the envelopes of the family of a straight lines $y + tx = 2at + at^3$ the parameter being t.

Solution:

Given that $y + tx = 2at + at^3$ — (1)

Differentiate w.r.t to 't'

$$0 + x \cdot 1 = 2a \cdot 1 + a \cdot 3t^2 \cdot 1$$

$$x = 2a + 3at^2$$

$$3at^2 = x - 2a$$

$$t^2 = \frac{x - 2a}{3a} \quad \text{— (2)}$$

(1) $\Rightarrow$ $y = 2at + at^3 - tx$

$$y = t(2a + at^2 - x)$$

$$y = t\left(2a + \cancel{a}\left(\frac{x-2a}{3\cancel{a}}\right) - x\right)$$

$$y = t/3\,[6a + x - 2a - 3x]$$

$$y = t/3\,[4a - 2x]$$

$$y = 2/3\,t\,[2a - x]$$

$$\Rightarrow 3y = 2t\,[2a - x]$$

Squaring on both side

$$9y^2 = 4t^2(2a - x)^2$$

$$9y^2 = 4\left(\frac{x-2a}{3a}\right)(2a-x)^2$$

$$27ay^2 = 4(x-2a)(2a-x)^2$$

$$= -4(2a-x)(2a-x)^2$$

$27ay^2 = -4(2a-x)^3$ which is the required equation of the envelope.

<u>Note</u>:

$At^2 \pm Bt + C = 0$ be the quadratic equation of t. Then the envelope of the equation is $B^2 = 4ac$.

Ex - 2

Find the envelope of the family of circles $(x-a)^2 + y^2 = 2a$ where a is the parameter

Solution

Given that $(x-a)^2 + y^2 = 2a$

$$x^2 - 2ax + a^2 + y^2 = 2a$$

$$a^2 - 2ax - 2a + x^2 + y^2 = 0$$

$$a^2 - (2x-2)a + x^2 + y^2 = 0$$

$$A = 1 \quad B = -(2x+2) \quad C = x^2 + y^2$$

The envelope $B^2 = 4AC$

$$[-(2x+2)]^2 = 4(1)(x^2+y^2)$$

$$\cancel{4}(x+1)^2 = \cancel{4}(x^2+y^2)$$

$$(x+1)^2 = x^2 + y^2$$

$$\cancel{x^2} + 2x + 1 = \cancel{x^2} + y^2$$

$$\boxed{y^2 = 2x + 1}$$

Ex-3

Find the envelope of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{k^2-a^2} = 1$ where a is the parameter.

Solution:

Given that $\frac{x^2}{a^2} + \frac{y^2}{k^2-a^2} = 1$

$$\frac{x^2(k^2-a^2) + a^2y^2}{a^2(k^2-a^2)} = 1$$

$$x^2k^2 - a^2x^2 + a^2y^2 = a^2k^2 - a^4$$

$$(a^2)^2 + (y^2-x^2-k^2)a^2 + x^2k^2 = 0$$

$$A = 1 \quad B = (y^2-x^2-y^2) \quad C = x^2k^2$$

$$B^2 = 4AC$$

$$(y^2-x^2-k^2)^2 = 4 \cdot 1\, x^2k^2$$

$$y^2-x^2-k^2 = \pm 2xk$$

$$y^2 = x^2 \pm 2xk + k^2$$

$$y^2 = (x \pm k)^2$$

$$\pm y = x \pm k$$

$$\boxed{x \pm y = \pm k}$$

Ex - 4

Find the envelope of the circle drawn on the radium vectors of the ellips $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as diameter.

Soln

The coordinator of any point P on the ellipses are $(a\cos\theta, b\sin\theta)$

The equation of the circle on CP as diameter is $x^2+y^2=0$

$$x(x-a\cos\theta)+y(y-b\sin\theta)=0$$

$$x^2-xa\cos\theta+y^2-yb\sin\theta=0$$

$$x^2+y^2-a\cos\theta\, x-b\sin\theta\, y=0 \quad \text{—} (1)$$

This is the required circle equation to find the envelope of this circle.

There θ is a parameter.

(1) is

Partial differential w.r.t θ we got

$$xa\sin\theta+by\cos\theta=0$$

$$xa\sin\theta=by\cos\theta$$

$$\frac{\sin\theta}{by}=\frac{\cos\theta}{ax} \quad \text{—} (2)$$

$\Rightarrow$ $x=a\cos\theta$ $\quad$ $y=b\sin\theta$

$\cos\theta=x/a$ $\quad$ $\sin\theta=y/b$

$$\frac{y/b}{by} = \frac{x/a}{ax}$$

$$\frac{1}{b^2} = \frac{1}{a^2} \qquad a = b$$

Now $\sqrt{a^2x^2 + b^2y^2} = \sqrt{a^2a^2\cos^2\theta + b^2b^2\sin^2\theta}$

$$= \sqrt{a^4\cos^2\theta + a^4\sin^2\theta}$$

$$= \sqrt{a^4(\cos^2\theta + \sin^2\theta)}$$

$$= \sqrt{a^4}$$

$$= a^2$$

$$\frac{\sin\theta}{by} = \frac{\cos\theta}{ax} = \frac{1}{\sqrt{a^2x^2 + b^2y^2}}$$

$$\frac{\sin\theta}{by} = \frac{1}{\sqrt{a^2x^2 + b^2y^2}}, \quad \frac{\cos\theta}{ax} = \frac{1}{\sqrt{a^2x^2 + b^2y^2}}$$

$$\sin\theta = \frac{by}{\sqrt{a^2x^2+b^2y^2}}, \quad \cos\theta = \frac{ax}{\sqrt{a^2x^2+b^2y^2}}$$

$$x^2+y^2 - ax\left(\frac{ax}{\sqrt{a^2x^2+b^2y^2}}\right) - by\left(\frac{by}{\sqrt{a^2x^2+b^2y^2}}\right) = 0$$

$$x^2+y^2 - \left(\frac{a^2x^2+b^2y^2}{\sqrt{a^2x^2+b^2y^2}}\right) = 0$$

$$x^2+y^2 - \sqrt{a^2x^2+b^2y^2} = 0$$

$$x^2+y^2 = \sqrt{a^2x^2+b^2y^2}$$

$$(x^2+y^2)^2 = a^2x^2+b^2y^2$$

$$x^4+2x^2y^2+y^4-a^2x^2-b^2y^2=0$$

Ex-5:

Find the envelopes of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ where the parameter are related by the equation $a^2+b^2=c^2$ where c is a constant.

Solution:

Given that $\frac{x}{a}+\frac{y}{b}=1$ — (1) and $a^2+b^2=c^2$ — (2)

Let (a) and (b) be a function of t differentiating

$\frac{x}{a}+\frac{y}{b}=1$ w.r.to t

$$\frac{-x}{a^2}\frac{da}{dt} - \frac{y}{b^2}\frac{db}{dt} = 0$$

$$\frac{x}{a^2}\frac{da}{dt} = \frac{-y}{b^2}\frac{db}{dt} \quad \text{— (3)}$$

Differentiate $a^2+b^2=c^2$ w.r.to t

$$2a\frac{da}{dt}+2b\frac{db}{dt}=0$$

$$\frac{da}{dt} = \frac{-b}{a}\frac{db}{dt} \quad \text{— (4)}$$

Substitute (4) in (3)

$$(3) \Rightarrow \frac{x}{a^2}\left(\frac{-b}{a}\cancel{\frac{db}{dt}}\right) = \frac{-y}{b^2}\cancel{\frac{db}{dt}}$$

$$\Rightarrow \frac{x}{a^3} = \frac{y}{b^3}$$

$$\Rightarrow \frac{x/a}{a^2} = \frac{y/b}{b^2} = \frac{x/a + y/b}{a^2+b^2} = \frac{1}{c^2}$$

$$\Rightarrow \frac{x}{a^3} = \frac{1}{c^2}, \quad a^3 = xc^2 \quad a = (xc^2)^{1/3}$$

$$\Rightarrow \frac{y}{b^3} = \frac{1}{c^2}, \quad b^3 = yc^2; \quad b = (yc^2)^{1/3}$$

$$\therefore (2) \Rightarrow a^2 + b^2 = c^2$$

$$(xc^2)^{2/3} + (yc^2)^{2/3} = c^2$$

$$x^{2/3} c^{4/3} + y^{2/3} c^{4/3} = c^2$$

$$x^{2/3} + y^{2/3} = c^2 \cdot c^{-4/3}$$

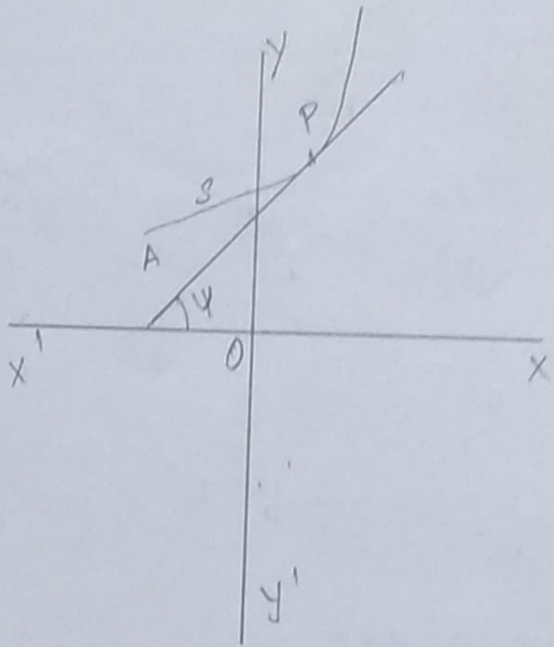
$$x^{2/3} + y^{2/3} = c^{2/3}$$

Radius of the Curvature ::

Curvature:

A curve has a definite direction at every point on it. At any particular point, the direction of the curve is the same as that of the tangent to the curve at that point. The direction usually changes from point to point and the tangent line rotates as the point moves along the curve.

Curvature



Let P be the radius of the curvature

Then the curvature = $1/\rho$

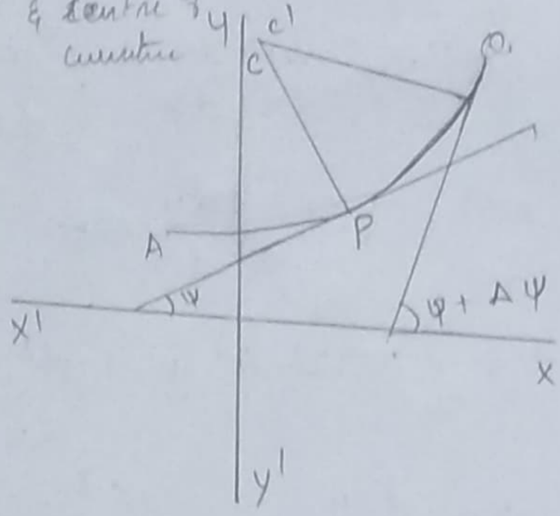
The radius of the curvature

Circle, radius and centre of curvature:

The circle whose centre is C and radius PC has the same tangent and the same curvature as the curve has at P.

This circle is called the circle of curvature at P. So it can be defined as that circle which touches the given curve at the point, has a radius equal to the radius of curvature at the point and lies on the same side of the tangent as the curve. Its radius is PC, the radius of curvature and its centre is C, the centre of curvature at the point P. The radius of curvature is often denoted by ρ and so the curvature is $1/\rho$

circle, radius & centre of curvature



Let ρ be the radius of the curvature

Then the curvature $= 1/\rho$

The radius of the curvature

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{d^2y/dx^2}$$

Ex-1:

What is the radius of the curvature of the curve $x^4 + y^4 = 2$ at the point $(1, 1)$?

Soln.

Given that $x^4 + y^4 = 2$ w.r.t x

$$4x^3 + 4y^3 \frac{dy}{dx} = 0 \quad \text{—— (1)}$$

Put $x = 1 \quad y = 1$

$$4 + 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -4/4 = -1$$

Again Differentiate (1) w.r.t x

$$12x^2 + 4\left[3y^2 \frac{dy}{dx}\frac{dy}{dx} + y^3 \frac{d^2y}{dx^2}\right] = 0$$

Put $x = 1 \quad y = 1 \quad \frac{dy}{dx} = -1$

$$12 + 4\left\{3(1)(-1)^2 + 1\frac{d^2y}{dx^2}\right\} = 0$$

$$3 + \frac{d^2y}{dx^2} = \frac{-\cancel{12}\,3}{\cancel{4}}$$

$$\frac{d^2y}{dx^2} = -3 - 3 = -6$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$= \frac{[1+1]^{3/2}}{-6}$$

$$\rho = \frac{2^{3/2}}{-6} = \frac{\cancel{2}\sqrt{2}}{-\cancel{6}\,3}$$

$$\rho = \frac{-\sqrt{2}}{3}$$

The length of the normal $\Big\} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Ex - 2

Show that the radius of curvature at any point of the catenary $y = c\cosh\frac{x}{c}$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x

Solution:

Given that $y = c\cosh\frac{x}{c}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \cancel{c}\sinh\frac{x}{c}\left(\frac{1}{\cancel{c}}\right) = \sinh\frac{x}{c}$$

Again w.r.t x

$$\frac{d^2y}{dx^2} = \cosh\frac{x}{c}\cdot\frac{1}{c} = \frac{1}{c}\cosh\frac{x}{c}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left\{1+\sinh^2\frac{x}{c}\right\}^{3/2}}{1/c\,\cosh\frac{x}{c}}$$

$$\rho = \frac{\left(\cosh^2\frac{x}{c}\right)^{3/2}}{1/c\,\cosh x/c}$$

$$\rho = \frac{\cosh^{\cancel{3}2}\frac{x}{c}}{\cancel{\frac{1}{c}\cosh\frac{x}{c}}}$$

$$\rho = c\cosh^2 x/c$$

$$\rho = \cancel{c}\,\frac{y^2}{c^{\cancel{2}}} = y^2/c \quad (\text{by})\ (1)$$

The length of the normal $= y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$

$$= y\left(1+\left(\sinh\frac{x}{c}\right)^2\right)^{1/2}$$

$$= y\left(1+\sinh^2\frac{x}{c}\right)^{1/2}$$

$$= y\left(\cosh^{\cancel{2}}\frac{x}{c}\right)^{\cancel{1/2}}$$

$$= y\left(\cosh\frac{x}{c}\right)$$

$$= y\left(\frac{y}{c}\right) = \frac{y^2}{c}$$

<u>Ex-3</u>

If a curve is defined by the parametric equation $x = f(\theta)$ and $y = \phi(\theta)$. Prove that the curvature is

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{\left((x')^2 + y'^2\right)^{3/2}} \quad \text{where}$$

dashes denote differentiation with respect to θ

Solution:

$$\frac{dx}{d\theta} = x' \qquad \frac{dy}{d\theta} = y'$$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dx} \cdot \dfrac{d\theta}{d\theta} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{y'}{x'}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\frac{1}{dx/d\theta}$$

$$= d/d\theta\left(\frac{y'}{x'}\right)\frac{1}{x'}$$

$$= \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\frac{1}{x'}$$

$$= \frac{x'y'' - y'x''}{(x')^2} \times \frac{1}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{(x')^3}$$

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{\left(1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

$$= \frac{\dfrac{x'y'' - y'x''}{(x')^3}}{\left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2}}$$

$$= \frac{x'y'' - y'x''}{(x')^3 \left(\frac{x'^2 + y'^2}{x'^2}\right)^{3/2}}$$

$$= \frac{x'y'' - y'x''}{(x')^3 \frac{(x'^2 + y'^2)^{3/2}}{(x')^3}}$$

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

$$\therefore \rho = \frac{[(x')^2 + (y')^2]^{3/2}}{x'y'' - y'x''}$$

Ex-4:

Prove that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ is $4a \cos\frac{\theta}{2}$

Solution:

Given that

$$x = a(\theta + \sin\theta)$$

$$y = a(1 - \cos\theta$$

Differentiate w.r.t θ

$x' = a(1 + \cos\theta)$ $x'' = a(-\sin\theta) = -a\sin\theta$

$y' = a(0 - (-\sin\theta)) = a\sin\theta$

$y'' = a\cos\theta$

$$\rho = \frac{\left[(x')^2+(y')^2\right]^{3/2}}{x'y''-y'x''}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a(1+\cos\theta)a\cos\theta - a\sin\theta(-a\sin\theta)}$$

$$= \frac{(a^{\cancel{2}})^{3/\cancel{2}}\left[1+2\cos\theta+\cos^2\theta+\sin^2\theta\right]^{3/2}}{a^2\left[\cos\theta+\cos^2\theta+\sin^2\theta\right]}$$

$$= \frac{a^{\cancel{3}}\left[1+2\cos\theta+1\right]^{3/2}}{\cancel{a^2}\left[\cos\theta+1\right]}$$

$$= \frac{a\left[2(1+\cos\theta)\right]^{3/2}}{\left[1+\cos\theta\right]}$$

$$= 2^{3/2}a(1+\cos\theta)^{3/2-1}$$

$$= 2^{3/2}a(1+\cos\theta)^{1/2}$$

$$= 2^{3/2}a\left[2\cos^2\theta/2\right]^{1/2}$$

$$= 2^{3/2}a\,2^{1/2}\cos\theta/2$$

$$= 2^{\cancel{4}/\cancel{2}\;2}a\cos\theta/2$$

$$\rho = 4a\cos\theta/2$$

Ex-5

Find ρ at the point t of the curve

$x = a(\cos t + t\sin t)$ $y = a(\sin t - t\cos t)$

Solution:

Given that

$$x = a(\cos t + t\sin t)$$
$$y = a(\sin t - t\cos t)$$

Differentiate w.r.t t

$$x' = a[-\cancel{\sin t} + (1)\cancel{\sin t} + t\cos t)$$
$$x' = at\cos t$$
$$y' = a[\cos t - (1)\cos t - t(-\sin t)]$$
$$y' = at\sin t$$
$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{at\sin t}{at\cos t} = \tan t$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t)$$
$$= \frac{d}{dt}(\tan t)\frac{dt}{dx}$$
$$= \sec^2 t\,\frac{1}{dx/dt}$$
$$= \sec^2 t\,\frac{1}{at\cos t}$$
$$\frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left(1+\tan^2 t\right)^{3/2}}{\frac{\sec^3 t}{at}}$$

$$\rho = at\left(\frac{(\sec^{\cancel{2}} t)^{3/\cancel{2}}}{\sec^3 t}\right)$$

$$\rho = at\,\frac{\cancel{\sec t^3}}{\cancel{\sec^3 t}}$$

$$\boxed{\rho = at}$$

Centre of the curvature :-

Let the centre of curvature of the curve $y = f(x)$ corresponding to the point $P(x, y)$ be X and Y

$$X = x - \frac{y_1 (1 + y_1^2)}{y_2}$$

The locus of the centre for a curve is called the evoluvate of the curve.

Ex-1

Find the co-ordinates of the centre curvature of the curve $xy = 2$ at the point $(2, 1)$

Soln

Given that $xy = 2$

$$y = \frac{2}{x}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = y_1 = \frac{-2}{x^2}$$

Again differentiate w.r.t. x

$$\frac{d^2y}{dx^2} = y_2 = \frac{4}{x^3}$$

At the point $(2,1)$

$$y_1 = \frac{-2}{x^2} = \frac{-2}{2^2} = -\frac{1}{2}$$

$$y_2 = \frac{4}{x^3} = \frac{4}{2^3} = \frac{1}{2}$$

$$X = x - \frac{y_1(1+y^2)}{y_2}$$

$$X = 2 + \frac{\cancel{\tfrac{1}{2}}\left(1+\left(-\tfrac{1}{2}\right)^2\right)}{\cancel{\tfrac{1}{2}}}$$

$$X = 2 + \left(1 + \frac{1}{4}\right)$$

$$X = 2 + \left(\frac{5}{4}\right)$$

$$X = \frac{8+5}{4}$$

$$\boxed{x = \frac{13}{4}}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$Y = 1 + \frac{\left(1+\left(-\tfrac{1}{2}\right)^2\right)}{\tfrac{1}{2}}$$

$$Y = 1 + \frac{\left(1+\tfrac{1}{4}\right)}{\tfrac{1}{2}}$$

$$Y = 1 + \frac{5}{\cancel{4}\,2} + \frac{\cancel{2}}{1}$$

$$Y = \frac{7}{2}$$

The centre of the curvature is $\left(\frac{13}{4}, \frac{7}{2}\right) = \left(3\frac{1}{4}, 3\frac{1}{2}\right)$

Ex. 2

Show that in the parabola

$y^2 = 4ax$ at the point t

$$\rho = -2a(1+t^2)^{3/2}$$

$$x = 2a + 3at^2 \quad y = -2at^3$$

Deduce the equation of the evolute

Soln

The co-ordinates of the parabola

is $x = at^2 \quad y = 2at$

Differentiate w.r.t t

$$x' = 2at \quad y' = 2a$$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{1}{dx/dt}$$

$$= \frac{-1}{t^2} \cdot \frac{1}{2at}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{2at^3}$$

The radius of curvature is

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left(1+\left(\frac{1}{t}\right)^2\right)^{3/2}}{-\frac{1}{2at^3}}$$

$$\rho = -2at^3\left(1+\frac{1}{t^2}\right)^{3/2}$$

$$\rho = -2at^3\left[\frac{t^2+1}{t^2}\right]^{3/2}$$

$$\rho = -2a\cancel{t^3}\frac{(t^2+1)^{3/2}}{(t^{\cancel{2}})^{\cancel{3}/\cancel{2}}}$$

$$\rho = -2a(1+t^2)^{3/2}$$

The centre of the curvature

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$X = at^2 - \frac{1/t\left(1+\frac{1}{t^2}\right)}{\left(-\frac{1}{2at^3}\right)}$$

$$X = at^2 + 2at^{\cancel{3}\,2}\,\cancel{1/t}\left(1+\frac{1}{t^2}\right)$$

$$X = at^2 + 2a\cancel{t^2}\left(\frac{t^2+1}{\cancel{t^2}}\right)$$

$$X = at^2 + 2at^2 + 2a$$

$$X = 2a + 3at^2 \quad \text{--- (1)}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$y = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2}at^3}$$

$$y = 2at - 2at^{\cancel{3}}\left(\frac{t^2+1}{\cancel{t^2}}\right)$$

$$y = 2at\,[\cancel{1} - t^2 - \cancel{1}]$$

$$y = -2at^3 \quad \text{——} \ (2)$$

(1) $\Rightarrow$ $\frac{x-2a}{3a} = t^2$

(2) $\Rightarrow$ $y = -2a\left[\frac{x-2a}{3a}\right]^{3/2}$

Squaring on both side, we get

$$y^2 = 4a^2\left[\frac{x-2a}{3a}\right]^3$$

$$y^2 (3a)^3 = 4a^2 (x-2a)^3$$

$$27a^{\cancel{3}}y^2 = 4\cancel{a^2}(x-2a)^3$$

$$27ay^2 = 4(x-2a)^3$$

The locus of (x,y) is

$$27ay^2 = 4(x-2a)^3$$

which is a semi cubical parabola.

Ex - 3

Find the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Soln

WKT that parametric form of the ellipse is

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$x' = -a\sin\theta \qquad y' = b\cos\theta$$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{+b\cos\theta}{-a\sin\theta} = -b/a \cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{-b}{a}\cot\theta\right)\frac{1}{x'}$$

$$= \frac{-b}{a}(-\operatorname{cosec}^2\theta)\frac{1}{-a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2}\operatorname{cosec}^3\theta$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$X = a\cos\theta - \frac{\left(\frac{-b}{a}\right)\cot\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-b/a^2 \operatorname{cosec}^3\theta}$$

$$X = a\cos\theta - a\sin^{2}\theta\,\frac{\cos\theta}{\cancel{\sin\theta}}\left(\frac{a^2+b^2\cot^2\theta}{a^2}\right)$$

$$X = a\cos\theta - \frac{\sin^2\theta\cos\theta}{a}\left(a^2 + b^2\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$X = a^2\cos\theta - \frac{\cancel{\sin^2\theta}.\cos\theta}{a}\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{\cancel{\sin^2\theta}}\right]$$

$$X = \frac{a^2\cos\theta - \cos\theta\ (a^2(1-\cos^2\theta) + b^2\cos^3\theta}{a}$$

$$X = \frac{a^2\cancel{\cos\theta} - a^2\cancel{\cos}\theta + a^2\cos^3\theta - b^2\cos^3\theta}{a}$$

$$X = \frac{(a^2-b^2)}{a}(\cos^3\theta) \quad — ①$$

$$Y = \cancel{y} + \frac{(1+y_1^2)}{y^2}$$

$$Y = b\sin\theta + \frac{\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)}{-b/a^2\ \operatorname{cosec}^3\theta}$$

$$Y = b\sin\theta - \frac{a^{\cancel{2}}}{b}\sin^3\theta\left(\frac{a^2+b^2\cot^2\theta}{\cancel{a^2}}\right)$$

$$Y = b\sin\theta - \frac{\sin^3\theta}{b}\left(a^2 + b^2\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$Y = b\sin\theta - \frac{\sin^3\theta}{b}\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{\sin^2\theta}\right)$$

$$Y = \frac{b^2\sin\theta - \sin\theta\ [a^2\sin^2\theta + b^2(1-\sin^2\theta)]}{b}$$

$$y = \frac{b^2 \cancel{\sin\theta} - a^2 \sin^3\theta - b^2 \cancel{\sin\theta} + b^2 \sin^3\theta}{b}$$

$$y = -\frac{(a^2-b^2)}{b} \sin^3\theta \quad \text{--- (2)}$$

$$(1) \Rightarrow \frac{xa}{a^2-b^2} = \cos^3\theta \Rightarrow \cos\theta = \left(\frac{xa}{a^2-b^2}\right)^{1/3}$$

$$(3) \Rightarrow \frac{-yb}{(a^2-b^2)} = \sin^3\theta \Rightarrow \sin\theta = \left(\frac{-yb}{a^2-b^2}\right)^{1/3}$$

WKT $\sin^2\theta + \cos^2\theta = 1$

$$\left(\frac{-yb}{a^2-b^2}\right)^{2/3} + \left(\frac{xa}{a^2-b^2}\right)^{2/3} = 1$$

$$\left(\frac{+yb}{a^2-b^2}\right)^{2/3} + \left(\frac{xa}{a^2-b^2}\right)^{2/3} = 1$$

$$(yb)^{2/3} + (xa)^{2/3} = (a^2-b^2)^{2/3}$$

Ex-4

Show that evolute of the cycloid $x = a(\theta - \sin\theta)$ $y = a(1-\cos\theta)$ is another cycloid.

Soln:

Given that $x = a(\theta - \sin\theta)$

$y = a(1-\cos\theta)$

Differentiate w.r.t θ

$x' = a(1-\cos\theta)$

$y' = a\sin\theta$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{\cancel{2}\sin\theta/2\ \cos\theta/2}{\cancel{2}\sin^2\theta/2}$$

$$\frac{dy}{dx} = \cot\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\cot\theta/2\right)\frac{1}{x'}$$

$$= -\operatorname{cosec}^2\theta/2\ \tfrac{1}{2}\ \frac{1}{a(1-\cos\theta)}$$

$$= \frac{-\operatorname{cosec}^2\theta/2}{2a(1-\cos\theta)}$$

$$= \frac{-\operatorname{cosec}^2\theta/2}{2a(2\sin^2\theta/2)}$$

$$\frac{d^2y}{dx^2} = \frac{-\operatorname{cosec}^4\theta/2}{4a}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$X = a(\theta-\sin\theta) - \frac{\cot\theta/2\left(1+\cot^2\theta/2\right)}{\frac{-\operatorname{cosec}^4\theta/2}{4a}}$$

$$X = a(\theta-\sin\theta) + 4a\sin^4\theta/2\ \cot\theta/2\left(\operatorname{cosec}^2\theta/2\right)$$

$$X = a(\theta-\sin\theta) + 4a\sin^{\cancel{4}}\theta/2 \cdot \frac{\cos\theta/2}{\cancel{\sin\theta/2}} \cdot \frac{1}{\cancel{\sin^2\theta/2}}$$

$$X = a(\theta-\sin\theta) + 4a\sin\theta/2\ \cos\theta/2$$

$$X = a(\theta - \sin\theta) + 2a \cdot 2\sin\theta/2 \cos\theta/2$$

$$X = a(\theta - \sin\theta) + 2a\sin\theta.$$

$$X = a(\theta - \sin\theta) + 2a\sin\theta.$$

$$X = a\theta - a\sin\theta + 2a\sin\theta$$

$$X = a\theta + a\sin\theta$$

$$X = a(\theta + \sin\theta)$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$Y = y + \frac{1 + \cot^2\theta/2}{\left(-\frac{\operatorname{cosec}^4\theta/2}{4a}\right)}$$

$$Y = y - 4a\sin^4\theta/2\,(\operatorname{cosec}^2\theta/2)$$

$$Y = y - 4a\sin^{\cancel{4}2}\theta/2 \cdot \frac{1}{\cancel{\sin^2\theta/2}}$$

$$Y = a(1-\cos\theta) - 4a\sin^2\theta/2$$

$$Y = 2a\sin^2\theta/2 - 4a\sin^2\theta/2$$

$$Y = -2a\sin^2\theta/2$$

$$Y = -a(2\sin^2\theta/2).$$

$$Y = -a(2\sin^2\theta/2).$$

$$Y = -a(1-\cos\theta)$$

The locus of (X, Y) is $X = a(\theta + \sin\theta)$ $Y = -a(1-\cos\theta)$

This is also a cycloid

Radius of curvature when the curve is given in polar coordinates.

Let $r = f(\theta)$

The radius of the curvature

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)}$$

Ex - 1

Find the radius of curvature of the cardioid $r = a(1 - \cos\theta)$

Soln

Given that $r = a(1 - \cos\theta)$

Differential w.r.t θ

$$\frac{dr}{d\theta} = a(0 + \sin\theta) = a\sin\theta$$

$$\frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\therefore \rho = \frac{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2} = \left[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}$$

$$= \left[a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta\right]^{3/2}$$

$$= \left[a^2 - 2a^2\cos\theta + a^2(1)\right]^{3/2}$$

$$= \left[2a^2 - 2a^2 \cos\theta\right]^{3/2}$$

$$= \left[2a^2(1-\cos\theta)\right]^{3/2}$$

$$= \left[2a^2 \cdot 2\sin^2\theta/2\right]^{3/2}$$

$$= \left[2^2 a^2 \sin^2\theta/2\right]^{3/2}$$

$$= 2^3 a^3 \sin^3\theta/2 = 8a^3 \sin^3\theta/2$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta)\,a\cos\theta$$

$$= a^2 - 2a^2\cos\theta + a^2\cos^2\theta + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta$$

$$= a^2 - 3a^2\cos\theta + 2a^2\sin^2\theta + 2a^2\cos^2\theta$$

$$= a^2 - 3a^2\cos\theta + 2a^2(1)$$

$$= 3a^2 - 3a^2\cos\theta$$

$$= 3a^2(1-\cos\theta)$$

$$= 3a^2 \cdot 2\sin^2\theta/2$$

$$= 6a^2\sin^2\theta/2$$

$$\therefore \rho = \frac{4\cancel{8}\, a^{\cancel{3}} \sin^{\cancel{3}}\theta/2}{3\,\cancel{6}\, \cancel{a^2}\, \cancel{\sin^2\theta/2}}$$

$$\rho = \tfrac{4}{3}\, a \sin\theta/2 \quad \text{—— (1)}$$

given $r = a(1-\cos\theta)$

$$r = a\, 2\sin^2\theta/2$$

$$r/2a = \sin^2\theta/2$$

$$\sin\theta/2 = \sqrt{\frac{r}{2a}} \quad \text{—— (2)}$$

Sub ① in ②

$$\rho = \frac{\cancel{4}^{2\sqrt{2}}}{3} \cancel{a}^{\sqrt{a}} \sqrt{\frac{r}{2\cancel{a}}}$$

$$\rho = \frac{2}{3}\sqrt{2ar}$$

Ex - 2

Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ is $\frac{a^n r^{-n+1}}{n+1}$

Soln

Given that $r^n = a^n \cos n\theta$

Taken log on both side

$$n \log r = n \log a + \log \cos n\theta$$

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-\sin n\theta . n)$$

$$\frac{dr}{d\theta} = -\frac{\cancel{n}r}{\cancel{n}} . \frac{\sin \theta}{\cos n\theta}$$

$$\frac{dr}{d\theta} = -r \tan n\theta$$

$$\frac{d^2r}{d\theta^2} = -r \sec^2 n\theta . n - \tan n\theta \frac{dr}{d\theta}$$

$$\frac{d^2r}{d\theta^2} = -nr \sec^2 n\theta - \tan n\theta (-r \tan n\theta)$$

$$\frac{d^2r}{d\theta^2} = -nr \sec^2 n\theta + r \tan^2 n\theta$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2} = \left[r^2 + r^2\tan^2 n\theta\right]^{3/2}$$

$$= \left[r^2(1+\tan^2 n\theta)\right]^{3/2}$$

$$= \left(r^2\sec^2 n\theta\right)^{3/2}$$

$$= r^3 \sec^3 n\theta$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right) = r^2 + 2r^2\tan^2 n\theta - r(nr\sec^2 n\theta + r\tan^2 n\theta)$$

$$= r^2 + 2r^2 + an^2 n\theta + nr^2\sec^2 n\theta - r^2\tan^2 n\theta$$

$$= r^2 + r^2\tan^2 n\theta + nr^2\sec^2 n\theta$$

$$= r^2(1+\tan^2 n\theta) + nr^2\sec^2 n\theta$$

$$= r^2\sec^2 n\theta + nr^2\sec^2 n\theta$$

$$= (n+1)r^2\sec^2 n\theta$$

$$\therefore \rho = \frac{r^{\cancel{3}}\sec^{\cancel{3}} n\theta}{(n+1)\cancel{r^2}\cancel{\sec^2 n\theta}}$$

$$\rho = \frac{r\sec n\theta}{n+1}$$

$$\rho = \frac{r}{(n+1)\cos n\theta}$$

$$\rho = \frac{r}{(n+1)\frac{r^n}{a^n}}$$

$$\rho = \frac{a^n \cdot r^{-n+1}}{n+1} \quad \text{———} \quad (1)$$

Case(i) put $n = 2$ in (1), we get

$$P = \frac{a^2 r^{-2+1}}{2+1} = \frac{a^2 r^{-1}}{3} = \frac{a^2}{3r}$$

which is a Bernouillis lemnis cate case(ii)

Case (ii) put $n = -2$ in (1) we get

$$P = \frac{a^{-2} r^{2+1}}{-2+1} = \frac{-r^3}{a^2}$$ which is a

rectangular hyperbola.

Case(iii) put $n = 1/2$ in (1) we get

$$P = \frac{a^{1/2} r^{-1/2+1}}{1/2+1} = \frac{\sqrt{ar}}{3/2} = \frac{2}{3}\sqrt{ar}$$

which is a cordioid.

Case (iv) put $n = -1/2$ in (1) we get

$$P = \frac{a^{-1/2} r^{1/2+1}}{-1/2+1} = \frac{2r\sqrt{r}}{\sqrt{a}}$$

which is a parabola

Case (v) put $n = 1$ in (1) we get

$$P = \frac{a^1 r^{-1+1}}{1+1} = \frac{a}{2}$$

which is a circlus.

Unit - III

Integration

Formulae:

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$
2. $\int \frac{dx}{x} = \log x + c$
3. $\int e^x \, dx = e^x$
4. $\int \sin x \, dx = -\cos x$
5. $\int \cos x \, dx = \sin x$
6. $\int \sec^2 x \, dx = \tan x$
7. $\int \operatorname{cosec}^2 x \, dx = -\cot x$
8. $\int \sec x \tan x \, dx = \sec x$
9. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$
10. $\int \cosh x \, dx = \sinh x$
11. $\int \sinh x \, dx = \cosh x$
12. $\int \frac{dx}{1+x^2} = \tan^{-1} x$ or $-\cot^{-1} x$
13. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$ or $-\cos^{-1} x$
14. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x$
15. $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x$
16. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x$ or $-\operatorname{cosec}^{-1} x$

Results

1. $\int c\, f(x) \, dx = c \int f(x) \, dx$, where c is a constant
2. $\int (u \pm v) \, dx = \int u \, dx \pm \int v \, dx$, where u and v are functions of x.

Integration by parts

If u and v are functions of x, then $\int u \, dv = uv - \int v \, du$

Reduction formulae:

① $I_n = \int x^n e^{ax}\, dx$, where n is a positive integer $\hookrightarrow$ ①

Sol Put $u = x^n$ $\quad dv = e^{ax}\, dx$

$$\int dv = \int e^{ax}\, dx$$

$$v = \frac{e^{ax}}{a}$$

Using in ①

$$I_n = \int \underset{u}{x^n}\, \underset{dv}{e^{ax}\, dx}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$= x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}\, n x^{n-1} dx$$

$$= \frac{e^{ax}}{a} x^n - \frac{n}{a} \int e^{ax} x^{n-1} dx$$

$$I_n = \frac{e^{ax}}{a} x^n - \frac{n}{a} I_{n-1}$$

Note: The auxiliary integral is of the same type as the integral but with index n reduced by 1. Such a formula is called a reduction formula and by successive applications.

② Find the reduction formula for $I_n = \int x^n \cos ax\, dx$ (n is a positive integer)

Sol $I_n = \int x^n \cos ax\, dx \rightarrow$ ①

Put $u = x^n$, $dv = \cos ax\, dx$

$$\int dv = \int \cos ax\, dx$$

$$v = \frac{\sin ax}{a}$$

Using in ①

$$I_n = \int x^n \cos ax\, dx$$

$$= x^n \frac{\sin ax}{a} - \int \frac{\sin ax}{a}\, n x^{n-1} dx$$

$$= x^n \frac{\sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax\, dx$$

$\hookrightarrow$ ②

Now, from $\int x^{n-1} \sin ax \, dx$

Put $u = x^{n-1}$ $\quad dv = \sin ax \, dx$

$\int dv = \int \sin ax \, dx$

$v = -\frac{\cos ax}{a}$

$$\therefore \int x^{n-1} \sin ax \, dx = x^{n-1}\left(-\frac{\cos ax}{a}\right) - \int -\frac{\cos ax}{a}(n-1)x^{n-2} \, dx$$

$$= -\frac{x^{n-1}\cos ax}{a} + \frac{n-1}{a}\int x^{n-2} \cos ax \, dx$$

Using in ②

$$I_n = \frac{x^n \sin ax}{a} - \frac{n}{a}\left[-\frac{x^{n-1}\cos ax}{a} + \frac{n-1}{a}\int x^{n-2}\cos ax \, dx\right]$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax \, dx$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2} I_{n-2}$$

<u>P.P</u> Establish a reduction formula for $\int x^n \sin ax \, dx$

<u>Problems</u>

$\int x^2 e^{-x} dx \rightarrow$ ①

<u>Sol</u> Put $u = x^2$ $\quad dv = e^{-x} dx$

$\int dv = \int e^{-x} dx$

$v = \frac{e^{-x}}{-1} = -e^{-x}$

Using in ①

$$\int x^2 e^{-x} dx = x^2(-e^{-x}) - \int (-e^{-x})\, 2x \, dx$$

$$= -x^2 e^{-x} + 2\int x e^{-x} dx \rightarrow ②$$

From ② Now $\int x e^{-x} dx$

Put $u = x$ $\quad dv = e^{-x} dx$

$\int dv = \int e^{-x} dx$

$v = \frac{e^{-x}}{-1} = -e^{-x}$

$$\int x e^{-x}\,dx = x(-e^{-x}) - \int (-e^{-x})\,dx$$
$$= -xe^{-x} + \int e^{-x}\,dx$$
$$= -xe^{-x} - e^{-x}$$

using in ②, we have

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2[-xe^{-x} - e^{-x}]$$
$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}$$
$$= -e^{-x}(x^2 + 2x + 2)$$

<u>Verification</u>

$$\int x^2 e^{-x}\,dx = x^2(-e^{-x}) - (2x)(+e^{-x}) + (2)(-e^{-x})$$
$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} \quad \text{[Bernoulli's formula]}$$
$$= -e^{-x}(x^2 + 2x + 2)$$

<u>P.P</u> $\int x^3 e^{2x}\,dx$ Ans: $\frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3)$

Evaluate $\int x \cos 2x\,dx$

<u>Sol</u> $\int x \cos 2x\,dx \rightarrow$ ①

Put $u = x$ $dv = \cos 2x\,dx$

$$\int dv = \int \cos 2x\,dx$$
$$V = \frac{\sin 2x}{2}$$

using in ①

$$\int x \cos 2x\,dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2}\,dx$$
$$= \frac{x \sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx$$
$$= \frac{x \sin 2x}{2} - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$$
$$= \frac{x \sin 2x}{2} + \frac{1}{4}\cos 2x$$

<u>Verification</u> $\int x \cos 2x\,dx$

$$= x\left(\frac{\sin 2x}{2}\right) - (1)\left(-\frac{\cos 2x}{4}\right)$$
$$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$$

If $I_n = \int_0^{\pi/2} x^n \cos x \, dx$, Show that $I_n + n(n-1) I_{n-2} = \left(\frac{\pi}{2}\right)^n$

Sol

$$I_n = \int_0^{\pi/2} x^n \cos x \, dx \rightarrow (1)$$

Put $u = x^n$ $\quad dv = \cos x \, dx$

$$\int dv = \int \cos x \, dx$$

$$v = \sin x$$

using in (1) and by integration by parts.

$$I_n = \int_0^{\pi/2} x^n \cos x \, dx$$

$$= \left(x^n \sin x\right)_0^{\pi/2} - \int_0^{\pi/2} \sin x \, n x^{n-1} \, dx$$

$$= \left[\left(\pi/2\right)^n \sin \pi/2 - 0\right] - n \int_0^{\pi/2} x^{n-1} \sin x \, dx$$

$$= \left(\frac{\pi}{2}\right)^n - n \int_0^{\pi/2} x^{n-1} \sin x \, dx \rightarrow (2)$$

From (2), $\int_0^{\pi/2} x^{n-1} \sin x \, dx$

Put $u = x^{n-1}$ $\quad dv = \sin x \, dx$

$$\int dv = \int \sin x \, dx$$

$$v = -\cos x$$

$$\therefore \int_0^{\pi/2} x^{n-1} \sin x \, dx = \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)(n-1) x^{n-2} \, dx$$

$$= \left[\left(\pi/2\right)^n \left(-\cos \frac{\pi}{2}\right) - 0\right] + (n-1) \int_0^{\pi/2} x^{n-2} \cos x \, dx$$

$$= (n-1) I_{n-2}$$

using in (2)

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$$

$$I_n + n(n-1) I_{n-2} = \left(\frac{\pi}{2}\right)^n$$

Est.

Establish the reduction formula for $I_n = \int \sin^n x\, dx$ (n being a Positive integer)

Sol

$$I_n = \int \sin^n x\, dx$$
$$= \int \sin^{n-1} x \sin x\, dx$$
$$= \int \underbrace{\sin^{n-1} x}_{u}\, d(\underbrace{-\cos x}_{v})$$
$$= -\sin^{n-1} x \cos x - \int (-\cos x)(n-1) \sin^{n-2} x \cos x\, dx$$
$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$
$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1-\sin^2 x)\, dx$$
$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$
$$I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - n I_n + I_n$$
$$n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

(or) $$\boxed{I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}}$$

Result:

$$\int_0^{\pi/2} \sin^n x\, dx = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$
$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$
$$= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$
$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\int_0^{\pi/2} \sin^{n-4} x\, dx$$
$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots$$

If n is even, we have $\int_0^{\pi/2} dx = (x)_0^{\pi/2} = \pi/2$

If n is odd, we have $\int_0^{\pi/2} \sin x\, dx = (-\cos x)_0^{\pi/2} = 1$

$\therefore \int_0^{\pi/2} \sin^n x\, dx =$

$$\boxed{\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\,\frac{\pi}{2},}$$ when n is even

$$\boxed{\frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{2}{3},}$$ when n is odd

Problem ① $\int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{\cancel{6}_2} \frac{\cancel{3}}{4} \frac{1}{2} \pi/2 = \frac{5\pi}{32}$

② $\int_0^{\pi/2} \sin^7 x\, dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{48}{105}$

③ Evaluate $\int_0^1 x(1-x^2)^{1/2} dx \longrightarrow ①$

Sol Put $x = \sin\theta$ $\quad dx = \cos\theta\, d\theta$

Limit $\quad x = 0 \quad \theta = 0$

$x = 1 \quad \theta = \pi/2$

using in ①

$$\int_0^1 x(1-x^2)^{1/2} dx = \int_0^{\pi/2} \sin\theta\,(1-\sin^2\theta)^{1/2} \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin\theta\,(\cos\theta) \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\, d(-\cos\theta)$$

$$= \left[-\frac{\cos^3\theta}{3} \right]_0^{\pi/2}$$

$$= -\frac{1}{3}\left[\cos^3 \pi/2 - \cos 0 \right]$$

$$= -\frac{1}{3}[0 - 1] = 1/3$$

Establish the reduction formula for $I_n = \int \cos^n x\, dx$
(n being a positive integer)

$$I_n = \int \cos^n x\, dx$$

$$= \int \cos^{n-1} x \cos x\, dx$$

$$= \int \cos^{n-1} x\, d(\sin x)$$

$$= \cos^{n-1} x \sin x - \int \sin x\, (n-1) \cos^{n-2} x\, (-\sin x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x\, dx.$$

$$= \cos^{n-1} x \sin x + (n-1) \int (1-\cos^2 x) \cos^{n-2} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, dx - (n-1) \int \cos^n x\, dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\cancel{I_n} = \cos^{n-1} x \sin x + (n-1) I_{n-2} - n I_n + \cancel{I_n}.$$

$$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

(or) $$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-1}$$

<u>Result</u>: $$\int_0^{\pi/2} \cos^n x\, dx = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$$

$$= 0 + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$$

$$= \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \int_0^{\pi/2} \cos^{n-4} x\, dx$$

$$= \frac{n-1}{n} \, \frac{n-3}{n-2} \, \frac{n-5}{n-4} \cdots$$

If n is even $\int_0^{\pi/2} \cos x\, dx = (\sin x)_0^{\pi/2} = 1$

The ultimate integral is $\int_0^{\pi/2} dx = (x)_0^{\pi/2} = \pi/2$, when n is even

If n is odd, we have

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}$$

Problem ① $\int_0^{\pi/2} \cos^8 x\,dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}$

② $\int_0^{\pi/2} \cos^5 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$

③ $\int_0^{\pi/2} \sin^4 x\,dx = \frac{4-1}{4}\cdot\frac{4-3}{4-2}$

$= \frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2$

$= \frac{3\pi}{16}$

④ $\int_0^{\pi/2} \cos^7 x\,dx = \frac{7-1}{7}\cdot\frac{7-3}{7-2}\cdot\frac{7-5}{7-4}$

$= \frac{\cancel{6}^{\,2}}{7}\cdot\frac{4}{5}\cdot\frac{2}{\cancel{3}_{\,1}} = \frac{16}{35}$

Obtain the reduction formula for $I_{m,n} = \int \sin^m x \cos^n x \, dx$ (m, n being a positive integers)

Proof: $I_{m,n} = \int \sin^m x \cos^n x \, dx$

$= \int \sin^m x \cos^{n-1} x \cos x \, dx$ $\quad \because \cos x \, dx = d(\sin x)$

$= \int \sin^m x \cos^{n-1} x \, d(\sin x)$ $\quad \sin^m x \, d(\sin x) = d\left(\frac{\sin^{m+1} x}{m+1}\right)$

$\cancel{\int \sin^m} * U \quad \underline{dv}$ $\quad \downarrow \quad \frac{(m+1)\sin^m x \cos x}{m+1}$

$= \int \cos^{n-1} x \, d\left(\frac{\sin^{m+1} x}{m+1}\right)$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} - \int \frac{\sin^{m+1} x}{m+1} d(\cos^{n-1} x)$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} - \frac{1}{m+1} \int \sin^{m+1} x \, (n-1) \cos^{n-2} x \, (-\sin x) dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x \, dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \sin^2 x \, dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x (1 - \cos^2 x) dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \, dx - \frac{n-1}{m+1} \int \sin^m x \cos^n x \, dx$

$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$

$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$

$I_{m,n}\left(1 + \frac{n-1}{m+1}\right) = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$

$$I_{m,n}\left(\frac{m+1+n-1}{m+1}\right) = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$(m+n)\, I_{m,n} = \underline{\cos^{n-1}x \sin^{m+1}x} + (n-1) I_{m,n-2}$$

$$\therefore I_{m,n} = \frac{1}{m+n}\left[\cos^{n-1}x \sin^{m+1}x + (n-1) I_{m,n-2}\right] \rightarrow \text{(I)}$$

Note: The power of $\cos x$ has been reduced by 2.

We may, by a similar arguement, arrive at the reduction formula in the form

$$(m+n)\, I_{m,n} = -\sin^{m-1}x \cos^{n+1}x + (m-1) I_{m-2,n}$$

ie, $$I_{m,n} = \frac{1}{m+n}\left[-\sin^{m-1}x \cos^{n+1}x + (m-1) I_{m-2,n}\right] \rightarrow \text{(II)}$$

Case i) Let m or n be an odd integer, say n

From (I) $$I_{m,1} = \int \sin^m x \cos x \, dx$$

$$= \int \sin^m x \, d(\sin x) \qquad \left(\because \int x\, dx = \frac{x^2}{2}\right)$$

$$= \frac{\sin^{m+1} x}{m+1}$$

If m is odd then from II

$$I_{1,n} = \int \sin x \cos^n x \, dx$$

$$= \int \cos^n x \, d(-\cos x)$$

$$= -\int \cos^n x \, d(\cos x)$$

$$= -\frac{\cos^{n+1} x}{n+1}$$

If both m and n are odd, reduce the smaller index

Results: $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$ (m, n being positive integers)

Now $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \frac{(n-1)}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx$$

$$= \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx$$

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2}\int_0^{\pi/2} \sin^m x \cos^{n-4} x \, dx$$

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots I_{m,1} \text{ or } I_{m,0}$$

according as n is odd or even

i) If n is odd, $I_{m,1} = \int_0^{\pi/2} \sin^m x \cos x \, dx$

$$= \int_0^{\pi/2} \sin^m x \, d(\sin x)$$

$$= \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2}$$

$$= \frac{1}{m+1}\left[\sin^{m+1} x\right]_0^{\pi/2}$$

$$= \frac{1}{m+1}[1-0] = \frac{1}{m+1}$$

When n is odd

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

ii) If n is even

$$I_{m,0} = \int_0^{\pi/2} \sin^m x \, dx = \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

When n is even

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

Problem: ① $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \frac{5-1}{6+5} \cdot \frac{5-3}{6+5-2}$

$= \frac{4}{11} \cdot \frac{2}{9} \frac{1}{7}$

$= \frac{8}{693}$

$99 \times 7 = 693$

② $\int_0^{\pi/2} \cancel{\cos} \sin^6 x \cos^4 x \, dx = \frac{6-1}{6+4} \cdot \frac{6-3}{6+4-2} \cdot \frac{6-5}{6+4-4} \frac{4-1}{4} \frac{4-3}{4-2}$

$= \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \frac{3}{4} \frac{1}{2} \pi/2$

$= \frac{3\pi}{512}$

③ $\int_0^{\pi/2} \sin^8 x \cos^4 x \, dx$

Sol $m = 8 \quad n = 4$

$\int_0^{\pi/2} \sin^8 x \cos^4 x \, dx = \frac{4-1}{8+4} \cdot \frac{4-3}{8+4-2} \cdot \frac{8-1}{8} \cdot \frac{8-3}{8-2} \cdot \frac{8-5}{8-4} \cdot \frac{8-7}{8-6}$

$= \frac{\cancel{3}}{\cancel{12}_4} \cdot \frac{1}{\cancel{10}_2} \cdot \frac{7}{8} \cdot \frac{\cancel{5}^1}{\cancel{6}_2} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{2} \cdot \pi/2$

$= \frac{35\cancel{\pi}}{1024} \quad \frac{7\pi}{2048}$

④ $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$

Sol $m = 7 \quad n = 5$

$\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx = \frac{5-1}{7+5} \cdot \frac{5-3}{7+5-2} \cdot \frac{1}{8}$

$= \cancel{\frac{4}{12} \cdot \frac{2}{10}}$

$= \frac{8}{120}$

$= \frac{\cancel{4}}{12} \cdot \frac{\cancel{2}}{10} \cdot \frac{1}{8}$

$= \frac{1}{120}$

Evaluate

$\int \sin^6 x \cos^3 x \, dx \rightarrow (1)$

Sol Put $y = \sin x$ $\quad dy = \cos x \, dx \rightarrow (2)$

From (1) $\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \cos^2 x \cos x \, dx$

$= \int \sin^6 x (1 - \sin^2 x) \cos x \, dx$

$= \int \cancel{\sin} \, y^6 (1 - y^2) \, dy$ by (2)

$= \int (y^6 - y^8) \, dy$

$= \dfrac{y^7}{7} - \dfrac{y^9}{9}$

$= \dfrac{\sin^7 x}{7} - \dfrac{\sin^9 x}{9}$ by (2)

Evaluate $\int \sin^9 x \cos^5 x \, dx \rightarrow (1)$

Sol From (1) $\int \sin^9 x \cos^5 x \, dx = \int \sin^9 x \cos^4 x \cos x \, dx \rightarrow (2)$

$\cancel{= \int \sin}$

~~Put $y = \sin x$ $dy = \cos x \, dx$~~

$= \int \sin^9 x (1 - \sin^2 x)^2 \cos x \, dx$

Put $y = \sin x$ $\quad dy = \cos x \, dx$

$\therefore \int \sin^9 x \cos^5 x \, dx = \int y^9 (1 - y^2)^2 \, dy$

$= \int y^9 (1 - 2y^2 + y^4) \, dy$

$= \int (y^9 - 2y^{11} + y^{13}) \, dy$

$= \dfrac{y^{10}}{10} - \dfrac{2y^{12}}{12} + \dfrac{y^{14}}{14}$

$= \dfrac{y^{10}}{10} - \dfrac{y^{12}}{6} + \dfrac{y^{14}}{14}$

$= \dfrac{\sin^{10} x}{10} - \dfrac{\sin^{12} x}{6} + \dfrac{\sin^{14} x}{14}$

P.P $\int \sin^7 x \cos^3 x\, dx$ ~~Ans:~~

$\int \sin^4 x \cos^3 x\, dx$ ~~Ans~~

Obtain the reduction formula for $I_n = \int \tan^n x\, dx$ (n being a positive integer)

$\sec^2 x - \tan^2 x = 1$

$\tan^2 x = \sec^2 x - 1$

Sol $\S$. $I_n = \int \tan^n x\, dx$

$= \cancel{\int \tan^n x} \int \tan^{n-2} x . \tan^2 x\, dx$

$= \int \tan^{n-2} x\, (\sec^2 x - 1)\, dx$

$= \int \tan^{n-2} x\, \sec^2 x\, dx - \int \tan^{n-2} x\, dx$

$= \int \tan^{n-2} x\, d(\tan x) - \int \tan^{n-2} x\, dx$

$I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2} \rightarrow (1)$

Result: i, When n is even, then the integral is $\int dx = x$

(ii) When n is odd, we get $\int \tan x\, dx = \log \sec x$

Problem Evaluate $\int \tan^4 x\, dx$

Sol $\S$ $I_n = \int \tan^n x\, dx$

$= \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

Put $n = 4$

$I_4 = \dfrac{\tan^{4-1} x}{4-1} - \int \tan^{4-2} x\, dx$

$= \dfrac{\tan^3 x}{3} - \int \tan^2 x\, dx$

$= \dfrac{\tan^3 x}{3} - \int (\sec^2 x - 1)\, dx$

$= \dfrac{\tan^3 x}{3} - \int \sec^2 x\, dx + \int dx$

$$= \frac{\tan^3 x}{3} - \tan x + x$$

Obtain the reduction formula for $I_n = \int \cot^n x\, dx$ (n being a positive integer)

Sol $I_n = \int \cot^n x\, dx$ $\qquad \operatorname{cosec}^2 x - \cot^2 x = 1$

$= \int \cot^{n-2} x \cdot \cot^2 x\, dx$ $\qquad \cot^2 x = \operatorname{cosec}^2 x - 1$

$= \int \cot^{n-2} x\, (\operatorname{cosec}^2 x - 1)\, dx$

$= \int \cot^{n-2} x \operatorname{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$

$= \int \cot^{n-2} x\, d(-\cot x) - \int \cot^{n-2} x\, dx$

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

Obtain the reduction formula for $I_n = \int \sec^n x\, dx$ (n being a positive integer)

Sol $I_n = \int \sec^n x\, dx$

$= \int \sec^{n-2} x \sec^2 x\, dx$ $\qquad \because d(\tan x) = \sec^2 x$

$= \int \underset{u}{\sec^{n-2} x}\, \underset{v}{d(\tan x)}$

$= \sec^{n-2} x \tan x - \int \tan x\, (n-2) \sec^{n-3} x \sec x \tan x\, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x\, [\sec^2 x - 1]\, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx$

$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$(n-2) I_n + I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$I_n (n-2+1) = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

Problem Evaluate $I = \int \sec^3 x \, dx$

Formula: $\int \sec x \, dx = \log(\sec x + \tan x)$

Sol $I = \int \sec^3 x \, dx$

$= \int \underset{u}{\sec x} \, \underset{v}{\sec^2 x} \, dx$

$= \int \sec x \; d(\tan x)$

$= \sec x \tan x - \int \tan x \sec x \tan x \, dx$

$= \sec x \tan x - \int \tan^2 x \sec x \, dx$

$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$= \sec x \tan x - I + \log(\sec x + \tan x)$

$2I = \sec x \tan x + \log(\sec x + \tan x)$

$I = \frac{1}{2}[\sec x \tan x + \log(\sec x + \tan x)]$

Evaluate $\int \sec^6 x \, dx$

$\sec^2 x = 1 + \tan^2 x$

Sol $\int \sec^6 x \, dx = \int \sec^4 x \, (\sec^2 x) \, dx$

$= \int (\sec^2 x)^2 \, d(\tan x)$

$= \int (1 + \tan^2 x)^2 \, d(\tan x)$

$= \int (1 + t^2)^2 \, dt$ where $t = \tan x$

$= \int (1 + 2t^2 + t^4) \, dt$

$= \int dt + 2\int t^2 \, dt + \int t^4 \, dt$

$= t + 2\frac{t^3}{3} + \frac{t^5}{5}$

$= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x$

Obtain the reduction formula for $I_n = \int \operatorname{cosec}^n x \, dx$ (n being a positive integer)

Sol $I_n = \int \operatorname{cosec}^n x \, dx$

$= \int \underset{u}{\operatorname{cosec}^{n-2} x} \; \underset{v}{\operatorname{cosec}^2 x} \, dx$ $\quad \because d(\cot x) = -\operatorname{cosec}^2 x \, dx$

$= \int \operatorname{cosec}^{n-2} x \, d(-\cot x)$

$= -\operatorname{cosec}^{n-2} x \cot x - \int (-\cot x)(n-2)\operatorname{cosec}^{n-3} x \cdot -\operatorname{cosec} x \cot x \, dx$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$

$\because \operatorname{cosec}^2 x - \cot^2 x = 1$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x (\operatorname{cosec}^2 x - 1) dx$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^n x + \cancel{(n-2)}$
$\quad (n-2)\int \operatorname{cosec}^{n-2} x \, dx$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$

$(n-2)I_n + I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

$I_n(n-2+1) = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

$(n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

Result: i) If n is an odd integer, we have $\int \operatorname{cosec} x \, dx = -\log(\operatorname{cosec} x + \cot x)$

ii) If n is an even integer, we have $\int dx = x$

Problem Evaluate $\int \csc^4 x \, dx$.

Sol $\int \csc^4 x = \int \csc^2 x \csc^2 x \, dx$

$d(\cot x) = -\csc^2 x$

$= \int \csc^2 x \, d(-\cot x)$

$\csc^2 x - \cot^2 x = 1$

$= -\int \csc^2 x \, d(\cot x)$

$= -\int (1+\cot^2 x) \, d(\cot x)$

$= -\int (1+y^2) \, dy$ Put $y = \cot x$

$= -\int dy - \int y^2 dy$

$= -y - y^3/3$

$= -\cot x - \dfrac{\cot^3 x}{3}$

Evaluate $\int \csc^5 x \, dx$

Sol W.K.T

$(n-1) I_n = -\csc^{n-2} x \cot x + (n-2) I_{n-2} \rightarrow ①$

Put $n = 5$ in ①

$(5-1) I_5 = -\csc^{5-2} x \cot x + (5-2) I_{5-2}$

$4 I_5 = -\csc^3 x \cot x + 3 \int I_3$

$I_5 = -\dfrac{\csc^3 x \cot x}{4} + \dfrac{3}{4} \int \csc^3 x \, dx \rightarrow ②$

Now, $\int \csc^3 x \, dx$

Put $n = 3$ in ①

$(3-1) I_3 = -\csc^{3-2} \cot x + (3-2) I_{3-2}$

$2 I_3 = -\csc x \cot x + I_1$

$I_3 = \dfrac{-\csc x \cot x}{2} + \dfrac{1}{2} \int \csc x \, dx$

$= \dfrac{-\csc x \cot x}{2} + \dfrac{1}{2} \log(\csc x + \cot x)$

: 20 :

$$= -\frac{\operatorname{cosec}^3 x \cot x}{4} - \frac{3}{4}\left[-\frac{\operatorname{cosec} x \cot x}{2} - \frac{1}{2}\log(\operatorname{cosec} x + \cot x)\right]$$

$$= -\frac{\operatorname{cosec}^3 x \cot x}{4} - \frac{3}{8}\operatorname{cosec} x \cot x - \frac{3}{8}\log(\operatorname{cosec} x + \cot x)$$

$I_{m,n} = \int x^m (\log x)^n \, dx$ (where m and n are positive integers) Hence or otherwise evaluate $\int x^4 (\log x)^3 dx$

Sol

$$I_{m,n} = \int \underset{u}{(\log x)^n} \, d\left(\underset{v}{\frac{x^{m+1}}{m+1}}\right)$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \int \frac{x^{m+1}}{m+1} \, n (\log x)^{n-1} \frac{1}{x} dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1} dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

Now $I_{m,0} = \int x^m dx = \frac{x^{m+1}}{m+1}$ $\quad \because \int x^m (\log x)^0 dx = \int x^m dx$

Now $\int (\log x)^3 x^4 dx = \int (\log x)^3 \, d\left(\frac{x^5}{5}\right)$

$$= \frac{x^5}{5}(\log x)^3 - \int \frac{x^5}{5} \, 3 (\log x)^2 \frac{1}{x} dx$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int (\log x)^2 x^4 dx.$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\left[\int \underset{u}{(\log x)^2} \, d\left(\underset{v}{\frac{x^5}{5}}\right)\right]$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\left[(\log x)^2\left(\frac{x^5}{5}\right) - \int \frac{x^5}{5}(2)(\log x)\frac{1}{x}dx\right]$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 x^5 + \frac{6}{25}\int \underset{u}{(\log x)} \, \underset{v}{x^4} \, dx$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 x^5 + \frac{6}{25}\left[\int \log x \, d\left(\frac{x^5}{5}\right)\right]$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 x^5 + \frac{6}{25}\left[(\log x)\frac{x^5}{5} - \int \frac{x^5}{5}\frac{1}{x}dx\right]$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 x^5 + \frac{6}{125}(\log x)x^5 - \frac{6}{125}\int x^4 dx$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 x^5 + \frac{6}{125}(\log x)x^5 - \frac{6}{125}\frac{x^5}{5}$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 x^5 + \frac{6}{125}\log x \, x^5 - \frac{6}{625}x^5$$

$$= x^5\left[\frac{1}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 + \frac{6}{125}\log x - \frac{6}{625}\right]$$

If $\int_0^{\pi/2} \cos^m x \cos nx \, dx = f(m,n)$, Prove that

$$f(m,n) = \frac{m}{m+n} f(m-1, n-1)$$ Hence, Prove that

$$f(n,n) = \frac{\pi}{2^{n+1}}$$

Sol $\int \cos^m x \cos nx \, dx = \int \overset{u}{\cos^m x} \, d\left(\overset{v}{\frac{\sin nx}{n}}\right)$

$$= \frac{\cos^m x \sin nx}{n} - \int \frac{\sin nx}{n} \, m \cos^{m-1} x \, (-\sin x) \, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin x \sin nx \, dx$$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\cos(nx - x) = \cos nx \cos x + \sin nx \sin x$

$\cos(n-1)x - \cos nx \cos x = \sin nx \sin x$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \{\cos(n-1)x - \cos nx \cos x\} \, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos(n-1)x \, dx - \frac{m}{n} \int \cos^{m-1} x \cos nx \cos x \, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos(n-1)x \, dx - \frac{m}{n} \int \cos^m x \cos nx \, dx$$

Now $f(m,n) = \int_0^{\pi/2} \cos^m x \cos nx \, dx$ $\qquad \left(1 + \frac{m}{n}\right)$ $\quad \frac{m+n}{n}$

$$\int \cos^m x \cos nx \, dx + \frac{m}{n} \int \cos^m x \cos nx \, dx = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos(n-1)x \, dx$$

$$\frac{m+n}{n} \int \cos^m x \cos nx \, dx = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos(n-1)x \, dx$$

$$\int \cos^m x \cos nx \, dx = \frac{1}{m+n} \left[\cos^m x \sin nx + m \int \cos^{m-1} x \cos(n-1)x \, dx\right]$$

Now, $f(m,n) = \int_0^{\pi/2} \cos^m x \cos nx \, dx$

$$= \frac{1}{m+n}\left\{ \left(\cos^m x \sin nx\right)_0^{\pi/2} + m\int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x \, dx \right\}$$

$$= \frac{1}{m+n}\left[0 + m\int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x \, dx \right]$$

$$f(m,n) = \frac{m}{m+n} f(m-1, n-1)$$

Put $m=n$, we get $f(n,n) = \frac{n}{n+n} f(n-1, n-1)$

$$= \frac{1}{2} f(n-1, n-1)$$

Repeating the same method, we have

$$f(n,n) = \frac{1}{2^2} f(n-2, n-2)$$

$$= \frac{1}{2^n} f(0,0)$$

$$= \frac{1}{2^n}\int_0^{\pi/2} \cos^0 x \cos 0x \, dx$$

$$= \frac{1}{2^n}\int_0^{\pi/2} dx$$

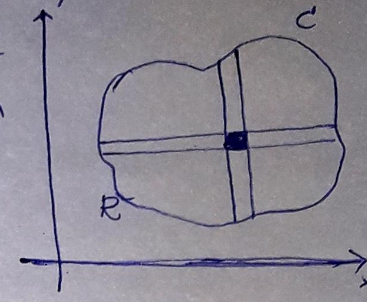
$$= \frac{1}{2^n}\left(x\right)_0^{\pi/2}$$

$$= \frac{1}{2^n}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2^{n+1}}$$

Multiple Integrals

Double Integral (Defn)



Let $f(x, y)$ be a Continuous and single-Valued function of x and y within a region R bounded by a closed curve C and upon the boundary C

The region R is called the region of integration Corresponding to the interval of integration (a, b) in the case of the single integral.

This integral is written as $\iint_R f(x,y)\,dx\,dy$

Problem ① Evaluate $\int_0^a \int_0^b (x^2+y^2)\,dx\,dy$

Sol
$$\int_0^a\int_0^b (x^2+y^2)\,dx\,dy = \int_{y=0}^{a}\int_{x=0}^{b} (x^2+y^2)\,dx\,dy$$
$$= \int_{y=0}^{a} \left(\frac{x^3}{3}+y^2x\right)_{x=0}^{b} dy$$
$$= \int_{y=0}^{a} \left(\frac{b^3}{3}+y^2b\right) dy$$
$$= \left(\frac{b^3}{3}y+\frac{y^3}{3}b\right)_{x=0}^{a}$$
$$= \frac{ab^3}{3}+\frac{ba^3}{3}$$
$$= \frac{ab}{3}(a^2+b^2)$$

② Evaluate $\int_0^3\int_1^2 xy(x+y)\,dy\,dx$

Sol
$$\int_0^3\int_1^2 xy(x+y)\,dy\,dx = \int_{x=0}^{3}\int_{y=1}^{2}(x^2y+xy^2)\,dy\,dx$$
$$= \int_{x=0}^{3}\left(\frac{x^2y^2}{2}+\frac{xy^3}{3}\right)_{y=1}^{2} dx$$
$$= \int_{x=0}^{3}\left[\frac{x^2}{2}(4-1)+\frac{x}{3}(8-1)\right]dx$$

$$= \int_{x=0}^{3} \left[\frac{3}{2} x^2 + \frac{7}{3} x \right] dx$$

$$= \left[\frac{3}{2}\left(\frac{x^3}{3}\right) + \frac{7}{3}\left(\frac{x^2}{2}\right) \right]_0^3$$

$$= \frac{\cancel{3}}{2}\left[\frac{3^3}{\cancel{3}}\right] + \frac{7}{\cancel{3}}\left[\frac{3^{\cancel{2}}}{2}\right]$$

$$= \frac{27}{2} + \frac{21}{2}$$

$$= \frac{48}{2} = 24$$

P.P $\int_0^a \int_0^b xy(x-y)\,dy\,dx$. Ans $\frac{a^2b^2(a-b)}{6}$

(4) Evaluate $\int_1^2 \int_1^x xy^2\,dy\,dx$

Sol $\int_1^2 \int_1^x xy^2\,dy\,dx = \int_{x=1}^{2} \int_{y=1}^{x} xy^2\,dy\,dx$

$$= \int_{x=1}^{2} \left(\frac{xy^3}{3}\right)_{y=1}^{x} dx$$

$$= \int_{x=1}^{2} \frac{x}{3}(x^3-1)\,dx$$

$$= \frac{1}{3}\int_{x=1}^{2} (x^4 - x)\,dx$$

$$= \frac{1}{3}\left[\frac{x^5}{5} - \frac{x^2}{2}\right]_{x=1}^{2}$$

$$= \frac{1}{3}\left\{\left(\frac{2^5}{5} - \frac{2^2}{2}\right) - \left(\frac{1}{5} - \frac{1}{2}\right)\right\}$$

$$= \frac{1}{3}\left[\left(\frac{32}{5} - \frac{4}{2}\right) - \left(\frac{2-5}{10}\right)\right]$$

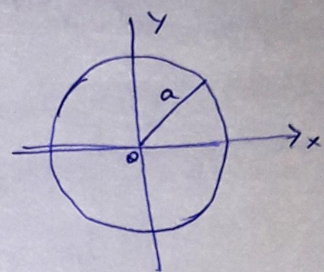
$$= \frac{1}{3}\left[\left(\frac{64-20}{10}\right) - \left(\frac{2-5}{10}\right)\right]$$

$$= \frac{1}{3}\left[\frac{44}{10} + \frac{3}{10}\right] = \frac{47}{30}$$

P.P (5) Evaluate $\int_0^a \int_0^x (x^2+y^2)\, dy\, dx$ Ans: $\frac{a^4}{3}$

Evaluate $\iint xy\, dx\, dy$ taken over the Positive quadrant of the Circle $x^2+y^2=a^2$

Sol



Treating x as a Constant y varies from 0 to $\sqrt{a^2-x^2}$

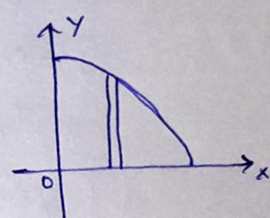
Now $x^2+y^2=a^2$

$y^2=a^2-x^2$

$y=\pm\sqrt{a^2-x^2}$

$y=0 \quad y=\sqrt{a^2-x^2}$

$x=0 \quad x=a$



$$\therefore \iint xy\, dy\, dx = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= \int_{x=0}^{a} \left[x\left(\frac{y^2}{2}\right) \right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a} \frac{x}{2}(a^2-x^2)\, dx$$

$$= \frac{1}{2}\int_{x=0}^{a} (a^2x - x^3)\, dx$$

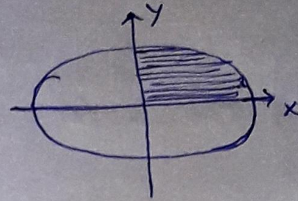
$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4} \right]_{x=0}^{a}$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4} \right]$$

$$= \frac{a^4}{2}\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= \frac{a^4}{2}\left(\frac{1}{4}\right) = \frac{a^4}{8}$$

Find the Value of $\iint xy\, dx\, dy$ taken over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Sol

Treating x as a Constant

y Varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

Now, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm b\sqrt{\left(1 - \frac{x^2}{a^2}\right.}$$

$\therefore \quad y = 0 \qquad y = b\sqrt{1 - \frac{x^2}{a^2}}$

$x = 0 \qquad x = a$

$$\therefore \iint xy\, dx\, dy = \int_{x=0}^{a} \int_{y=0}^{b} xy\, dy\, dx$$

$$= \int_{x=0}^{a} x\left(\frac{y^2}{2}\right) dx$$

$$= \int_{x=0}^{a} \frac{x}{2}\left[b^2\left(1 - \frac{x^2}{a^2}\right)\right] dx$$

$$= \frac{b^2}{2}\int_{x=0}^{a} x\left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{b^2}{2}\int_{x=0}^{a}\left[x - \frac{x^3}{a^2}\right] dx$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2} - \frac{x^4}{4a^2}\right]_{x=0}^{a}$$

$$= \frac{b^2}{2}\left[\frac{a^2}{2} - \frac{a^{4}}{4a^2}\right]$$

$$= \frac{b^2}{2}\left[\frac{a^2}{2} - \frac{a^2}{4}\right]$$

$$= \frac{a^2b^2}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{a^2b^2}{2}\left(\frac{1}{4}\right)$$

$$= \frac{a^2b^2}{8}$$

Evaluate $\iint x^3 y \, dx\, dy$ over the region for which x, y are each ≥ 0 and $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$

Sol

$y = 0 \qquad y = b\sqrt{1 - \frac{x^2}{a^2}}$

$x = 0 \qquad x = a$

$$\iint (x^3 y)\, dx\, dy = \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} x^3 y \, dy\, dx$$

$$= \int_{x=0}^{a} x^3 \left(\frac{y^2}{2}\right)_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \int_{x=0}^{a} \frac{x^3}{2} \left[b^2 \left(1 - \frac{x^2}{a^2}\right) \right] dx$$

$$= \frac{b^2}{2} \int_{x=0}^{a} x^3 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{b^2}{2} \int_{x=0}^{a} \left(x^3 - \frac{x^5}{a^2}\right) dx$$

$$= \frac{b^2}{2} \left[\frac{x^4}{4} - \frac{x^6}{6a^2} \right]_{x=0}^{a}$$

$$= \frac{b^2}{2} \left[\frac{a^4}{4} - \frac{a^6}{6a^2} \right]$$

$$= \frac{b^2}{2} \left[\frac{a^4}{4} - \frac{a^4}{6} \right]$$

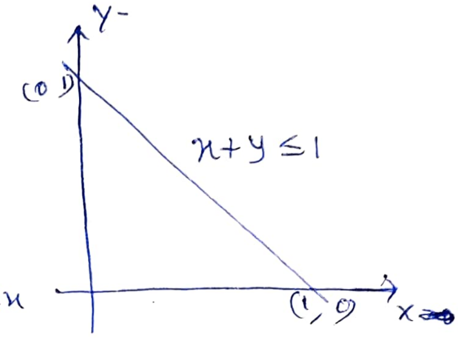
$$= \frac{a^4 b^2}{2} \left[\frac{3-2}{12} \right]$$

$$= \frac{a^4 b^2}{2} \left[\frac{1}{12} \right] = \frac{a^4 b^2}{24}$$

Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region for which x, y are each ≥ 0 and $x+y \leq 1$

<u>Sol</u> This region is the triangle formed by the lines $x=0$, $y=0$

$x+y=1$



Now $\quad y=0, \quad y=1-x$

$\qquad x=0 \quad x=1$

$$\iint (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} (x^2+y^2)\,dy\,dx$$

$$= \int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_{x=0}^{1-x} dx$$

$$= \int_{x=0}^{1}\left[x^2(1-x)+\frac{(1-x)^3}{3}\right] dx$$

$$= \int_{x=0}^{1}\left[(x^2-x^3)+\frac{1}{3}(1-3x+3x^2-x^3)\right] dx$$

$$= \frac{1}{3}\int_{x=0}^{1}\left[3x^2-3x^3+1-3x+3x^2-x^3\right] dx$$

$$= \frac{1}{3}\int_{x=0}^{1}\left[-4x^3+6x^2-3x+1\right] dx$$

$$= \frac{1}{3}\left[-\frac{\cancel{4}x^4}{\cancel{4}}+\frac{\overset{2}{\cancel{6}}x^3}{\cancel{3}}-\frac{3x^2}{2}+x\right]_{x=0}^{1}$$

$$= \frac{1}{3}\left[-\cancel{1}+2-\frac{3}{2}+\cancel{1}\right]$$

$$= \frac{1}{3}\left[\frac{4-3}{2}\right]$$

$$= \frac{1}{6}$$

Evaluate $\displaystyle\int_0^{\pi/2}\int_1^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$

$\underline{\underline{\text{Sol}}}$

$$\int_0^{\pi/2}\int_1^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2} = \int_{\theta=0}^{\pi/2} d\theta \; \cancel{\tfrac{1}{2}} \int_{r=1}^{\infty} \frac{r\,dr}{(r^2+a^2)^2}$$

$$\frac{2r\,dr}{2}$$

$$= \int_{\theta=0}^{\pi/2} d\theta \; \frac{1}{2} \int_{r=1}^{\infty} \frac{d(r^2+a^2)}{(r^2+a^2)^2}$$

$$\frac{(r^2+a^2)^{-2+1}}{-1}$$

$$= \int_{\theta=0}^{\pi/2} d\theta \left[\frac{1}{2}\left(-\frac{1}{r^2+a^2}\right)\right]_{r=1}^{\infty}$$

$$= \int_{\theta=0}^{\pi/2} d\theta \left[-\frac{1}{2}\left(\frac{1}{0} - \frac{1}{1+a^2}\right)\right]$$

$$= \int_{\theta=0}^{\pi/2} d\theta \left[\frac{1}{2(1+a^2)}\right]$$

$$= \frac{1}{2(1+a^2)} \int_{\theta=0}^{\pi/2} d\theta$$

$$= \frac{1}{2(1+a^2)} \left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4(1+a^2)}$$

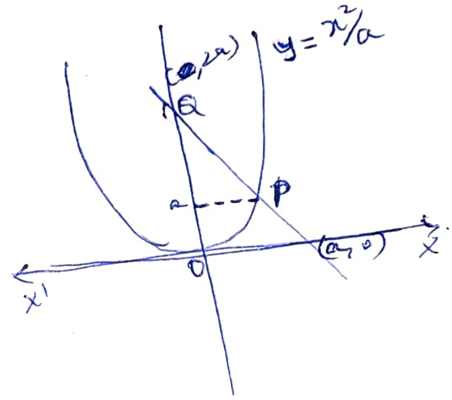
Change the order of integration in the integral

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dx\, dy \text{ and evaluate}$$

Sol Given Limit: $y = \frac{x^2}{a}$ to $y = 2a - x$

$x = 0$ to $x = a$

The region of integration is OPQ



$y = 2a - x$

$x + y = 2a$

$x = 0$ $\quad \frac{x^2}{a} = y$

$x^2 = ay$

$x = \sqrt{ay}$

$y = a$

$y = 0$

~~x = a = y = a~~

$x = a$ $\quad y = 2a - x$

$x = 2a - y$

$y = a$

$\cancel{x = a}$

$x = a$ $\quad y = 2a - x$

$x = 2a - y$

$y = 2a$

$y = a$

$$\therefore \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dx\, dy = \int_{y=0}^{a} \int_{x=0}^{\sqrt{ay}} xy\, dx\, dy + \int_{y=a}^{2a} \int_{x=0}^{2a-y} xy\, dx\, dy$$

$$= \int_{y=0}^{a} \left[\frac{yx^2}{2}\right]_{x=0}^{\sqrt{ay}} dy + \int_{y=a}^{2a} \left[\frac{x^2 y}{2}\right]_{x=0}^{2a-y} dy$$

$$= \int_{y=0}^{a} \left[\frac{yay}{2}\right] dy + \int_{y=a}^{2a} \left[\frac{(2a-y)^2 y}{2}\right] dy$$

$$= \int_{y=0}^{a} \left[\frac{y^2 a}{2}\right] dy + \int_{y=a}^{2a} (4a^2 - 4ay + y^2) y\, dy$$

$$= \left(\frac{y^3 a}{6}\right)_{y=0}^{a} + \int_{y=a}^{2a} (4a^2y - 4ay^2 + y^3)\,dy$$

$$= \frac{a^3 a}{6} + \left[\cancel{4}^{2} a^2 \frac{y^2}{\cancel{2}_1} - 4a\frac{y^3}{3} + \frac{y^4}{4}\right]_{y=a}^{2a}$$

$$= \frac{a^4}{6} + \left[\cancel{4}2a^2(2a-a)^2 - \frac{4a}{3}(2a-a)^3 + \frac{(2a-a)^4}{4}\right]$$

$$= \frac{a^4}{6} + \left[2a^2a^2 - \frac{4a}{3}(a)^3 + \frac{a^4}{4}\right]$$

$$= \frac{a^4}{6} + 2a^4 - \frac{4a^4}{3} + \frac{a^4}{4}$$

$$\begin{array}{r|l} 3 & 6, 3, 4 \\ 2 & 2, 1, 4 \\ & 1, 1, 2 \end{array}$$

$$= \frac{2a^4 + 24a^4 - 16a^4 + 3a^4}{12}$$

$$\begin{array}{r} 26 \\ 16 \\ \hline 10 \end{array}$$

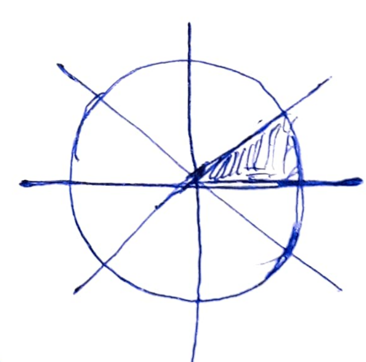
$$= \frac{a^4}{12}(2 + 24 - 16 + 3)$$

$$= \frac{13a^4}{12}$$

Triple Integral

Evaluate $\iiint xyz\,dx\,dy\,dz$ taken through the positive octant of the sphere

Sol To cover the whole positive octant of sphere ~~x^2y~~ $x^2+y^2+z^2=a^2$



$z=0 \qquad z = \sqrt{a^2-x^2-y^2}$

$y=0 \qquad y = \sqrt{a^2-x^2}$

$x=0 \qquad x = a$

$$\iiint xyz\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left(\frac{xyz^2}{2} \right)_{z=0}^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,(a^2-x^2-y^2)\, dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (xya^2 - x^3y - xy^3)\, dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left[\frac{xy^2a^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left[\frac{x(a^2-x^2)a^2}{2} - \frac{x^3}{2}(a^2-x^2) - \frac{x}{4}(a^2-x^2)^2 \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left[\frac{xa^4 - x^3a^2}{2} + \frac{-x^3a^2 + x^5}{2} - \frac{x}{4}(a^4+x^4-2a^2x^2) \right] dx$$

$$= \frac{1}{4} \int_{x=0}^{a} \left[a^4x - x^3a^2 - x^3a^2 + x^4 - \frac{x}{2}(a^4+x^4-2a^2x^2) \right] dx$$

$$= \frac{1}{8} \int_{x=0}^{a} \left[2a^4x - 4x^3a^2 + 2x^5 - a^4x - x^5 + 2a^2x^3 \right] dx$$

$$= \frac{1}{8} \left[\cancel{\frac{2a^4x^2}{2} - \frac{4x^4a^2}{4} + \frac{2x^5}{5} - \frac{a^5}{5} - \frac{a^5}{5} + \frac{2a^2x^3}{3}} \right]_{x=0}^{a}$$

$$= \frac{1}{8} \left[\cancel{\frac{2a^4a^2}{2} - \frac{4a^4a^2}{4}} \right.$$

$$= \frac{1}{8} \left[\frac{2a^4x^2}{2} - \frac{4x^4a^2}{4} + \frac{2x^6}{6} - \frac{a^4x^2}{2} - \frac{x^6}{6} + \frac{2a^2x^4}{4} \right]_{x=0}^{a}$$

$$= \frac{1}{8} \left[\frac{2a^4a^2}{2} - \frac{4a^4a^2}{4} + \frac{2a^6}{6} - \frac{a^4a^2}{2} - \frac{a^6}{6} + \frac{2a^2a^4}{4} \right]$$

$$= \frac{1}{8} \left[\frac{2a^6}{2} - \frac{4a^6}{4} + \frac{2a^6}{6} - \frac{a^6}{2} - \frac{a^6}{6} + \frac{2a^6}{4} \right]$$

$$= \frac{a^6}{8} \left[1 - 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{6} + \frac{1}{2} \right] = \frac{a^6}{48}$$

P.P Evaluate $\int_0^1 \int_0^2 \int_0^3 (x+y+z)\, dx\, dy\, dz$, Ans 18

UNIT- IV

TRIGONOMETRY

Basic Formula:

$$nC_r = \frac{n!}{r(n-r)!}$$

$$nP_r = n(n-1)(n-3)\dots[n-(r-1)]$$

$$(x+a)^n = nC_0 x^n + nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 \dots + nC_n a^n$$

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$(a-b)^2 = a^2 + b^2 - 2ab$$

$$(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

1 Expantions of $\cos(n\theta)$ and $\sin(n\theta)$

Let's invoke De Moivre's theoreme here.

We have

$$\cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n$$

Since n is a positive integer, the binomial theorem - N choose K holds for $(\cos(\theta) + i\sin(\theta))^n$.

Hence, by expanding, we have

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + n\cos^{n-1}\theta(i\sin\theta) + \frac{n(n-1)}{2!}\cos^{n-2}\theta(i^2\sin^2\theta) + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}(i\sin\theta)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta(i\sin\theta)^4$$

Since $i^2 = -1$, $i^4 = 1$, $i^3 = -i$, $i^5 = i$

$$= \cos^n\theta + i n \cos^{n-1}\theta \sin\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta \sin^2\theta - \frac{i n(n-1)(n-2)}{3!}\cos^{n-3}\theta \sin^3\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\sin^4\theta$$

$$= \left[\cos^n\theta - \frac{n(n-1)}{1\times 2}\cos^{n-2}\theta . \sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1\times 2\times 3\times 4}\cos^{n-4}\theta \sin^4 + \cdots\right]$$
$$+ i\left[n\cos^{n-1}\theta . \sin\theta - \frac{n(n-1)(n-2)}{1\times 2\times 3}\cos^{n-3}\theta \sin^3\theta + \cdots\right]$$

By equating the real and imaginary parts, we obtain

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1\times 2}\cos^{n-2}\theta . \sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1\times 2\times 3\times 4}\cos^{n-1}\theta , \sin^4\theta + \cdots$$

$$\sin n\theta = n\cos^{n-1}\theta . \sin\theta - \frac{n(n-1)(n-2)}{1\times 2\times 3}\cos^{n-3}\theta . \sin^3\theta + \cdots$$

Corollary I

$$\frac{\sin n\theta}{\sin\theta} = \frac{n\cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{2!}\cos^{n-3}\theta \sin^3\theta}{\sin\theta}$$

$$= n\cos^{n-1}\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta \sin^2\theta$$

$$\frac{\sin n\theta}{\sin\theta} = n\cos^{n-1}\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta \sin^2\theta$$

$$\frac{\sin n\theta}{\sin\theta} = n\cos^{n-1}\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta(1-\cos^2\theta)$$

Similarly, in the expansion of $\cos n\theta$ by putting

$\sin^2\theta = 1 - \cos^2\theta$

$\cos n\theta$ can be expressed in a series containing powers of $\cos\theta$

Corollary 2

Co efficient of $\cos^{n-1}\theta$ in the expansion of

$$\frac{\sin n\theta}{\sin\theta} = nC_1 + nC_3 + nC_5 + \cdots = 2^{n-1}$$

Corollary 3

Co efficient of $\cos^{n}\theta$ in the expansion of

$$\cos n\theta = nC_0 + nC_2 + \cdots = 2^{n-1}.$$

② Expansions of $\tan n\theta$ in the powers of $\tan\theta$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$= \frac{nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta}{\cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta}$$

$\cos^n\theta$ divide on both side

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta}{1 - nC_2 + \tan^2\theta \cdots}$$

Example

Expansion of $\cos 8\theta$ in terms of $\sin\theta$

Soln

By De Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = \cos 8\theta + i\sin 8\theta$$

$$= \cos^8\theta + 8C_1 \cos^7\theta \, i \sin\theta + 8C_2 \cos^6\theta \, i \sin^2\theta +$$
$$8C_3 \cos^5\theta \, i^3 \sin^3\theta + 8C_4 \cos^4\theta \, i^4 \sin^4\theta +$$
$$8C_5 \cos^3\theta \, i \sin^5\theta + 8C_6 \cos^2\theta \, i^6 \sin^6\theta +$$
$$8C_7 \cos\theta \, i^7 \sin^7\theta + i^8 \sin^8\theta .$$

$$= \cos^8\theta + 8i \cos^7\theta \sin\theta - 28 \cos^6\theta \sin^2\theta$$
$$- i\, 8C_3 \cos^5\theta \sin^3\theta + 8C_4 \cos^4\theta \sin^4\theta +$$
$$i\, 8C_5 \cos^3 \sin^5\theta - 8C_6 \cos^2\theta \sin^6\theta - i \cos\theta \sin^7\theta$$
$$+ \sin^8\theta$$

$$= (\cos^8\theta - 8C_2 \cos^6\theta \sin^2\theta + 8C_4 \cos^4\theta \sin^4\theta$$
$$- 8C_6 \cos^2\theta \sin^6\theta + \sin^8\theta) + i(8i \cos^7\theta \sin\theta + \cdots$$

Equaling real and imaginary part

$$\cos 8\theta = \cos^8\theta - 28 \cos^6\theta \sin^2\theta + 70 \cos^4\theta \sin^4\theta$$
$$- 28 \cos^2\theta \sin^2\theta + \sin^8\theta$$

$$\cos 8\theta = (1-\sin^2\theta)^4 - 28(1-\sin^2\theta)^3 \sin^2\theta +$$
$$70(1-\sin^2\theta)^2 \sin^4\theta - 28(1-\sin^2\theta)\sin^6\theta$$
$$+ \sin^8\theta$$

$$= [1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta]$$
$$- 28\sin^2\theta [1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta]$$
$$+ 70\sin^4\theta [1 + \sin^4\theta - 2\sin^2\theta] - 28\sin^6\theta$$
$$+ 28\sin^8\theta + \sin^8\theta$$

$= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta - 28\sin^2\theta$
$+ 84\sin^4\theta - 84\sin^6\theta + 28\sin^8\theta$
$+ 70\sin^4\theta + 70\sin^8\theta - 140\sin^6\theta$
$- 28\sin^6\theta + 28\sin^8\theta + \sin^8\theta$

$= 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$

Express $\dfrac{\sin 6\theta}{\sin\theta}$ in terms of $\cos\theta$

Soln

By De Moivre's theorem

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

$= \cos^6\theta + 6C_1\cos^5\theta\, i\sin\theta + 6C_2\cos^4\theta\, i^2\sin^2\theta +$
$6C_3\cos^3\theta\, i^3\sin^3\theta + 6C_4\cos^2\theta\, i^4\sin^4\theta +$
$6C_5\cos\theta\, i^5\sin^5\theta + i^6\sin^6\theta$

$= \cos^6\theta + i\,6\cos^5\theta\,\cancel{i}\sin\theta - 15\cos^4\theta\sin^2\theta -$
$20\,i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + 6\cos\theta\, i\sin^5\theta$
$+ \sin^6\theta$

$= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + \ldots) + i(6\cos^5\theta\sin\theta -$
$20\cos^3\theta\sin^3\theta +$
$6\cos\theta\sin^5\theta)$

Equating the real and imaginary part

$$\sin 6\theta = 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$$

divide $(\sin\theta)$ on both side

$$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta \sin^2\theta + 6\cos\theta \sin^4\theta$$

$$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$$

$$= (\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta(1-2\cos^2\theta + \cos^4\theta)$$

$$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta - 12\cos^3\theta + 6\cos^5\theta$$

$$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$$

$$\therefore \frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$$

Expansion for $\tan[\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n]$

Soln

We know that $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)$

$$= \cos[\theta_1 + \theta_2 + \cdots + \theta_n] + i\sin[\theta_1 + \theta_2 + \cdots + \theta_n]$$

Now

$$\cos\theta_1 + i\sin\theta_1 = \cos\theta_1(1 + i\tan\theta_1)$$

$$\cos\theta_2 + i\sin\theta_2 = \cos\theta_2(1 + i\tan\theta_2)$$

$$\cos\theta_3 + i\sin\theta_3 = \cos\theta_3(1 + i\tan\theta_3)$$

Multiplying all, then we get

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)$$

$$= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n (1+i\tan\theta_1)(1+i\tan\theta_2)\cdots(1+i\tan\theta_n)$$

$$\cos[\theta_1+\theta_2+\cdots\theta_n] + i\sin[\theta_1+\theta_2\cdots+\theta_n] =$$

$$\cos\theta_1 \cos\theta_2 \cdots \cos\theta_n [1+\Sigma i\tan\theta_1 + i^2 \Sigma \tan\theta_1 \tan\theta_2 + \Sigma i^3 \tan\theta_1 \tan\theta_2 \tan\theta_3]$$

$$= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n [1+iS_1 + i^2 S_2 + i^3 S_3 + \cdots]$$

$$= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n [(1-S_2) + i[S_1 - S_3]\cdots]$$

Equalizing the real and imaginary

$$\cos[\theta_1+\theta_2+\cdots\theta_n] = \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n (1-S_2\cdots) - (1)$$

$$\sin[\theta_1+\theta_2+\cdots\theta_n) = \cos\theta_1 \cos\theta_2 \cdots \cos\theta_n (S_1 - S_3 \cdots) - (2)$$

divide (2) by (1)

$$\tan(\theta_1+\theta_2\cdots\theta_n) = \frac{S_1 - S_3 \cdots}{1 - S_2 \cdots}$$

If we take $\theta_1 = \theta_2 = \theta_3 \cdots \theta_n = \theta$.

$$\tan\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta \cdots}{1 - nC_2 \tan^2\theta \cdots}$$

Expansion for $\cos^n\theta$ and $\sin^n\theta$ in terms of multiple angles of θ.

Soln

We know that

formula.

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

$$\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$$

now

we see that

$\cos^n\theta$ and $\sin^n\theta$

- Can be expressed in terms of cosines of multiple of θ as series of multiple of θ

Formula

Let $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta$$

$$x - \frac{1}{x} = 2i\sin\theta$$

$$x^n = \cos n\theta + i\sin n\theta$$

$$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x^n - \frac{1}{x^n} = 2i\sin n\theta$$

Expansion of $\cos^n\theta$ when n is a positive integer

$$2\cos n\theta = x^n + 1/x^n$$

$$(2\cos\theta)^n = (x + 1/x)^n$$

$$2^n \cos^n\theta = x^n + nC_1\, x^{n-1} \frac{1}{x^1} + nC_2\, x^{n-2} \frac{1}{x^2} + \ldots + nC_{n-1}\, x \frac{1}{x^{n-1}} + \frac{1}{x^n}$$

$$2^n\cos^n\theta = \left(x^n + \frac{1}{x^n}\right) + nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right)$$

$$2^n\cos^n\theta = 2\cos n\theta + nC_1\, 2\cos(n-2)\theta + nC_2\, 2\cos(n-4)\theta + \ldots$$

$$2^{n-1}\cos^n\theta = \cos n\theta + nC_1 \cos(n-2)\theta + nC_2\cos(n-4)\theta + \ldots$$

Expand $\cos^6\theta$ and $\cos^5\theta$ in series of cosines of multiple of θ

Soln

Let $x = \cos\theta + i\sin\theta$

Then $2\cos\theta = x + \frac{1}{x}$

$$(2\cos\theta)^6 = (x + 1/x)^6$$

$$2^6\cos^6\theta = x^6 + 6x^5\, 1/x + 15x^4\, 1/x^2 + 20x^3 \frac{1}{x^3} + 15x^2\, 1/x^4 + 6x\, 1/x^5 + \frac{1}{x^6}$$

$$2^6\cos^6\theta = \left(x^6 + 1/x^6\right) + 6\left(x^4 + 1/x^4\right) + 15\left(x^2 + 1/x^2\right) + 20$$

$2^6 \cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 4\theta) + 20$

$2^{6-1} \cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 4\theta + 10.$

$2^5 \cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 4\theta + 10$

Expand $\underline{\cos^5\theta}$

$(2\cos\theta)^5 = (x + 1/x)^5$

$2^5 \cos^5\theta = x^5 + 5x^4 1/x + 10x^3 1/x^2 + 10x^2 1/x^3 + 5x 1/x^4 + 1/x^5$

$= (x^5 + 1/x^5) + 5(x^3 + 1/x^3) + 10(x + 1/x)$

$2^5 \cos^5\theta = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$

$2^5 \cos^5\theta = 2(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

$2^4 \cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$

$\cos^5\theta = \frac{1}{16}[\cos 5\theta + 5\cos 3\theta + 10\cos\theta]$

Expansion of $\sin^n\theta$ n is positive integer

Soln

Given that $\sin^n\theta$

$x - 1/x = 2i\sin\theta$

$(2i\sin\theta)^n = (x - 1/x)^n$

$2^n i^n \sin^n\theta = x^n - nC_1 x^{n-1} 1/x + nC_2 x^{n-2} 1/x^2 + \cdots$

$\cdots nC_{n-1} x \frac{1}{x^{n-1}}$

Case (i) 'n' is even

This number of term in the expansion is add. Sign of the term are alternatively positive and negative and the last term is positive

$$2^n(-1)^{n/2}\sin^n\theta = \left(x^n+\frac{1}{x^n}\right)-\left(nC_1\, x^{n-2}+\frac{1}{x^{n-2}}\right)+ nC_2\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+\cdots$$

$$2^n(-1)^{n/2}\sin^n\theta = 2\cos n\theta - nC_1\, 2\cos(n-2)\theta + nC_2\, 2\cos(n-4)\theta+\cdots$$

Case (ii) n is add

$$(2i\sin\theta)^n = x^n - nC_1\, x^{n-2} + nC_2\, x^{n-4}+\cdots -\frac{1}{x^n}$$

$$2^n i^n \sin^n\theta = \left(x^n-\frac{1}{x^n}\right)-nC_1\left(x^{n-2}-\frac{1}{x^{n-2}}\right)+ nC_2\left(x^{n-4}-\frac{1}{x^{n-4}}\right)\cdots$$

$$2^n i^n\sin^n\theta = 2i\sin n\theta - nC_1\, 2i\sin(n-2)\theta + nC_2\, 2i\sin(n-4)\theta\cdots$$

? $$\Rightarrow 2^{n-1} i^{n-1}\sin^n\theta = \sin n\theta - nC_1\sin(n-2)\theta + nC_2\sin(n-4)\theta\cdots$$

Expand $\sin^3\theta \cos^5\theta$ in a series are sines multiple of θ

<u>Soln</u>

$$x = \cos\theta + i\sin\theta$$

$$1/x = \cos\theta - i\sin\theta$$

$$x - \frac{1}{x} = 2i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta$$

$$(2i\sin\theta)^3 (2\cos\theta)^5 = \left(x - \frac{1}{x}\right)^3 \left(x + \frac{1}{x}\right)^5$$

$$= \left(x - \frac{1}{x}\right)^3 \left(x + \frac{1}{x}\right)^3 \left(x + \frac{1}{x}\right)^2$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3 \left(x + \frac{1}{x}\right)^2$$

$$= \left(x^6 - 3x^{\cancel{4}\,2}\frac{1}{x^{\cancel{2}}} + 3x^{\cancel{2}}\frac{1}{x^{\cancel{4}\,2}} - \frac{1}{x^6}\right)\left(x^2 + 2 + \frac{1}{x^2}\right)$$

$$= \left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)\left(x^2 + 2 + \frac{1}{x^2}\right)$$

$$= x^8 - 3x^4 + 3 - \frac{1}{x^4} + 2x^6 - 6x^2 + \frac{6}{x^2} - \frac{2}{x^6} + x^4 - 3 + \frac{3}{x^4} - \frac{1}{x^8}$$

$$= \left(x^8 - \frac{1}{x^8}\right) + 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) - 6\left(x^2 - \frac{1}{x^2}\right)$$

$$2^8 i^3 \sin^3\theta \cos^5\theta = 2i\sin 8\theta + 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(i\sin 2\theta)$$

$$2^{\cancel{8}\,7} i^{\cancel{3}\,2} \sin^3\theta \cos^5\theta = \cancel{2}\cancel{i}\,[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta]$$

$$\sin^3\theta \cos^5\theta = \frac{1}{2^7}[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta$$

Expand $\sin^4\theta \cos^2\theta$ in a series of cosines of multiples of θ

Soln $\sin^4\theta \cos^2\theta$

$x - \frac{1}{x} = 2i\sin\theta$

$x + \frac{1}{x} = 2\cos\theta$

$(2i\sin\theta)^4 (2\cos\theta)^2 = (x - 1/x)^4 (x + 1/x)^2$

$= \left(x - \frac{1}{x}\right)^2 (x - 1/x)^2 (x + 1/x)^2$

$= (x - 1/x)^2 \left(x^2 - \frac{1}{x^2}\right)^2$

$= (x^2 - 2 + 1/x^2)(x^4 - 2 + 1/x^4)$

$= x^6 - 2x^4 + x^2 - 2x^2 + 4 - 2/x^2 + 1/x^2 - \frac{2}{x^4} + 1/x^6$

$= \left(x^6 + \frac{1}{x^6}\right) - 2\left(x^4 + 1/x^4\right) - \left(x^2 + 1/x^2\right) + 4$

$2^6 i^4 \sin^4 \cos^2\theta = 2\cos 6\theta - 2(2\cos 4\theta) - (2\cos 2\theta) + 4$

$2^{\cancel{6}\,5} \sin^4\theta \cos^2\theta = \cancel{2}[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$

$\sin^4\theta \cos^2\theta = \frac{1}{2^5}[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$

Expansion of $\sin\theta$ and $\cos\theta$ in a series ascending power of θ

Soln $\cos n\alpha = \cos^n\alpha - \frac{n(n-1)}{2!}\cos^{n-2}\alpha \sin^2\alpha +$

put $n\alpha = \theta$ that $\alpha = \frac{\theta}{n}$

$$\cos\theta = \cos^n \theta/n - \frac{n(n-1)}{2!}\cos^{n-2}\theta/n \sin^2\theta/n +$$

$$\frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta/n \sin^4\theta/n + \cdots$$

$$= \cos^n\theta/n - \frac{n(n-1)}{2!}\frac{\theta^2}{n^2}\cos^{n-2}\theta/n\left[\frac{\sin\theta/n}{\theta/n}\right]^2 +$$

$$\frac{n(n-1)(n-2)(n-3)}{4!}\frac{\theta^4}{n^4}\cos^{n-4}\frac{\theta}{n}\left[\frac{\sin\theta/n}{\theta/n}\right] + \cdots$$

$$\cos\theta = \cos^n\theta/n - \frac{(1-1/n)}{2!}\theta^2\cos^{n-2}\theta/n\left(\frac{\sin\theta/n}{\theta/n}\right)^2 +$$

$$\frac{(1-1/n)(1-2/n)(1-3/n)}{4!}\theta^4\cos^{n-4}\theta/n\left[\frac{\sin\theta/n}{\theta/n}\right]^4 + \cdots$$

As $n = \alpha$, $\theta/n \to 0$ and $\frac{\sin\theta/n}{\theta/n} \to 1$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots$$

Similarly

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots$$

Corollary

$$\tan\theta = \frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots}{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots}$$

$$= \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right)\left[1 - \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\right)\right]^{-1}$$

$$= \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right)\left[1 + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\right) + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\right)^2 \cdots\right.$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}\right)\left[1 + \left(\frac{\theta^2}{2} - \frac{\theta^4}{24}\right) + \frac{\theta^4}{4}\right]$$

$$= \theta + \frac{\theta^3}{2} - \frac{\theta^5}{24} + \frac{\theta^5}{4} - \frac{\theta^3}{6} - \frac{\theta^5}{12} + \frac{\theta^5}{120}$$

$$= \theta + \frac{\theta^3}{2} - \frac{\theta^3}{6} + \frac{\theta^5}{4} - \frac{\theta^5}{24} + \frac{\theta^5}{120} - \frac{\theta^5}{12}$$

$$= \theta + \frac{3\theta^3 - \theta^3}{6} + \frac{30\theta^5 - 5\theta^5 + \theta^5 - 10\theta^5}{120}$$

$$= \theta + \frac{\cancel{2}\theta^3}{\cancel{6}\,3} + \frac{\cancel{16}^{\,2}\theta^5}{\cancel{120}_{\,15}}$$

$$= \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15}$$

Find $\underset{\theta \to 0}{\text{limit}} \dfrac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta)}$

Soln

We have

$$\underset{\theta \to 0}{\text{lt}} \frac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta)}$$

$$= \frac{n\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\cdots\right) - \left(n\theta - \frac{n^3\theta^3}{3!} + \frac{n^5\theta^5}{5!} + \cdots\right]}{\theta\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \left[n\theta - \frac{n^3\theta^3}{n!} \cdots\right)\right]}$$

$$= \frac{\frac{\theta^3}{3!}(n^3-n)+\frac{\theta^5}{5!}(n-n^5)}{\theta\left[1-n\theta-\frac{\theta^2}{2!}+\frac{n^3\theta^3}{3!}+\frac{\theta^4}{4!}\cdots\right]}$$

when $\theta \to 0$ limit becomes 0

Find the $\lim\limits_{\theta \to 0} \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$

Soln

$$= \lim_{\theta \to 0} \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$$

$$= \lim_{\theta \to 0} \frac{\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}-\frac{1}{\cos\theta}+1}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}$$

$$= \lim_{\theta \to 0} \frac{\left(\theta-\frac{\theta^3}{3!}\cdots\right)-\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\cdots\right)+1}{\left(\theta-\frac{\theta^3}{3!}+\cdots\right)+1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-1}$$

$$= \lim_{\theta \to 0} \frac{\theta+\frac{\theta^2}{2!}-\frac{\theta^3}{3!}-\frac{\theta^4}{4!}}{\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\frac{\theta^4}{4!}}$$

$$= \lim_{\theta \to 0} \frac{\cancel{\theta}\left(1+\frac{\theta}{2!}-\frac{\theta^2}{3!}-\frac{\theta^3}{4!}\right)}{\cancel{\theta}\left(1-\frac{\theta}{2!}-\frac{\theta^2}{3!}+\frac{\theta^3}{4!}\right)}$$

$= 1$ as $\theta \to 0$

If $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$ show that $\theta = 1^\circ 58'$ approximately.

Soln

$\frac{\sin\theta}{\theta} \longrightarrow 1$ as $\theta \to 0$ θ is measured in radius

As $\frac{5045}{5046}$ is approximately equal to 1

θ is very small

$$\frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}}{\theta}$$

$$\frac{\sin\theta}{\theta} = \frac{\cancel{\theta}\left(1 - \theta^2/3! + \theta^4/5!\right)}{\cancel{\theta}}$$

$$\frac{\sin\theta}{\theta} \Rightarrow 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} = \frac{5045}{5046}$$

$$\frac{\theta^2}{3!} - \frac{\theta^4}{5!} = 1 - \frac{5045}{5046}$$

$$\frac{\theta^2}{6} - \frac{\theta^4}{120} = \frac{1}{5046}$$

First degree approximately

$$\frac{\theta^2}{6} = 1/5046$$

$$\theta^2 = \frac{\cancel{6}\,1}{\cancel{5046}\ 841}$$

$$\theta^2 = 1/841$$

$$\theta = 1/29$$

$\theta = 1/29$ of a radius

$= 1/29$ of $57^\circ 17' 44.8''$

$= 1^\circ 58'$ approximately.

UNIT- V

HYPERBOLIC FUNCTION

Hyperbolic functions:

Definition: The hyperbolic functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} ; \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} ; \quad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} ; \quad \operatorname{sech} x = \frac{1}{\cosh x}$$

<u>Note</u>: The following are immediate consequences of definition.

1. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

We observe that $\cosh x > 1$ for all x

3. $\cosh 0 = 1$ and $\sinh 0 = 0$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Relations between Hyperbolic function :-

(i) $\cos^2 hx - \sin^2 hx = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{1}{4}\left[(e^x + e^{-x})^2 - (e^x - e^{-x})^2\right]$

$= \frac{1}{4}\left[\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}\right]$

$= \frac{1}{4}[4]$

$= 1$

(ii) $2 \sin hx \cosh x = 2\left(\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)\right)$

$= \frac{e^{2x} - e^{-2x}}{2}$

$= \sin h 2x$

(iii) $\cos^2 hx + \sin^2 hx = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{1}{4}\left((e^x + e^{-2})^2 + (e^x - e^{-x})^2\right)$

$= \frac{1}{4}\left[e^{2x} + e^{-2x} + \cancel{2} + e^{2x} + e^{-2x} - \cancel{2}\right]$

$= \frac{2}{4}\left[e^{2x} + e^{-2x}\right]$

$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$

The series of $\sinh x$ and $\cosh x$ are derived below.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \longrightarrow (1)$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \longrightarrow (2)$$

Substracting both (1) - (2)

$$e^x - e^{-x} = 0 + 2x + \frac{2x^3}{3!} + \dots$$

$$e^x - e^{-x} = 2\left(x + \frac{x^3}{3!}\right)$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!}$$

$$\sinh x = x + \frac{x^3}{3!} + \dots$$

add (1) and (2)

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots$$

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right)$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Relations

$$\cosh 2x = 2\cos^2 hx - 1$$

$$\cosh 2x = 1 + 2\sin^2 hx$$

$$\cos^2 hx = \frac{1}{2}(\cosh 2x + 1)$$

$$\sin^2 hx = \tfrac{1}{2}(\cosh 2x - 1)$$

$$\sinh(i\theta) = i \sin h\theta$$

$$\cosh(i\theta) = \cos h\theta$$

$$\tanh(i\theta) = i \tan h\theta$$

Using these relations, we can derive relations between hyperbolic functions corresponding to relations between circular trignometric functions.

Ex-1

$$\sin^2\theta + \cos^2\theta = 1$$

Put $\theta = ix$

$$\sin^2(ix) + \cos^2(ix) = 1$$

$$(i \sinh x)^2 + (\cosh x)^2 = 1$$

$$i^2 \sin^2 hx + \cos^2 hx = 1$$

$$\cos^2 hx - \sin^2 hx = 1$$

Ex-2

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Put $\theta = ix$

$$\cos 2(ix) = \cos^2(ix) - \sin^2(ix)$$

$$\cosh 2x = \cos^2 hx - i^2 \sin^2 hx$$

$$\cosh 2x = \cos^2 hx + \sin^2 hx$$

Ex-3

$\sin 2\theta = 2\sin\theta \cos\theta$ circular function

Put $\theta = ix$

$$\sin 2(ix) = 2\sin(ix)\cos(ix)$$

$$i \sin 2hx = 2i \sin hx \cos hx$$

$$\sin 2hx = 2\sin hx \cosh x$$

Ex-4 $1 + \tan^2\theta = \sec^2\theta$ circular function

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Put $\theta = ix$

$$1 + \frac{\sin^2(ix)}{\cos^2(ix)} = \frac{1}{\cos^2(ix)}$$

$$1 + \frac{i \sin^2 hx}{\cos^2 hx} = \frac{1}{\cos^2 hx}$$

$$1 - \tan^2 hx = \sec^2 hx$$

Ex-5

$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$

circular function.

Put $\theta = ix$ $\phi = iy$

$\sin(ix + iy) = \sin(ix)\cos(iy) + \cos(ix)\sin(iy)$

$i \sinh(x+y) = i \sinh x \cosh y + \cosh x \, i \sinh y$

$i \sinh(x+y) = i\,[\sinh x \cosh y + \cosh x \sinh y]$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Ex-6

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ circular function

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2\dfrac{\sin\theta}{\cos\theta}}{1-\dfrac{\sin^2\theta}{\cos^2\theta}}$$

$\Rightarrow$ Put $\theta = ix$

$$\frac{\sin 2ix}{\cos 2(ix)} = \frac{2\dfrac{\sin(ix)}{\cos(ix)}}{1-\dfrac{\sin^2(ix)}{\cos^2(ix)}}$$

$$\frac{i\sin 2hx}{\cos 2hx} = \frac{2\dfrac{i\sin hx}{\cos hx}}{1+\dfrac{\sin^2 hx}{\cos^2 hx}}$$

$$\cancel{i} \tan 2hx = \frac{2\cancel{i} \tanh x}{1+\tan^2 hx}$$

$$\tan 2hx = \frac{2 \tan hx}{1+\tan^2 hx}$$

INVERSE HYPERBOLIC FUNCTION.

Consider the function $y = \sin hx$.
This is a 1-1 onto map from $R \to R$.
Given any $y \in R$ there exists unique x such that $\sin hx = y$.
$\therefore$ we define $x = \sinh^{-1} y$

Similarly $y = \cos hx$ is a map from $R \to [1, \infty)$.
Both x and $-x$ have the same image under $\cos hx$. Hence, given any $y \in [1, \infty)$, we can find unique positive x such that $\cosh x = y$. We define $x = \cosh^{-1} y$ and x is called the principal value of $\cosh^{-1} y$.

The function $y = \tan hx$ is a map from $R \to (-1, 1)$.
Given any $y \in R$ there exists unique x such that $\tanh x = y$. We define $x = \tan h^{-1} y$.

Theorem

$$\sinh^{-1} x = \log_e\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \log_e\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

Proof

(i) Let $y = \sinh^{-1} x$

$$\sinh y = x$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y}$$

$$2x = \frac{e^{2y} - 1}{e^y}$$

$$2xe^y = e^{2y} - 1$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4(-1)(1)}}{2}$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Positive only

$$e^y = x + \sqrt{x^2+1}$$

$$y = \log_e(x + \sqrt{x^2+1})$$

(ii) Let $\cosh^{-1} x = y$

$$x = \cosh y$$

$$x = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + 1/e^y$$

$$2x = \frac{e^{2y}+1}{e^y}$$

$$2x e^y = e^{2y} + 1$$

$$e^{2y} - 2x e^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4(1)(1)}}{2}$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

Positive only

log on both side

$$y = \log_e (x + \sqrt{x^2-1})$$

(iii) Let $y = \tan h^{-1} x$

$$\tan hy = x$$

$$x = \sinh x / \cosh y$$

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$x = \frac{e^y - 1/e^y}{e^y + 1/e^y}$$

$$x = \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$x e^{2y} + x = e^{2y} - 1$$

$$1 + x = e^{2y} - x e^{2y}$$

$$1 + x = e^{2y}(1-x)$$

$$e^{2y} = \frac{1+x}{1-x}$$

log on both side

$$2y = \log_e \left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2} \log_e \left(\frac{1+x}{1-x}\right)$$

4 $\tan A = \tan\alpha \tanh\beta$

$\tan B = \cot\alpha \tanh\beta$ Prove that

$\tan(A+B) = \sinh 2\beta \operatorname{cosec} 2\alpha$

Soln

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\tan\alpha \tanh\beta + \cot\alpha \tanh\beta}{1 - (\tan\alpha \tanh\beta)(\cot\alpha \tanh\beta)}$$

$$= \frac{\tanh\beta(\tan\alpha + \cot\alpha)}{1 - \tanh^2\beta}$$

$$= \frac{\cancel{\tanh\beta}\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right)}{\cancel{\tanh\beta}\left(\frac{1}{\tanh\beta} - \tanh\beta\right)}$$

$$= \frac{\frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha}}{\frac{\cosh\beta}{\sinh\beta} - \frac{\sinh\beta}{\cosh\beta}}$$

$$= \frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha} \times \frac{\sinh\beta\cosh\beta}{\cosh^2\beta - \sinh^2\beta}$$

$$= \frac{(1)\sinh\beta\cosh\beta}{\sin\alpha\cos\alpha(1)}$$

$$= \frac{\cancel{\tfrac{1}{2}}\sinh 2\beta}{\cancel{\tfrac{1}{2}}\sin 2\alpha}$$

$$= \sinh 2\beta \operatorname{cosec} 2\alpha$$

Proved that $\tan(A+B) = \sinh 2\beta \operatorname{cosec} 2\alpha$

Express $\cosh^6\theta$ in terms of hyperbolic cosines of multiple of θ.

Soln

We have

$$\cos\theta + i\sin\theta = \cos(x+iy)$$

$$\cos\theta + i\sin\theta = \cos x\cos(iy) - \sin x\sin(iy)$$

$$\cos\theta + i\sin\theta = \cos x\cosh y - i\sin x\sinh y$$

equating real and imaginary part

$$\cos\theta = \cos x\cosh y$$

$$\sin\theta = -\sin x\sinh y$$

Squaring adding both side

$$\cos^2\theta + \sin^2\theta = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y$$

$$1 = \cos^2 x \cosh^2 y + (1-\cos^2 x)\sinh^2 y$$

$$1 = \cos^2 x \cosh^2 y + \sinh^2 y - \sinh^2 y \cos^2 x$$

$$1 = \cos^2 x \left[\cosh^2 y - \sin^2 hy\right] + \sinh^2 y$$

$$1 = \cos^2 x + \sinh^2 y$$

$$1 = \frac{1}{2}\left[1+\cos 2x\right] + \frac{1}{2}\left[\cos 2hy - 1\right]$$

$$2 = \cancel{1} + \cos 2x + \cos 2hy - \cancel{1}$$

$$2 = \cancel{1} + \cos 2x + \cos 2hy - \cancel{1}$$

$\therefore$ proved that

$$\cos 2x + \cos 2hy = 2$$

The Separate into real and imaginery parts $\tanh(1+i)$

<u>Soln</u>

It's know that

$$\tan(ix) = i \tanh x$$

Put $x = (1+i)$

$$= i \tanh(1+i)$$

$$= \tan i(1+i)$$

$$= \tan(i-1)$$

$$\therefore\ i\tanh(1+i) = \frac{\sin(i-1)}{\cos(i-1)}$$

$$= \frac{\sin(i-1)}{\cos(i-1)} \times \frac{\cos(i+1)}{\cos(i+1)}$$

$$= \frac{\sin(i-1)\cos(i+1)}{\cos(i+1)\cos(i-1)}$$

$$= \frac{2\cos(i+1)\sin(i-1)}{2\cos(i+1)\cos(i-1)}$$

$$= \frac{\sin(i+1+i-1) - \sin(i+1-i+1)}{\cos(i+1+i-1) + \cos(i+1-i+1)}$$

$$= \cancel{\cos}\ \frac{\sin(2i) - \sin(2)}{\cos(2i) + \cos(2)}$$

$$= \frac{i\sinh 2 - \sin 2}{\cosh 2 + \cos 2}$$

$$\tan(1+i) = \frac{i\sinh 2 - \sin 2}{i(\cosh 2 + \cos 2)}$$

Separate into real and imaginary parts $\tan^{-1}(x+iy)$

Soln

$$\tan^{-1}(x+iy) = \alpha + i\beta$$

$$x+iy = \tan(\alpha+i\beta)$$

we easily see that

$$\tan(\alpha - i\beta) = x - iy$$

Real part: ~~tan 2α~~

$$\tan 2\alpha = \tan(\alpha + i\beta + \alpha - i\beta)$$

$$= \frac{\tan(\alpha+i\beta) + \tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$= \frac{x+iy + x-iy}{1-(x+iy)(x-iy)}$$

$$\tan 2\alpha = \frac{2x}{1-(x^2+y^2)}$$

$$2\alpha = \tan^{-1}\left[\frac{2x}{1-(x^2+y^2)}\right]$$

$$\alpha = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1-(x^2+y^2)}\right]$$

Imaginary Part

$$\tan 2i\beta = \tan(\alpha + i\beta - \alpha + i\beta)$$

$$= \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta)\tan(\alpha - i\beta)}$$

$$= \frac{x + iy - (x - iy)}{1 + (x + iy)(x - iy)}$$

$$= \frac{\cancel{x} + iy - \cancel{x} + iy}{1 + x^2 + y^2}$$

$$\tan 2i\beta = \frac{2iy}{1+x^2+y^2}$$

$$\beta = \frac{1}{2i}\tan^{-1}\left(\frac{2iy}{1+x^2+y^2}\right)$$

Alternate

$$i\tan 2h\beta = \frac{2iy}{1+x^2+y^2}$$

$$\beta = \frac{1}{2}\tan h^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

If $\cosh u = \sec\theta$ Show that $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

Soln

$$\cosh u = \sec\theta$$

$$u = \cosh^{-1}(\sec\theta) \quad \left[\because \cosh^{-1}x = \log_e\left(x+\sqrt{x^2-1}\right)\right]$$

$$u = \log_e\left[\sec\theta + \sqrt{\sec^2\theta - 1}\right]$$

$$u = \log_e\left[\sec\theta + \sqrt{\tan^2\theta}\right)$$

$$u = \log_e\left\{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right]$$

$$u = \log_e\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$$u = \log_e\left[1+\frac{2\tan\theta/2}{1+\tan^2\theta/2}\right] \div \left[\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right]$$

$$u = \log_e\left(\frac{1+\tan^2\theta/2+2\tan\theta/2}{\cancel{1+\tan^2\theta/2}}\right) \times \left(\frac{\cancel{1+\tan^2\theta/2}}{1-\tan^2\theta/2}\right)$$

$$u = \log_e\frac{(1+\tan\theta/2)^{\cancel{2}}}{\cancel{(1+\tan\theta/2)}(1-\tan\theta/2)}$$

$$u = \log_e\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)$$

$$u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

29/1/08

Unit ...

Logrithom of Complex Quantities

If u & z are any 2 complex quantites such that $z = e^{u}$

Then u is called the Log z and we write $u = \log_e z$ (or)

simply $\boxed{u = \log z}$

B.W

To find $\log_e(x+iy)$

$$\log_e(x+iy) = \alpha + i\beta \quad \text{—— (I)}$$

$$x + iy = e^{\alpha + i\beta}$$

$$= e^{\alpha} \cdot e^{i\beta} \qquad e^{i\beta} = (\cos\beta + i\sin\beta)$$

$$= e^{\alpha}(\cos\beta + i\sin\beta)$$

$$= e^{\alpha}\cos\beta + i\, e^{\alpha}\sin\beta$$

Eqv./. Real & Imagy. parts

$$x = e^{\alpha}\cos\beta \quad \text{—— (1)}$$

$$y = e^{\alpha}\sin\beta \quad \text{—— (2)}$$

Squar./. & Adding eqn (1) & (2)

$$x^2 + y^2 = e^{2\alpha}(\cos^2\beta + \sin^2\beta)$$

$$x^2 + y^2 = e^{2\alpha}$$

$$e^{2\alpha} = x^2 + y^2$$

$$2\alpha = \log (x^2 + y^2)$$

$$\alpha = \frac{1}{2} \log (x^2 + y^2)$$

Diving eqn (2) / (1)

$$\frac{(2)}{(1)} \Rightarrow \frac{y}{x} = \frac{\sin B}{\cos B} = \tan B$$

$$B = \tan^{-1} (y/x)$$

Using in (I)

$$\log_e (x + iy) = \frac{1}{2} \log (x^2 + y^2) + i \tan^{-1} (y/x)$$

$$= \log (x^2 + y^2)^{1/2} + i \tan^{-1} (y/x)$$

$$= \log r + \theta$$

Where $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} (y/x)$

B. W : 2

Log

General Value of $\log_e (x + iy)$

$$\log_e (x + iy) = \alpha + iB \quad \text{— (I)}$$

$$x + iy = e^{\alpha + iB}$$

$$= e^{\alpha} \cdot e^{iB}$$

$$= e^{\alpha} \cdot \cancel{e}(\cos B + i \sin B)$$

$$= e^{\alpha} [\cos (2m\pi + B) + i \sin (2n\pi + B)]$$

$= e^{\alpha} \cdot e^{i(2n\pi + \beta)}$

$= e^{\alpha} \cdot e^{2n\pi i + i\beta}$

$x + iy = e^{\alpha + i\beta + 2n\pi i}$.

$\therefore$ $\alpha + i\beta + 2n\pi$ is the value of $\log(x+iy)$.

$\therefore$ The general value of $\log(x+iy)$ is denoted by

$\mathrm{Log}(x+iy) = \alpha + i\beta + 2n\pi i$,

where n is any integer.

$\therefore$ $\boxed{\mathrm{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi i}$

Result:

i) Put $y = 0$

$\mathrm{Log}\, x = \frac{1}{2}\log x^2 + 0 + 2n\pi i$

$= \log x + 2n\pi i$

ii) Let $y = 0$ and x, be +ve. ~~same~~ say

If $\mathrm{Log}(x_1 + iy) = \alpha + i\beta$

$x_1 = e^{\alpha}\cos\beta$

$y_1 = e^{\alpha}\sin\beta$

$e^{\alpha} \cos\beta = -x_1$

$e^{\alpha} \sin\beta = 0$

Squaring & Adding

$e^{2\alpha} = x_1^2$

$\cancel{2\alpha = \log x_1^2}$

$e^{\alpha} = x_1$

$x_1 \cos\beta = -x_1$ $\qquad$ $x_1 \sin\beta = 0$

$\cos\beta = -1$ $\qquad$ $\sin\beta = 0$

$\beta = \pi$

$\therefore \log(-x_1) = \log x_1 + i\pi$

$\therefore \operatorname{Log}(-x_1) = \log x_1 + i\pi + 2n\pi i$

$= \log x_1 + i(2n+1)\pi$

Put $x = 0$ in (II)

$\operatorname{Log}(iy) = \frac{1}{2}\log y^2 + i\tan^{-1}(\alpha) + 2n\pi i$

$= \log y + i\,\pi/2 + 2n\pi i$

$= \log y + i\pi(2n + 1/2)$

Find $\operatorname{Log}(1-i)$

$\operatorname{Log}(1-i) = \frac{1}{2}\log(1^2 + (-1)^2) + i\tan(-1/1) + 2n\pi i$

$= \frac{1}{2}\log(2) + i\tan^{-1}(-1) + 2n\pi i$

$= \frac{1}{2} \log 2 + i \, {}^{3\pi}/_{4} + 2n\pi i$

$= \frac{1}{2} \log 2 + i \left(\frac{3\pi}{4} + 2n\pi\right)$

If $\log \sin(\theta + i\varphi) = L + iB$. Prove that $2e^{2L} = \cosh 2\varphi - \cos 2\theta$.

Solution:

$L + iB = \log \sin(\theta + i\varphi)$

$= \log[\sin\theta \cos i\varphi + \cos\theta \sin i\varphi]$

$= \log[\sin\theta \cosh\varphi + i \cos\theta \sinh\varphi]$

$L + iB = \frac{1}{2} \log(\sin^2\theta \cosh^2\varphi + \cos^2\theta \sinh^2\varphi] + i \tan^{-1}\left(\frac{\cos\theta \sinh\varphi}{\sin\theta \cosh\varphi}\right)$

$L = \frac{1}{2} \log(\sin^2\theta \cosh^2\varphi + \cos^2\theta \sinh^2\varphi)$

$= \log(\sin^2\theta \cosh^2\varphi + \cos^2\theta \sinh^2\varphi]$

$e^{2L} = \sin^2\theta \cosh^2\varphi + \cos^2\theta \sinh^2\varphi$

$= \left(\frac{1 - \cos 2\theta}{2}\right) \cosh^2\varphi + \left(\frac{1 + \cos 2\theta}{2}\right) \sinh^2\varphi$

$= \frac{1}{2}\left[\underbrace{(\cosh^2\varphi + \sinh^2\varphi)}_{\cosh 2\varphi} - \cos 2\theta (\cosh^2\varphi - \sinh^2\varphi)\right.$

$e^{2L} = \frac{1}{2}[\cosh 2\varphi - \cos 2\theta]$

$$2e^{2L} = \cosh 2\varphi - \cos 2\theta$$

Hence the proof.

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Reduce the expantion of $\tan^{-1} x$ in powers of x from the expantion of $\log(a+ib)$

$$\log(a+ib) = \tfrac{1}{2}\log(a^2+b^2) + i\tan^{-1}(b/a)$$

Put $a=1$, $b=x$

$$\log(1+ix) = \tfrac{1}{2}\log(1+x^2) + i\tan^{-1}(x/1)$$

$$= \tfrac{1}{2}\log(1+x^2) + i\tan^{-1} x$$

$\tan^{-1} x$ is imaginary part of $\log(1+ix)$

$$\tan^{-1} x = \text{Imaginary part of } \log(1+ix)$$

$$= \text{Imag. part of } ix - \frac{(ix)^2}{2} + \frac{(ix)^3}{3} - \frac{(ix)^4}{4} + \frac{(ix)^5}{5} \cdots$$

$$= \text{Im. part of } x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

Reduce $(\alpha+i\beta)^{x+iy}$ to the form $A+iB$

Solution:

$$(\alpha+i\beta)^{(x+iy)} = e^{\log(\alpha+i\beta)^{(x+iy)}}$$

$$= e^{(x+iy)\operatorname{Log}(\alpha+i\beta)}$$

$$= e^{(x+iy)\left\{\frac{1}{2}\log(\alpha^2+\beta^2)+i\tan^{-1}(\beta/\alpha)+2n\pi i\right\}}$$

$$= e^{(x+iy)\left\{\log(\alpha^2+\beta^2)^{1/2}+i\tan^{-1}(\beta/\alpha)+2n\pi i\right\}}$$

$$= e^{(x+iy)\left\{\log r+i\theta+2n\pi i\right\}}$$

Where $r = \sqrt{\alpha^2+\beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$

$$= e^{\{x\log r+ix(\theta+2n\pi i)\}+\{iy\log r-y(\theta+2n\pi)\}}$$

$$= e^{x\log r-y(\theta+2n\pi)} \cdot e^{i\{y\log r+x(\theta+2n\pi)\}}$$

$$= e^{x\log r-y(\theta+2n\pi)}\left[\cos(y\log r+x(\theta+2n\pi)) + i\sin(y\log r+x(\theta+2n\pi))\right]$$

$$A = e^{x\log r-y(\theta+2n\pi)}\cos(y\log r+x(\theta+2n\pi))$$

$$B = e^{x\log r-y(\theta+2n\pi)}\sin(y\log r+x(\theta+2n\pi))$$

S.T $\log_i i = \frac{4n+1}{4m+1}$ where n & m are integers

Proof

Let $\log_i i = x + iy$

$$i = i^{(x+iy)}$$

Taking the general Logarithm on both sides

$$\text{Log } i = \text{Log } i^{(x+iy)}$$

$$= (x+iy) \text{ Log } i$$

$$(x+iy) = \frac{\text{Log}_e i}{\text{Log}_e i}$$

$$= \frac{\frac{1}{2}\log(0+1) + i\tan^{-1}(\alpha) + 2n\pi i}{\frac{1}{2}\log(0+1) + i\tan^{-1}(\alpha) + 2m\pi i}$$

$$= \frac{i\left(\pi/2 + 2n\pi\right)}{i\left(\pi/2 + 2m\pi\right)}$$

$$= \frac{(1/2 + 2n)}{(1/2 + 2m)}$$

$$x + iy = \frac{4n+1}{4m+1}$$

Find the value of $\log(1+i)$

Solution

w.k.t

$$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x) + 2n\pi i$$

$$\text{Log}(1+i) = \frac{1}{2}\log(1+1) + i\tan^{-1}(1/1) + 2n\pi i$$

$$= \frac{1}{2}\log 2 + i\tan^{-1}(1) + 2n\pi i$$

$$= \frac{1}{2}\log 2 + i\cdot\pi/4 + 2n\pi i$$

$$= \frac{1}{2}\log 2 + i\pi\left(\frac{1}{4} + 2n\right)$$

Find the value of $\text{Log}(4+3i)$

w.k.t

$$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x) + 2n\pi i$$

$$\text{Log}(4+3i) = \frac{1}{2}\log(16+9) + i\tan^{-1}(3/4) + 2n\pi i$$

$$= \frac{1}{2}\log 25 + i\tan^{-1}(3/4) + 2n\pi i$$

$$= \log 5 + i\tan^{-1}(3/4) + 2n\pi i$$

$$= \log 5 + i\, 36^{\circ}86 + 2n\pi i$$

Find value of $\text{Log}(\sqrt{3}+i)$

w.k.t

$$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2 + i\tan^{-1}(y/x) + 2n\pi i$$

$$\operatorname{Log}(\sqrt{3}+i) = \frac{1}{2}\log(3+1) + i\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + 2n\pi i$$

$$= \frac{1}{2}\log 4 + i\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + 2n\pi i$$

$$= \log 2 + i\,\frac{\pi}{6} + 2n\pi i$$

$$= \log 2 + i\left(\frac{\pi}{6} + 2n\pi\right)$$

Practise Problem

$$\operatorname{Log}(1+i)^{i}$$

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